AN OPEN QUESTION ON THE CONVERGENCE OF CERTAIN RELAXATION METHODS

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Abstract

The phenomenon discussed in this talk is shared by several dual iterative methods, such as the methods proposed by Kaczmarz, Hildreth, Agmon, Cryer, Mangasarian, Herman, Censor, and others. Casted as "row-action methods" these algorithms have been proved as a useful tool for solving large convex feasibility problems that arise in medical image reconstruction from projections, in inverse problems in radiation therapy, and in groundwater inverse problems. Consider a typical convex feasibility problem

$$\text{minimize } \ 1/2\|\mathbf{x}\|_2^2 \quad \text{subject to} \quad A\mathbf{x} \geq \mathbf{b},$$

whose dual problem has the form

$$\text{maximize } \ \mathbf{b}^T\mathbf{y} - 1/2\|A^T\mathbf{y}\|_2^2 \quad \text{subject to} \quad \mathbf{y} \geq \mathbf{0}.$$

If the feasible region is not empty both problems are solvable and the solution points satisfy $\mathbf{x}^* = A^T\mathbf{y}^*$. Maximizing the dual objective function by changing one variable at a time results in a row-action version of Hildreth's method. Mathematically this method can be viewed as a "restricted" SOR method for solving the system

$$AA^T\mathbf{y} = \mathbf{b} \ \text{ and } \ \mathbf{y} \geq \mathbf{0},$$

in which the variables are restricted to stay nonnegative. Let $\mathbf{y}_k \geq \mathbf{0}$ denote the current estimate of the solution at the end of the kth iteration, $k = 1, 2, \ldots,$ and let $\mathbf{x}_k = A^T\mathbf{y}_k$ denote the corresponding primal estimate. If the feasible region is not empty both sequences converge. The primal sequence converges toward $\mathbf{x}^*$, the unique solution of (9). The "final" rate of convergence (which is achieved as $\mathbf{x}_k$ approaches $\mathbf{x}^*$) depends on the active constraints at $\mathbf{x}^*$. Recent experiments expose cases where the "final" rate of convergence is fast, but the algorithm performs thousands of iterations before reaching the final stage. A close inspection of these cases reveals a highly surprising phenomenon: We see that for many consecutive iterations the sequence $\{\mathbf{y}_k\}$ follows the rule

$$\mathbf{y}_k = \mathbf{u}_k + k\mathbf{v},$$

where $\mathbf{v} \in Null(A^T)$ is a fixed vector that satisfies $\mathbf{b}^T\mathbf{v} > 0$, and $\{\mathbf{u}_k\}$ is a fast converging sequence. Consequently the sequence $\{\mathbf{x}_k\}$ actually "stuck" in the same point for several consecutive iterations. This "false convergence" phenomenon is repeated several times before reaching the vicinity of $\mathbf{x}^*$. The explanation relies on the behaviour of the SOR method when solving an inconsistent system. If $\mathbf{v} \geq \mathbf{0}$ the system $A\mathbf{x} \geq \mathbf{b}$ is inconsistent, but the sequence $\{\mathbf{x}_k\}$ converges in spite of that fact! Indeed preliminary experiments suggest that the sequence $\{\mathbf{x}_k\}$ converges even when the feasible region is empty. However, the validity of this conjecture remains an open problem.