A NEW PARTIAL ORDERING OF NONSINGULAR TOTALLY NONNEGATIVE AND OSCILLATORY MATRICES

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Abstract

Every nonsingular totally nonnegative matrix (i. e., matrix all square submatrices of which have nonnegative determinants) can be written in the "standard" form

$$B_1 B_2 \dots B_{n-1} D C_{n-1} C_{n-2} \dots C_1,$$

where each B_k (resp., C_k) is a lower (resp., upper) triangular tridiagonal nonnegative matrix with ones along the diagonal and zeros in the first $n-k-1$ positions in the first subdiagonal (resp., superdiagonal) and D is a diagonal matrix with positive diagonal entries.

This standard form need not be unique. We say that A_1 is *majorized* by A_2, $A_1 \prec A_2$, if for some standard forms for A_1 and A_2, the corresponding B_i's and C_j's in A_2 have non-zero entries in all positions in which the B_i's and C_j's in A_1 have non-zero entries. We thus obtain a partial ordering of nonsingular totally nonnegative matrices.

In a recent paper, T.L. Markham and the author have shown (even in a more general non-commutative setting) that $A_1 \prec A_2$ implies $QA_1 \prec QA_2$ as well as $A_1Q \prec A_2Q$ for any nonsingular totally nonnegative matrix Q.

For *oscillatory* matrices, i. e. for totally nonnegative matrices some power of which is already totally positive, we have the following:

THEOREM 1. A nonsingular $n \times n$ totally nonnegative matrix is oscillatory if and only if for each $k = 2, \dots, n$ there is at least one B_i which has an off-diagonal positive entry in the kth row and at least one C_j which has an off-diagonal positive entry in the kth column.

Basic oscillatory matrices are then defined as oscillatory matrices in which for each k *exactly one* of the matrices B_i and exactly one of the matrices C_j has such positive off-diagonal entry.

THEOREM 2. A matrix is oscillatory if and only if it is a product of a basic oscillatory matrix with a nonsingular totally nonnegative matrix.

THEOREM 3. Among nonsingular totally nonnegative matrices, basic oscillatory matrices are characterized by the fact that they are irreducible and have both subdiagonal and superdiagonal rank one.

Here, *subdiagonal* (resp., *superdiagonal*) rank of a square matrix is the maximum rank of any submatrix which has all entries in the subdiagonal (resp., superdiagonal) part of the matrix.