THEORY AND APPLICATIONS OF THE POLYNOMIAL NUMERICAL HULL

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Abstract

The behavior of a normal matrix is governed by its eigenvalues; that is, if A is a normal matrix and f is any analytic function, then the 2-norm of $f(A)$ is the maximum absolute value of f on the spectrum of A. The same holds for normal linear operators, except that now the spectrum may contain more than just the eigenvalues. This statement does not hold for nonnormal matrices and linear operators, and there is considerable interest in identifying sets in the complex plane that can be associated with nonnormal operators to provide the sort of information that the spectrum provides in the normal case.

In this talk we consider one such type of set called the *polynomial numerical hull of degree k*. These sets were first considered by Nevanlinna [*Convergence of Iterations for Linear Equations*, Birkhäuser, Basel, 1993] and further studied by the author [*Generalizations of the field of values useful in the study of polynomial functions of a matrix*, to appear in Lin. Alg. Appl.]. The polynomial numerical hull of degree k is defined as

$$\mathcal{G}_k(A) = \{z \in \mathbf{C} : \ \|p(A)\| \geq |p(z)| \ \ \forall p \in \mathcal{P}_k\},$$

where $\mathcal{P}_k$ is the set of polynomials of degree k or less. For $k = 1$, $\mathcal{G}_k(A)$ is the field of values of A, and for k greater than or equal to the degree of the minimal polynomial of A, $\mathcal{G}_k(A)$ is the spectrum of A. The interesting values of k are those between one and the degree of the minimal polynomial, and for these values $\mathcal{G}_k$ is intermediate between the field of values and the spectrum.

Knowledge of the polynomial numerical hulls of various degrees enables one to answer a number of questions that cannot be addressed through eigenvalues alone. In this talk we show how the polynomial numerical hulls of various degrees can be used to explain cutoff phenomena in Markov processes, such as the observation that it takes seven riffle shuffles to randomize a deck of cards. We also discuss the geometry of these sets, show many examples of numerically computed polynomial numerical hulls, and discuss a few examples in which these sets can be determined analytically.