ON THE CONVERGENCE OF KRYLOV SUBSPACE METHODS FOR SINGULAR SYSTEMS

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Abstract

Consider applying the Generalized Conjugate Residual (GCR(k)) method to the system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$, or the least squares problem $\min_{\boldsymbol{x}\in\mathbf{R}^n} \|\boldsymbol{b} - A\boldsymbol{x}\|_2$, where $A \in \mathbf{R}^{n\times n}$ is a nonsymmetric and singular matrix, and $\boldsymbol{x}, \boldsymbol{b} \in \mathbf{R}^n$. Let $\boldsymbol{x}_i$ be the approximate solution at the i-th step, and $\boldsymbol{x}_0$ be the initial approximate solution.

Following [1,2], let $R(A)$ be the range space of A, $r = \mathrm{rank}A$, $R(A)^\perp$ the orthogonal complement of $R(A)$, $\boldsymbol{q}_1, \cdots, \boldsymbol{q}_r$: the orthonormal basis of $R(A)$, $\boldsymbol{q}_{r+1}, \cdots, \boldsymbol{q}_n$: the orthonormal basis of $R(A)^\perp$, $Q_1 := [\,\boldsymbol{q}_1, \cdots, \boldsymbol{q}_r]$, $Q_2 := [\,\boldsymbol{q}_{r+1}, \cdots, \boldsymbol{q}_n]$, $A_{11} := {Q_1}^{\mathrm{T}} A Q_1$, and denote the symmetric part of a matrix A by $M(A)$. Then, we will show the following.

Theorem 1 The following (1)–(4) are equivalent.
(1) The GCR(k) method converges without break-down for arbitrary $\boldsymbol{b}$ and $\boldsymbol{x}_0$.
(2) $M(A_{11})$ is definite and $R(A)^\perp = \ker A$.
(3) $M(A)$ of A is definite in $R(A)$, and $R(A)^\perp = \ker A$.
(4) $M(A)$ is semi-definite, $\mathrm{rank}\, M(A) = \mathrm{rank}A$, and $R(A)^\perp = \ker A$. □

Theorem 2 If $\boldsymbol{b} \in R(A)$, the following (1)–(3) are equivalent.
(1) For arbitrary $\boldsymbol{x}_0$, the GCR(k) method does not break down and the residual converges to $\mathbf{0}$.
(2) $M(A_{11})$ is definite.
(3) $M(A)$ is definite in $R(A)$. □

We can further show that when the GCR method does not break down, the method converges in at most $r = \mathrm{rank}A$ iterations.

In the talk, we intend to present similar analysis for the GMRES(k) method.

References

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