Study of the BDF-DGFE method for the solution of the compressible Navier-Stokes equations

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Introduction

- **Our aim**: efficient numerical scheme for the solution of the compressible Navier-Stokes equations,

$$\frac{\partial \mathbf{w}}{\partial t} = \nabla \cdot \mathbf{G}(\mathbf{w}, \nabla \mathbf{w}), \qquad \mathbf{w} : \Omega \times (0,T) \to R^4, \qquad (1)$$

- space semi-discretization: $\mathbf{w}(x,t) \approx \mathbf{w}_h(t) \in \mathbf{S}_h, \ t \in (0,T)$

$$\frac{\partial}{\partial t}\mathbf{w}_h = F(t, \mathbf{w}_h, \nabla \mathbf{w}_h), \qquad \mathbf{w}_h : (0,T) \to \mathbf{S}_h, \qquad (2)$$

- system of ODEs (stiff)
- ODE solver: explicit, implicit, semi-implicit

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- full time-space discretization,

$$\left(\mathbf{M} + \tau_k \mathbf{C}(\mathbf{w}_h^k)\right) \mathbf{w}_h^{k+1} = \mathbf{F}(\mathbf{w}_h^k), \tag{3}$$

  - $\mathbf{w}_h^k \in R^{\mathrm{dof}}, \ k = 1, 2, \dots,$
  - $\mathbf{M}$ – mass matrix,
  - $\mathbf{C}$ – "disrete flux" matrix,
  - $\mathbf{F}$ – RHS (boundary conditions),
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Navier-Stokes equations

$$\frac{\partial \mathbf{w}}{\partial t} + \sum_{s=1}^{2} \frac{\partial}{\partial x_s} \mathbf{f}_s(\mathbf{w}) = \sum_{s=1}^{2} \frac{\partial}{\partial x_s} \mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w}) \text{ in } \Omega \times (0, T) \quad (4)$$

$$\mathbf{w} = (\rho, \rho v_1, \rho v_2, e)^T,$$

$$\mathbf{f}_s(\mathbf{w}) = (\rho v_s, \rho v_s v_1 + p\delta_{s1}, \rho v_s v_2 + p\delta_{s2}, (e + p)\, v_s)^T, \quad s = 1, 2,$$

$$\mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w}) = \left(0, \tau_{1s}^V, \tau_{2s}^V, \sum_{r=1}^{2} \tau_{rs}^V v_r + \frac{\gamma}{Re\ Pr} \frac{\partial \theta}{\partial x_s}\right)^T, \quad s = 1, 2,$$

$$\tau_{rs}^V = \frac{1}{Re}\left[\left(\frac{\partial v_s}{\partial x_r} + \frac{\partial v_r}{\partial x_s}\right) - \frac{2}{3} \mathrm{div}\mathbf{v}\,\delta_{rs}\right], \quad r, s = 1, 2,$$

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$$p = \theta\rho(\gamma - 1), \qquad p = (\gamma - 1)\left(e - \rho|\mathbf{v}|^2/2\right)$$

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Properties of the inviscid/viscous fluxes

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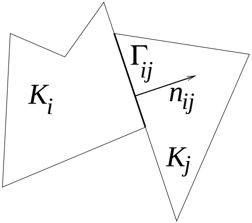
Discretization



- let $\mathcal{T}_h$, $h > 0$ be a partition of $\overline{\Omega}$
- $\mathcal{T}_h = \{K_i\}_{i \in I}$, K_i 2-dimensional polyhedra (triangles, quadrilaterals,...) $h = \max_{K \in \mathcal{T}_h} \text{diam}(K)$,
- $\Gamma_{ij} \equiv \partial K_i \cap \partial K_j$
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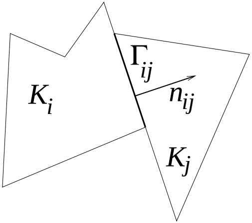

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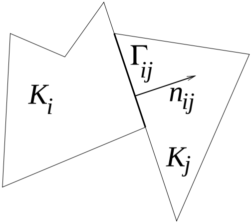

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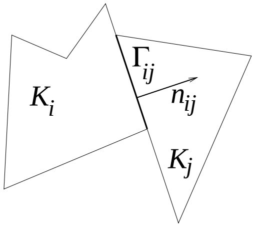

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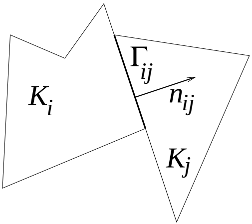

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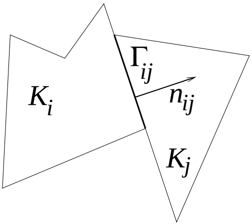

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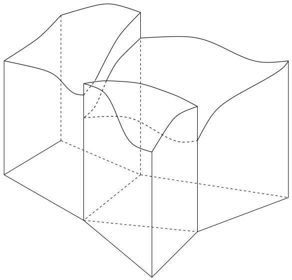
Discontinuous piecewise polynomial functions



- approximate solution $\mathbf{w}_h(t) : (0, T) \to \mathbf{S}_h$, where

$$\mathbf{S}_h \equiv [S_h]^4, \quad S_h \equiv \{v; v|_K \in P_k(K)\ \forall K \in \mathcal{T}_h\},$$

where $P_k(K)$ are polynomials of degree $\leq k$ on $K \in \mathcal{T}_h$.

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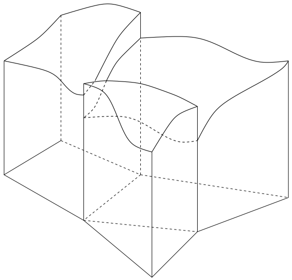

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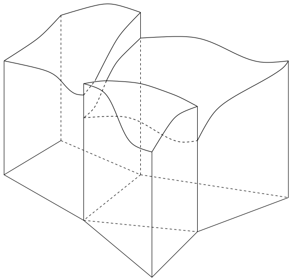

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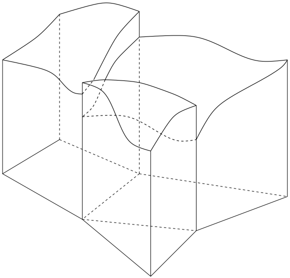

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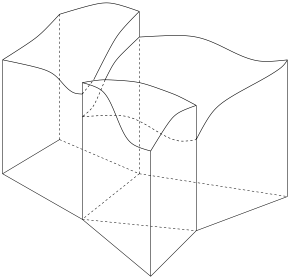

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Discontinuous piecewise polynomial functions (2)

- for $v \in S_h$, $i \in I$ and $j \in s(i)$ we put

$$\begin{aligned} v|_{\Gamma_{ij}} &= \text{the trace of } v|_{K_i} \text{ on } \Gamma_{ij}, \\ v|_{\Gamma_{ji}} &= \text{the trace of } v|_{K_j} \text{ on } \Gamma_{ji}, \\ \langle v \rangle_{\Gamma_{ij}} &= \frac{1}{2}\left(v|_{\Gamma_{ij}} + v|_{\Gamma_{ji}} \right), \\ [v]_{\Gamma_{ij}} &= v|_{\Gamma_{ij}} - v|_{\Gamma_{ji}}. \end{aligned}$$

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Space DGFE semi-discretization

Space semi-discretization

i) $\mathbf{w}_h : (0,T) \to \mathbf{S}_h$,

ii)

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{w}_h(t), \varphi_h) + \tilde{\mathbf{a}}_h(\mathbf{w}_h(t), \varphi_h) + \tilde{\mathbf{b}}_h(\mathbf{w}_h(t), \varphi_h) + \mathbf{J}_h(\mathbf{w}_h(t), \varphi_h) = 0, \quad \forall \varphi_h \in \mathbf{S}_h, \ t \in (0,T). \tag{5}$$

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Inviscid terms

$$\tilde{\mathbf{b}}_h(\mathbf{w}, \varphi) \equiv \sum_{i \in I} \left\{ -\int_{K_i} \sum_{s=1}^{2} \mathbf{f}_s(\mathbf{w}) \cdot \frac{\partial \varphi}{\partial x_s} \, \mathrm{d}x + \sum_{j \in S(i)} \int_{\Gamma_{ij}} \boldsymbol{H}(\mathbf{w}|_{\Gamma_{ij}}, \mathbf{w}|_{\Gamma_{ji}}, \mathbf{n}_{ij}) \cdot \varphi \, \mathrm{d}S \right\}, \tag{6}$$

where $\boldsymbol{H}$ is a *numerical flux.*

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$\theta = 1$ (SIPG), 0 (IIPG), -1 (NIPG).

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& - \sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \sum_{s=1}^{2} \Big(\mathbf{D}_s(\mathbf{w},\nabla\mathbf{w},\mathbf{w},\nabla\mathbf{w})\,(n_{ij})_s \cdot \varphi \\
& \qquad + \theta \mathbf{D}_s(\mathbf{w},\nabla\mathbf{w},\varphi,\nabla\varphi)\,(n_{ij})_s \cdot (\mathbf{w}-\mathbf{w}_B) \Big)\,\mathrm{d}S \Bigg\},
\end{aligned}
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$\theta = 1$ (SIPG), 0 (IIPG), -1 (NIPG).

Viscous terms

$$
\begin{aligned}
\tilde{\mathbf{a}}_h(\mathbf{w},\varphi) \equiv \sum_{K_i\in\mathcal{T}_h} \Bigg\{ \int_{K_i} \sum_{s=1}^{2} \mathbf{D}_s(\mathbf{w},\nabla\mathbf{w},\mathbf{w},\nabla\mathbf{w})\cdot\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x \\
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Interior and boundary penalties

$$\mathbf{J}_h(\mathbf{w},\varphi) = \sum_{K_i\in\mathcal{T}_h}\left\{\sum_{\substack{j\in s(i)\\ j<i}}\int_{\Gamma_{ij}}\sigma[\mathbf{w}]\cdot[\varphi]\,\mathrm{d}S + \sum_{j\in\gamma_D(i)}\int_{\Gamma_{ij}}\sigma(\mathbf{w}-\mathbf{w}_B)\cdot\varphi\,\mathrm{d}S\right\}, \tag{7}$$

$$\sigma|_{\Gamma_{ij}} = \frac{C_W}{|\Gamma_{ij}|\,Re}, \qquad C_W > 0.$$

Interior and boundary penalties

$$\mathbf{J}_h(\mathbf{w},\varphi) \equiv \sum_{K_i\in\mathcal{T}_h} \left\{ \sum_{\substack{j\in s(i)\\ j<i}} \int_{\Gamma_{ij}} \sigma[\mathbf{w}]\cdot[\varphi]\,\mathrm{d}S + \sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \sigma(\mathbf{w}-\mathbf{w}_B)\cdot\varphi\,\mathrm{d}S \right\}, \tag{7}$$

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Space DGFE semi-discretization (2)

Space semi-discretization

i) $\mathbf{w}_h : (0, T) \to \mathbf{S}_h$,

ii)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{w}_h(t), \varphi_h) \; &+ \; \tilde{\mathbf{a}}_h(\mathbf{w}_h(t), \varphi_h) \\ &+ \; \tilde{\mathbf{b}}_h(\mathbf{w}_h(t), \varphi_h) + \mathbf{J}_h(\mathbf{w}_h(t), \varphi_h) = 0, \\ &\qquad \forall \varphi_h \in \mathbf{S}_h, \;\; t \in (0, T). \end{aligned} \tag{8}$$

iii) $\mathbf{w}_h(0)$ satisfies the initial condition.

Time discretization

- semi-discrete problem (8) represents ODEs
- explicit method leads to a high restriction on time step (low speed flow)
- full implicit method leads to a system of nonlinear equations at each time step

Semi-implicit method

- linearize system (8)
- linear terms are treated implicitly
- nonlinear terms are treated explicitly

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Linearization of inviscid terms

Linearized inviscid form

$$\mathbf{b}_h(\bar{\mathbf{w}}_h, \mathbf{w}_h, \varphi_h) \equiv -\sum_{i\in I}\int_{K_i}\sum_{s=1}^{2}\mathbf{A}_s(\bar{\mathbf{w}}_h)\mathbf{w}_h\frac{\partial\varphi_h}{\partial x_s}\,\mathrm{d}x$$
$$+\sum_{i\in I}\sum_{j\in S(i)}\int_{\Gamma_{ij}}\sum_{s=1}^{2}\Big(\mathbf{P}^{+}(\langle\bar{\mathbf{w}}_h\rangle_{\Gamma_{ij}})\mathbf{w}_h|_{\Gamma_{ij}}$$
$$+\mathbf{P}^{-}(\langle\bar{\mathbf{w}}_h\rangle_{\Gamma_{ij}})\mathbf{w}_h|_{\Gamma_{ji}}\Big)\varphi_h|_{\Gamma_{ij}}\,\mathrm{d}S$$

- *consistency*: $\tilde{\mathbf{b}}_h(\mathbf{w}_h, \varphi_h) = \mathbf{b}_h(\mathbf{w}_h, \mathbf{w}_h, \varphi_h) \quad \forall \mathbf{w}_h, \varphi_h \in \mathbf{S}_h.$

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Linearization of viscous terms

$$\mathbf{a}_h(\bar{\mathbf{w}}_h, \mathbf{w}_h, \varphi_h) \equiv \sum_{K_i \in \mathcal{T}_h} \Big\{ \int_{K_i} \sum_{s=1}^{2} \mathbf{D}_s(\bar{\mathbf{w}}_h, \nabla\bar{\mathbf{w}}_h, \mathbf{w}_h, \nabla\mathbf{w}_h) \cdot \frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x$$

$$- \sum_{\substack{j \in s(i) \\ j<i}} \int_{\Gamma_{ij}} \sum_{s=1}^{2} \Big(\langle \mathbf{D}_s(\bar{\mathbf{w}}_h, \nabla\bar{\mathbf{w}}_h, \mathbf{w}_h, \nabla\mathbf{w}_h) \rangle \, (n_{ij})_s \cdot [\varphi]$$

$$+ \theta \langle \mathbf{D}_s(\bar{\mathbf{w}}_h, \nabla\bar{\mathbf{w}}_h, \varphi, \nabla\varphi) \rangle \, (n_{ij})_s \cdot [\mathbf{w}_h] \Big)\,\mathrm{d}S$$

$$- \sum_{j \in \gamma_D(i)} \int_{\Gamma_{ij}} \sum_{s=1}^{2} \Big(\mathbf{D}_s(\bar{\mathbf{w}}_h, \nabla\bar{\mathbf{w}}_h, \mathbf{w}_h, \nabla\mathbf{w}_h) \, (n_{ij})_s \cdot \varphi$$

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\end{aligned}
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- \sum_{\substack{j \in s(i) \\ j < i}} \int_{\Gamma_{ij}} \sum_{s=1}^{2} \Big(\langle \mathbf{D}_s(\bar{\mathbf{w}}_h, \nabla \bar{\mathbf{w}}_h, \mathbf{w}_h, \nabla \mathbf{w}_h) \rangle \, (n_{ij})_s \cdot [\varphi] \\
+ \theta \langle \mathbf{D}_s(\bar{\mathbf{w}}_h, \nabla \bar{\mathbf{w}}_h, \varphi, \nabla \varphi) \rangle \, (n_{ij})_s \cdot [\mathbf{w}_h] \Big) \, \mathrm{d}S \\
- \sum_{j \in \gamma_D(i)} \int_{\Gamma_{ij}} \sum_{s=1}^{2} \Big(\mathbf{D}_s(\bar{\mathbf{w}}_h, \nabla \bar{\mathbf{w}}_h, \mathbf{w}_h, \nabla \mathbf{w}_h) \, (n_{ij})_s \cdot \varphi \\
+ \theta \mathbf{D}_s(\bar{\mathbf{w}}_h, \nabla \bar{\mathbf{w}}_h, \varphi, \nabla \varphi) \, (n_{ij})_s \cdot (\mathbf{w}_h - \mathbf{w}_B) \Big) \, \mathrm{d}S \Bigg\},
\end{aligned}
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Linearization of viscous terms

$$\mathbf{a}_h(\bar{\mathbf{w}}_h, \mathbf{w}_h, \varphi_h) \equiv \sum_{K_i \in \mathcal{T}_h} \Bigg\{ \int_{K_i} \sum_{s=1}^{2} \mathbf{D}_s(\bar{\mathbf{w}}_h, \nabla\bar{\mathbf{w}}_h, \mathbf{w}_h, \nabla\mathbf{w}_h) \cdot \frac{\partial \varphi}{\partial x_s} \, \mathrm{d}x$$

$$- \sum_{\substack{j \in s(i) \\ j<i}} \int_{\Gamma_{ij}} \sum_{s=1}^{2} \Big(\langle \mathbf{D}_s(\bar{\mathbf{w}}_h, \nabla\bar{\mathbf{w}}_h, \mathbf{w}_h, \nabla\mathbf{w}_h) \rangle \, (n_{ij})_s \cdot [\varphi]$$

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Semi-implicit DGFE discretization

- let $t_0 < t_1 < \cdots < t_r$ be a partition of $(0, T)$, $\tau_k \equiv t_{k+1} - t_k$,
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Higher order semi-implicit DGFE scheme

General higher order scheme

$$\left(\frac{\sum_{l=0}^{n} \alpha_l \mathbf{w}_h^{k+1-l}}{\tau_k}, \varphi_h \right) + \mathbf{B}_h(\bar{\mathbf{w}}_h^{k+1}, \mathbf{w}_h^{k+1}, \varphi_h) = 0,$$

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α_l, β_l for constant time step

order	α_l				β_l		
1	1	-1			1		
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Linear algebra problem

- BDF-DGFEM scheme:

$$\left(\mathbf{M} + \tau_k \mathbf{B}^k(\mathbf{w}_h^k, \dots)\right) \mathbf{w}_h^{k+1} = \mathbf{F}^k(\mathbf{w}_h^k, \dots), \quad k = 0, 1, \dots \tag{9}$$

- iterative solver necessary for industrial applications
  $\Rightarrow$ restarted GMRES with block diagonal preconditioning
- L^2-orthogonal basis of $\mathbf{S}_h \Rightarrow \mathbf{M} \approx \mathbf{I}$
  $\Rightarrow$ for small τ_k, solution of (9) is very fast,
- for increasing τ_k, solution of (9) more expensive,
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$$\left(\mathbf{M} + \tau_k \mathbf{B}^k(\mathbf{w}_h^k, \dots)\right) \mathbf{w}_h^{k+1} = \mathbf{F}^k(\mathbf{w}_h^k, \dots), \quad k = 0, 1, \dots \tag{9}$$

- iterative solver necessary for industrial applications
  $\Rightarrow$ restarted GMRES with block diagonal preconditioning
- L^2-orthogonal basis of $\mathbf{S}_h \Rightarrow \mathbf{M} \approx \mathbf{I}$
  $\Rightarrow$ for small τ_k, solution of (9) is very fast,
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Choice of the time step

- ABDF – adaptive BDF [D., Kůs, IJNME (in press)]
- two n-step BDF of the same order of accuracy,

$$\sum_{l=0}^{n} \alpha^{\mathrm{I}}_{n,l}\, \mathbf{w}^{\mathrm{I}}_{k-l} = \tau_k F(\mathbf{w}^{\mathrm{I}}_k),$$

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- from $\|\mathbf{w}^{\mathrm{I}}_{k-l} - \mathbf{w}^{\mathrm{II}}_{k-l}\|$ we estimate local discretization error e_k
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Behaviour of the convergence

- steady state solution by $t \to \infty$, i.e. $\frac{\partial}{\partial t} w_h(t) = 0$

$$\mathbf{w}_h^{k+1} \approx \mathbf{w}_h^k : \quad \left(M + \tau_k \mathbf{B}^k(\mathbf{w}_h^k, \dots)\right) \mathbf{w}_h^{k+1} = \mathbf{F}^k(\mathbf{w}_h^k, \dots) \tag{11}$$

- when $\mathbf{w}_h^k$ is close to steady state solution, τ_k should be longer GMRES for (11) needs too much iterative loops
- often GMRES does not converge, i.e. $\#$ loops $> 1\,500$ loops
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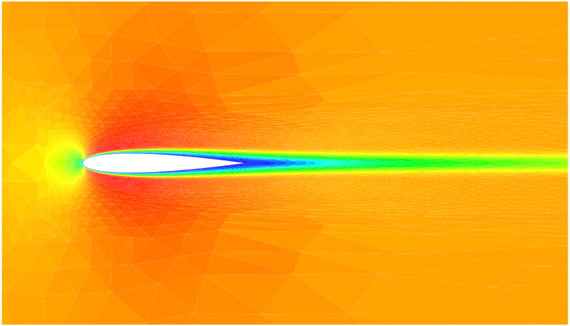
$$\mathbf{w}_h^{k+1} \approx \mathbf{w}_h^k : \quad \left(M + \tau_k \mathbf{B}^k(\mathbf{w}_h^k, \dots)\right) \mathbf{w}_h^{k+1} = \mathbf{F}^k(\mathbf{w}_h^k, \dots) \tag{11}$$

- when $\mathbf{w}_h^k$ is close to steady state solution,
  τ_k should be longer
  GMRES for (11) needs too much iterative loops
- often GMRES does not converge, i.e. $\#$ loops $>$ 1 500 loops
- artificial time restriction:
  if GMRES needs more than 750 loops
  then τ_k can not be increased

Steady-state flow

- NACA0012 profile, $M = 0.5$, $\alpha = 0^\circ$, $Re = 5\,000$.
- grid, 3 206 elements, P_1 – approximation



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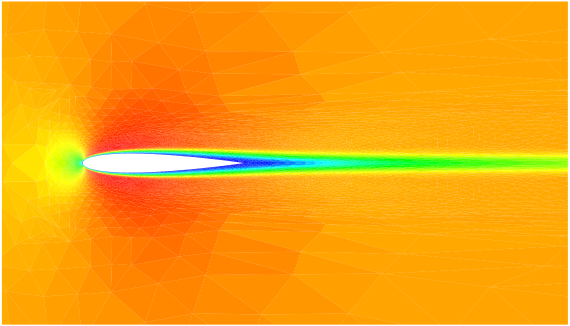

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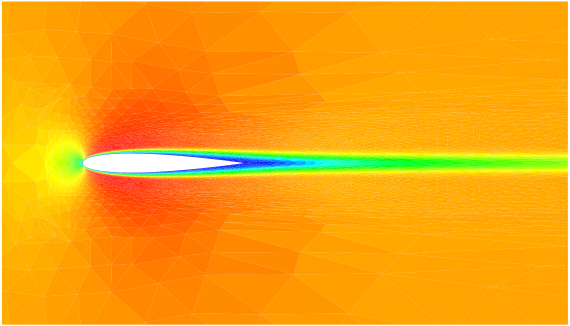

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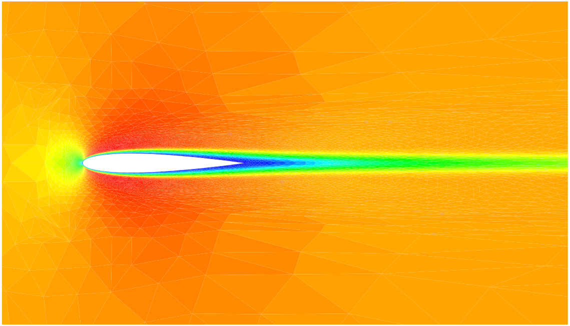

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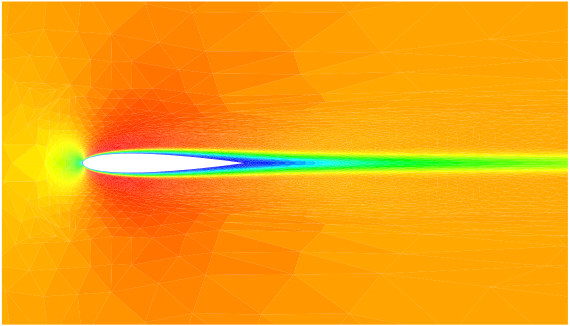

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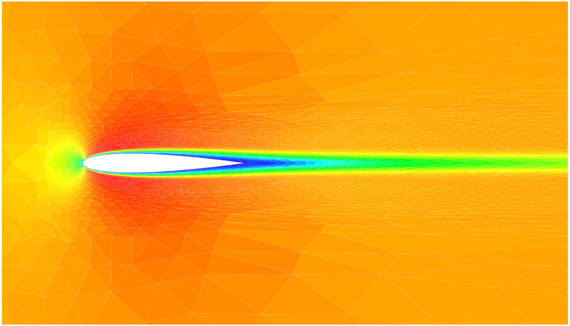

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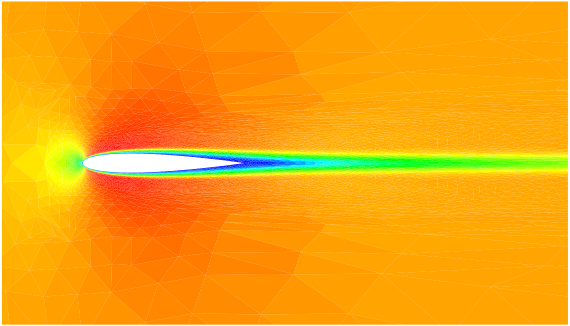
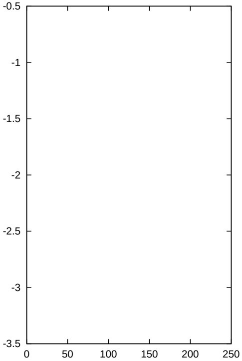
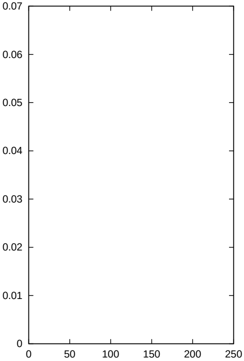


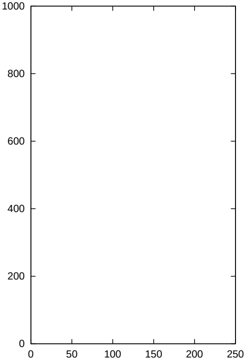
Steady-state flow (2)



$\frac{1}{\tau_k}\|\mathbf{w}_h^{k+1} - \mathbf{w}_h^k\|_{L^2(\Omega)}$

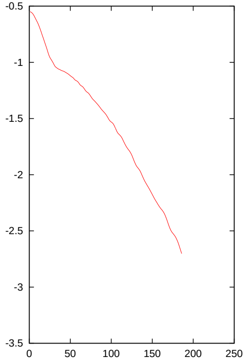


τ_k



GMRES loops

Steady-state flow (2)



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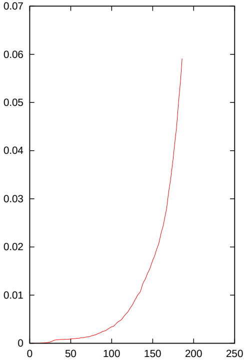

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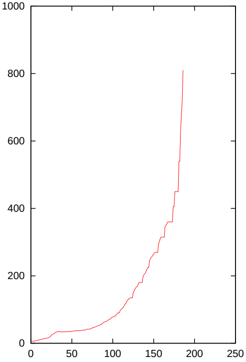

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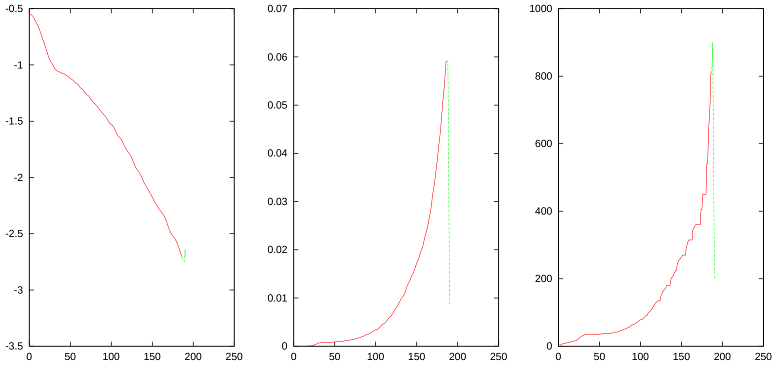

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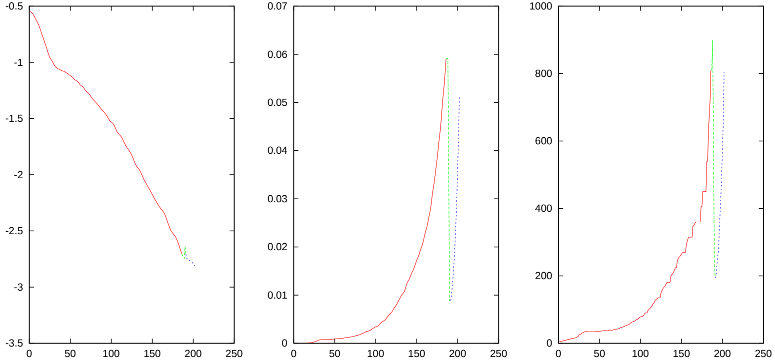

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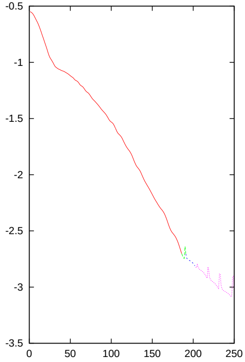

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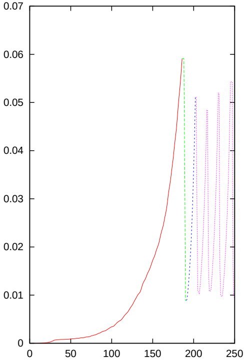

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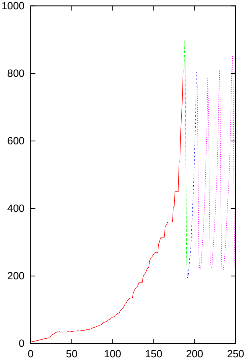

GMRES loops

Influence of the stopping criterion

- stopping criterion $\|Ax - b\|_{\ell^2} \leq \omega$
- influence of ω to
  - convergence to the steady state solution,
  - size of the time step given by ABDF,
  - computational time,
- computations for $t \in (5, 10)$ with different ω,

ω	$\frac{1}{\tau_k}\|\mathbf{w}_h^{k+1} - \mathbf{w}_h^k\|_{L^2(\Omega)}$	# time steps	CPU
1E-06	2.120E-04	483	4 612 s
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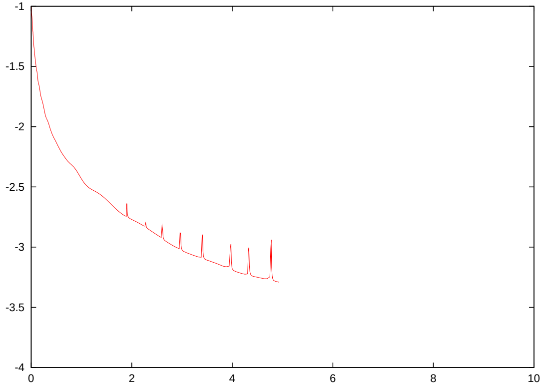
Influence of the stopping criterion – convergence



computation $t \in (0, 5)$, ω =1E-06, ω =1E-08, ω =1E-10, ω =1E-12

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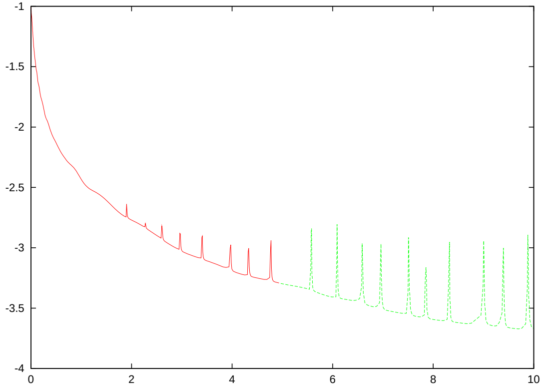

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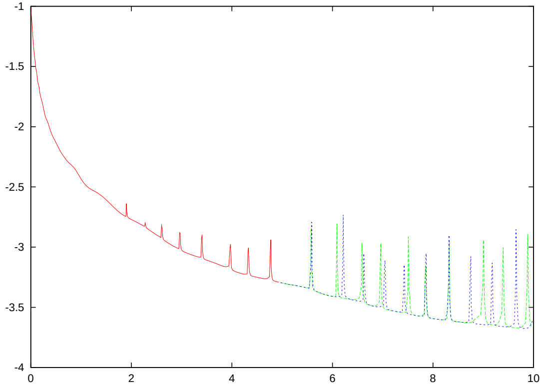

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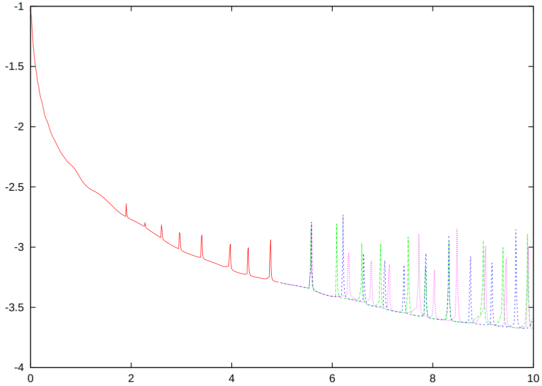

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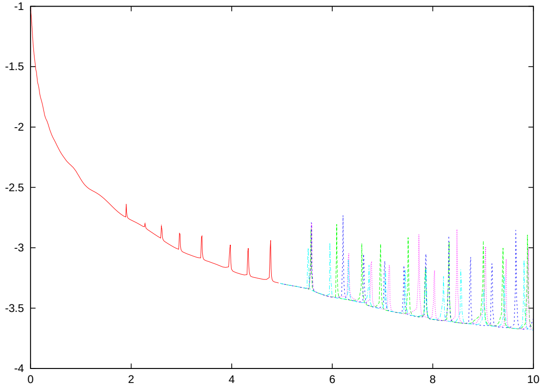

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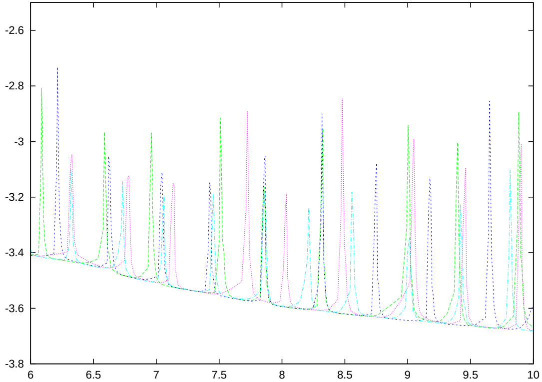

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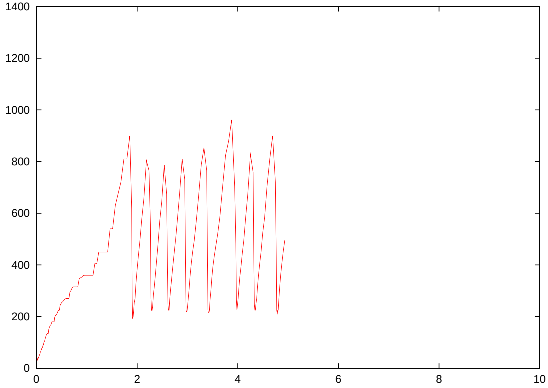
Influence of the stopping criterion – # GMRES loops



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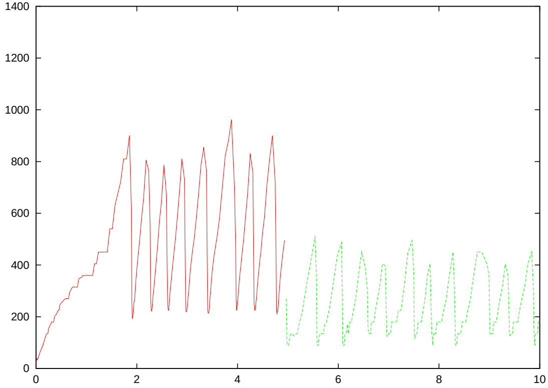

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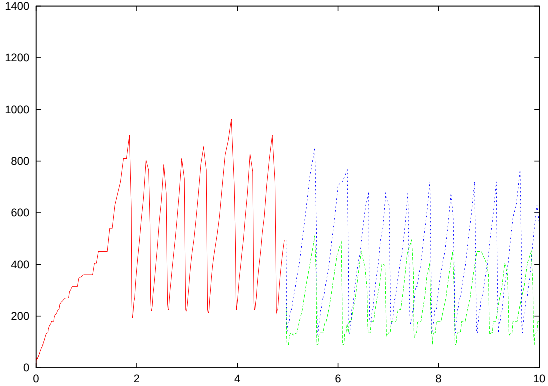

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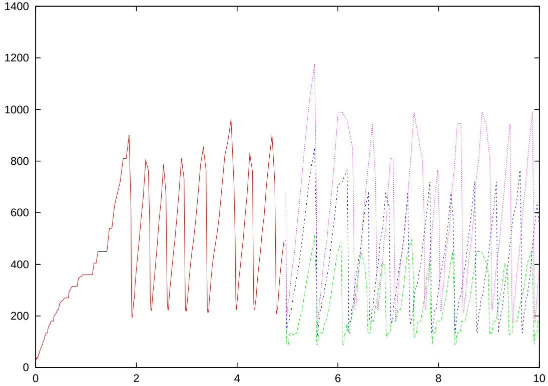

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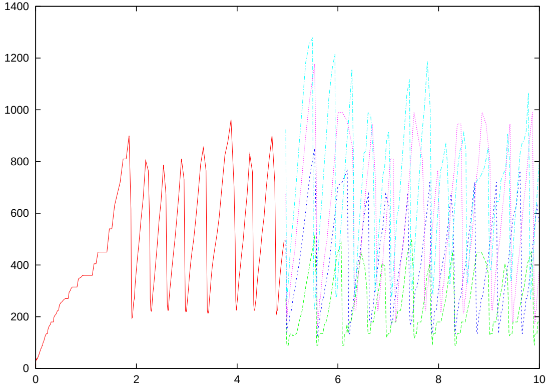

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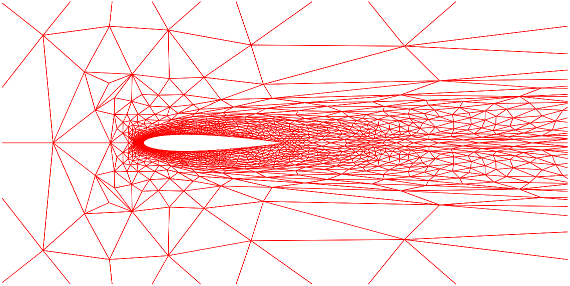
Unsteady flow around NACA0012 profile

- $M = 0.85$, $\alpha = 0°$, $Re = 10\,000$ ($\#\mathcal{T}_h = 3\,206$)



Mach

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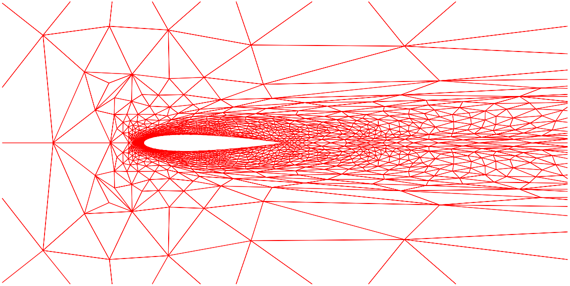

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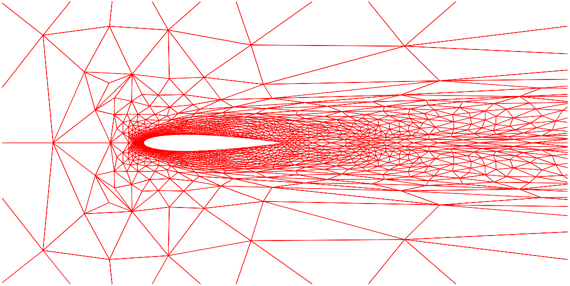

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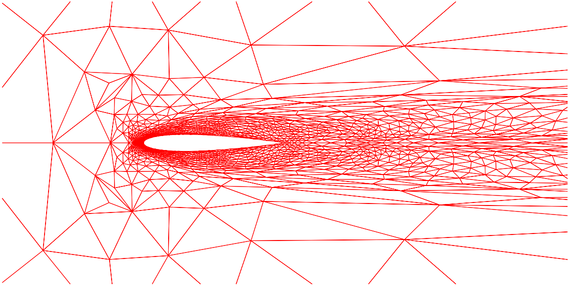

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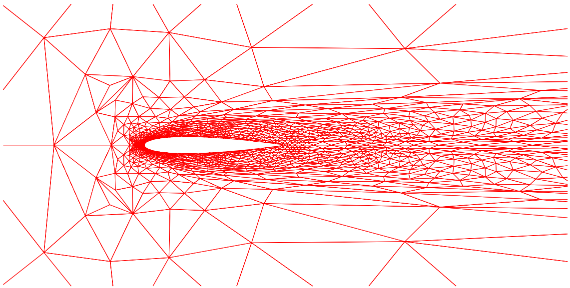

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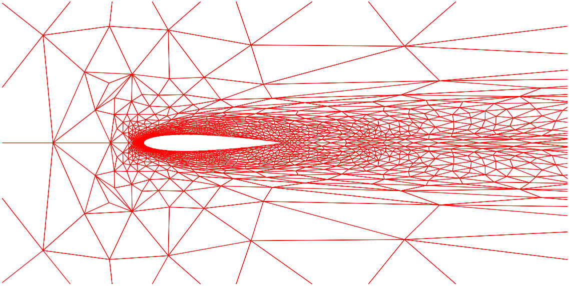
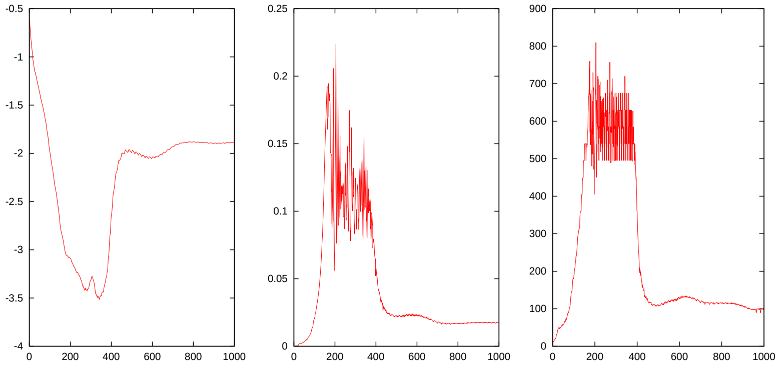


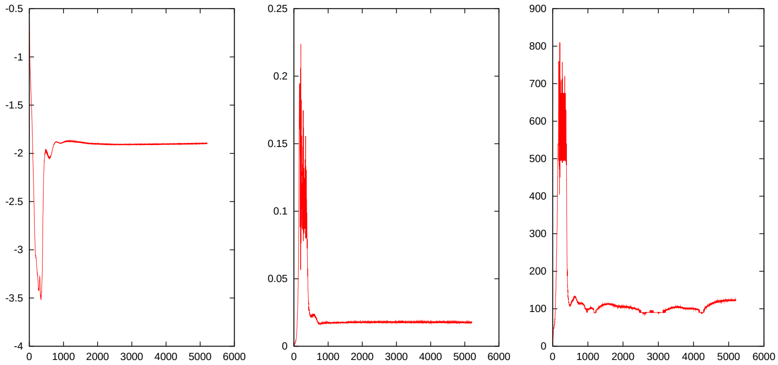
Mach

Unsteady flow around NACA0012 profile (2)



$\frac{1}{\tau_k}\|\mathbf{w}_h^{k+1} - \mathbf{w}_h^k\|_{L^2(\Omega)}$ τ_k # GMRES loops

Unsteady flow around NACA0012 profile (3)



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Conclusion and Outlook

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