On A Quadratic Eigenproblem Arising In The Analysis of Delay Equations

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Outline

- Time Delay System
- Polynomial Eigenvalue Problem
- Spectral Symmetry
- Structured Linearization
- Conclusions

Time Delay Systems (TDS)

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{m} A_k x(t - h_k),\ t > 0 \qquad (\Sigma)$$

$$x(t) = \varphi(t),\ t \in [-h_m, 0]$$

with $0 < h_1 < \ldots < h_m$ and $A_k \in \mathbb{R}^{n \times n}$.

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Definition

- *Eigenvalue s and eigenvector $v \neq 0$:*

$$\mathbb{M}(s)v := \left(-sI_n + A_0 + \sum_{k=1}^{m} A_k e^{-h_k s} \right) v = 0$$

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- *spectrum* $\sigma(\Sigma)$: set of all eigenvalues
- stable: $\sigma(\Sigma) \subset \mathbb{C}^-$

Critical System

Problem

For what $h_1, \ldots, h_m$ *is there an* ω *s.t*

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Definition

Σ is called critical iff $\sigma(\Sigma) \cap \imath\mathbb{R} \neq \emptyset$.

Example (Jarlebring 2005)

Two delay problem: $\dot{x}(t) = -x(t-h_1) - 2x(t-h_2)$

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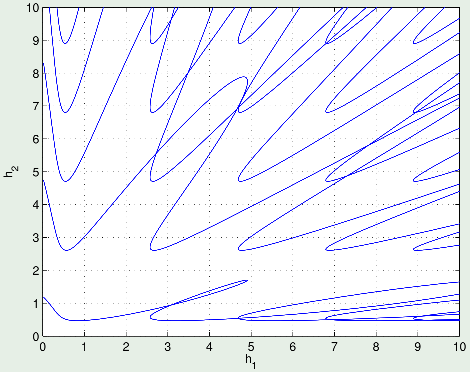
Critical curves:



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Hale & Huang 1993: Scalar two delays: Geometric classification
Chen & Gu & Nett 1995: Commensurate delays
Louisell 2001: Single delay, neutral, moderate size
Sipahi & Olgac 2003 : Small systems, few delays:
Form determinant + Routh table + Rekasius Substitution.

Given free parameters φ_k, $k = 1, \ldots, m-1$.

Theorem (Jarlebring 2005)

The point in delay space $(h_1, \ldots, h_m)$ is critical iff

$$h_k = \frac{\varphi_k + 2p\pi}{\omega}, \; k = 1, \ldots, m-1$$
$$h_m = \frac{\mathrm{Arg}\; s + 2q\pi}{\omega}$$

$$\left[s^2 I \otimes A_m + s \left(\sum_{k=0}^{m-1} I \otimes A_k e^{-\imath\varphi_k} + e^{\imath\varphi_k} A_k \otimes I \right) + A_m \otimes I \right] u = 0,$$

where $s = e^{\imath\omega}$, $u = \mathrm{vec}\; vv^* = v \otimes \bar{v}$ *and*

$$\omega = -\imath v^* \left(A_0 + \sum_{k=1}^{m-1} A_k e^{-\imath\varphi_k} + A_m s \right) v.$$

Quadratic eigenproblem

$$\left[s^2 (I \otimes A_m) + s \left(\sum_{k=0}^{m-1} I \otimes A_k e^{-\imath \varphi_k} + e^{\imath \varphi_k} A_k \otimes I \right) + (A_m \otimes I) \right] u = 0,$$

Quadratic eigenproblem

$$\left[s^2 \overbrace{(I \otimes A_m)}^{M} + s \overbrace{\left(\sum_{k=0}^{m-1} I \otimes A_k e^{-\imath\varphi_k} + e^{\imath\varphi_k} A_k \otimes I \right)}^{G} + \overbrace{(A_m \otimes I)}^{K} \right] u = 0,$$

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Quadratic Eigenvalue Problem

$$\begin{aligned} M &\in \mathbb{R}^{n^2 \times n^2} \\ G &\in \mathbb{C}^{n^2 \times n^2} \\ K &\in \mathbb{R}^{n^2 \times n^2} \end{aligned}$$

Theorem (Horn, Johnson)

There exists an involutary permutation matrix $P \in \mathbb{R}^{n^2 \times n^2}$ *such that* $B \otimes C = P(C \otimes B)P$ *for all* $B, C \in \mathbb{R}^{n \times n}$.

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In particular,

$$P = \sum_{i,j=1}^{n} E_{ij} \otimes E_{ij}^T = [E_{ij}^T]_{i,j=1}^n,$$

where $E_{ij} \in \mathbb{R}^{n \times n}$ has entry 1 in position i, j and all other entries are zero.

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$$P = \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

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and

$$A_k \otimes I = P(I \otimes A_k)P$$

such that

$$\begin{aligned} G &= \sum_{k=0}^{m-1} e^{-\imath\varphi_k}(I \otimes A_k) + e^{\imath\varphi_k}(A_k \otimes I) \\ &= P\left(\sum_{k=0}^{m-1}(A_k \otimes I)e^{-\imath\varphi_k} + e^{\imath\varphi_k}(I \otimes A_k)\right)P = P\overline{G}P. \end{aligned}$$

As M and K are real, this implies

$$Q(z) = z^2 M + zG + K = z^2 PKP + zP\overline{G}P + PMP = P(z^2\overline{K} + z\overline{G} + \overline{M})P,$$

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that is, $Q(z)$ is a matrix polynomial which satisfies

$$Q = P \cdot \mathrm{rev}(\overline{Q}) \cdot P,$$

with

$$\overline{Q}(z) = z^2\overline{M} + z\overline{G} + \overline{K},$$

and

$$\mathrm{rev}(Q(z)) := z^2 Q(\frac{1}{z}) = M + zG + z^2 K.$$

Problem Statement

We will consider

$$Q(\lambda)v = 0 \quad \text{with} \quad Q(\lambda) = \sum_{i=0}^{k} \lambda^i B_i, \qquad B_k \neq 0, \quad B_i \in \mathbb{C}^{n \times n},$$

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As $Q(\lambda) = \sum_{i=0}^{k} \lambda^i B_i$, this implies $B_i = P\overline{B}_{k-i}P,$ $i = 0, \ldots, k$.

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Questions to be answered:

- eigenvalue pairing
- structured linearizations

Structure reminds of:

- (anti-)palindromic: $\pm\mathrm{rev}(Q(\lambda)) = Q(\lambda)$
- $\star$-(anti-)palindromic: $\pm\mathrm{rev}(Q^\star(\lambda)) = Q(\lambda)$
- even, odd: $\pm Q(-\lambda) = Q(\lambda)$
- $\star$-even, $\star$-odd: $\pm Q^\star(-\lambda) = Q(\lambda)$

where $\star$ is used for transpose T in the real case and either T or conjugate transpose $*$ in the complex case.

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Recall

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Recall PCP-(anti-)palindromic (short PCP/anti-PCP)

$$Q(\lambda) = \pm P \cdot \mathrm{rev}(\overline{Q}(\lambda)) \cdot P.$$

Define even/odd equivalent PCP-even/odd

$$Q(\lambda) = \pm P \cdot \overline{Q}(-\lambda) \cdot P.$$

Spectral Symmetry

Let $Q(\lambda)v = 0$, and Q is PCP, then we have

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which implies

$$\mathrm{rev}(\overline{Q}(\lambda)) \cdot (Pv) = 0$$

and

$$Q(1/\overline{\lambda}) \cdot (P\overline{v}) = 0.$$

Hence, if λ is an eigenvalue with eigenvector v, then $1/\overline{\lambda}$ is an eigenvalue with eigenvector $P\overline{v}$.

Theorem

Let $Q(\lambda) = \sum_{i=0}^{k} \lambda^i B_i$, $B_k \neq 0$ *be a regular matrix polynomial, that is,* $\det Q(\lambda)$ *is not identically zero for all* $\lambda \in \mathbb{C}$.

1. *If* $Q(\lambda) = \pm P \cdot \mathrm{rev}(\overline{Q}(\lambda)) \cdot P$, *then the spectrum of* $Q(\lambda)$ *has the eigenvalue pairing* $(\lambda, 1/\overline{\lambda})$.
2. *If* $Q(\lambda) = \pm P \cdot \overline{Q}(-\lambda) \cdot P$, *then the spectrum of* $Q(\lambda)$ *has the eigenvalue pairing* $(\lambda, -\overline{\lambda})$

Moreover, the algebraic, geometric, and partial multiplicities of the two eigenvalues in each such pair are equal. (Here, we allow $\lambda = 0$ *and interpret* $1/\lambda$ *as the eigenvalue* ∞*.)*

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Idea of the proof of statement 1: $Q(\lambda)$ and its first companion form $C_1(\lambda) = \lambda X + Y$ have the same eigenvalues (including multiplicities). C_1 of a (anti-)PCP Q is strictly equivalent to $X^* + \lambda Y^*$.

Structure of $Q(\lambda)$	eigenvalue pairing
(anti)-palindromic, T-(anti)-palindromic	$(\lambda, 1/\lambda)$
$*$-palindromic, $*$-anti-palindromic	$(\lambda, 1/\bar{\lambda})$
(anti)-PCP-palindromic	$(\lambda, 1/\bar{\lambda})$
even, odd, T-even, T-odd	$(\lambda, -\lambda)$
$*$-even, $*$-odd	$(\lambda, -\bar{\lambda})$
PCP-even, PCP-odd	$(\lambda, -\bar{\lambda})$

Spectral symmetries

Cayley Transformations

The Cayley transformation for a matrix polynomial $Q(\lambda)$ of degree k with pole at $+1$ or -1, resp., is

$$\begin{aligned} \mathcal{C}_{+1}(Q)(\mu) &:= (1-\mu)^k Q(\frac{1+\mu}{1-\mu}), \\ \mathcal{C}_{-1}(Q)(\mu) &:= (\mu+1)^k Q(\frac{\mu-1}{\mu+1}). \end{aligned}$$

$Q(\lambda)$	$\mathcal{C}_{-1}(Q)(\mu) = (\mu+1)^k Q(\frac{\mu-1}{\mu+1})$	
	k even	k odd
palindromic $\star$-palindromic	even $\star$-even	odd $\star$-odd
anti-palindromic $\star$-anti-palindromic	odd $\star$-odd	even $\star$-even
PCP anti-PCP	PCP-even PCP-odd	PCP-odd PCP-even
even $\star$-even	palindromic $\star$-palindromic	
odd $\star$-odd	anti-palindromic $\star$-anti-palindromic	
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Cayley transformations

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PCP anti-PCP	PCP-even PCP-odd	
even $\star$-even	palindromic $\star$-palindromic	anti-palindromic $\star$-anti-palindromic
odd $\star$-odd	anti-palindromic $\star$-anti-palindromic	palindromic $\star$-palindromic
PCP-even PCP-odd	PCP anti-PCP	anti-PCP PCP

Cayley transformations

Linearization

The classical approach to solve $Q(\lambda)v = 0$ for

$$Q(\lambda) = \sum_{i=0}^{k} \lambda^i B_i, \qquad B_k \neq 0$$

is linearization, in which the given polynomial is transformed into a $kn \times kn$ matrix pencil $L(\lambda) = \lambda X + Y$ that satisfies

$$E(\lambda)L(\lambda)F(\lambda) = \begin{bmatrix} Q(\lambda) & 0 \\ 0 & I_{(k-1)n} \end{bmatrix},$$

where $E(\lambda)$ and $F(\lambda)$ are unimodular matrix polynomials. (A matrix polynomial is unimodular if its determinant is a nonzero constant, independent of λ).

Let $X_1 = X_2 = \mathrm{diag}(B_k, I_n, \ldots, I_n)$,

$$Y_1 = \begin{bmatrix} B_{k-1} & B_{k-2} & \cdots & B_0 \\ -I_n & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -I_n & 0 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} B_{k-1} & -I_n & & 0 \\ B_{k-2} & 0 & & \\ \vdots & \vdots & \ddots & -I_n \\ B_0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then $C_1(\lambda) = \lambda X_1 + Y_1$ and $C_2(\lambda) = \lambda X_2 + Y_2$ are the first and second companion forms for $Q(\lambda)$. These linearizations do not reflect the structure present in the matrix polynomial Q.

Structured Linearization

Source of linearizations: [Mackey, Mackey, Mehl, Mehrmann 2006]

$$\mathbb{L}_1(Q) = \left\{ L(\lambda) = \lambda X + Y : L(\lambda) \cdot (\Lambda \otimes I_n) = v \otimes Q(\lambda), v \in \mathbb{C}^k \right\},$$

$$\mathbb{L}_2(Q) = \left\{ L(\lambda) = \lambda X + Y : (\Lambda^T \otimes I_n) \cdot L(\lambda) = w^T \otimes Q(\lambda), w \in \mathbb{C}^k \right\}$$

where $\Lambda = [\lambda^{k-1} \;\; \lambda^{k-2} \;\; \cdots \;\; \lambda \;\; 1]^T.$

v is called right ansatz vector, w left ansatz vector.

$$\dim \mathbb{L}_1(Q) = \dim \mathbb{L}_2(Q) = k(k-1)n^2 + k$$

Structured PCP-Pencil

We have Q which satisfies

$$P \cdot \mathrm{rev}(\overline{Q}(\lambda)) \cdot P = Q(\lambda)$$

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We want a pencil $L(\lambda) \in \mathbb{L}_1(Q)$ such that

$$\widehat{P} \cdot \mathrm{rev}(\overline{L}(\lambda)) \cdot \widehat{P} = L(\lambda)$$

for some $kn \times kn$ real involution $\widehat{P}$.

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for some $kn \times kn$ real involution $\widehat{P}$.
It is not immediately obvious what to use for $\widehat{P}$.

Back to

$$(\lambda^2 M + \lambda G + K)v = 0,$$

with

$$M = P\overline{K}P \qquad \text{and} \qquad G = P\overline{G}P.$$

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does not work as there are no pencils in $\mathbb{L}_1(Q)$ satisfying

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unless the matrix G is very specifically tied to the leading coefficient M,

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unless the matrix G is very specifically tied to the leading coefficient M, e.g. for $v = [1 \;\; 1]^T$

$$G = P\overline{M}P + M = K + M.$$

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that is $v_1 = \overline{v}_2$ and

$$\lambda X + Y = \lambda \begin{bmatrix} v_1 M & -W_1 \\ \overline{v}_1 M & \overline{v}_1 G + P\overline{W}_1 P \end{bmatrix} + \begin{bmatrix} W_1 + v_1 G & v_1 P\overline{M}P \\ -P\overline{W}_1 P & \overline{v}_1 P\overline{M}P \end{bmatrix},$$

where W_1 is arbitrary, is a structured pencil in $\mathbb{L}_1(Q)$.

Structured Linearization

For regular Q and $L(\lambda) \in \mathbb{L}_1(Q)$ with $v \neq 0, v \in \mathbb{C}^2$

- select any nonsingular matrix T such that $Tv = \alpha e_1$
- apply $T \otimes I_n$ to $L(\lambda)$ to produce $\widetilde{L}(\lambda) = (T \otimes I_n) \cdot L(\lambda)$

$$\widetilde{L}(\lambda) = \lambda \widetilde{X} + \widetilde{Y} = \lambda \begin{bmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ 0 & -Z \end{bmatrix} + \begin{bmatrix} \widetilde{Y}_{11} & \widetilde{Y}_{12} \\ Z & 0 \end{bmatrix},$$

where $\widetilde{X}_{11}$ and $\widetilde{Y}_{12}$ are $n \times n$.

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- select any nonsingular matrix T such that $Tv = \alpha e_1$
- apply $T \otimes I_n$ to $L(\lambda)$ to produce $\widetilde{L}(\lambda) = (T \otimes I_n) \cdot L(\lambda)$

$$\widetilde{L}(\lambda) = \lambda \widetilde{X} + \widetilde{Y} = \lambda \begin{bmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ 0 & -Z \end{bmatrix} + \begin{bmatrix} \widetilde{Y}_{11} & \widetilde{Y}_{12} \\ Z & 0 \end{bmatrix},$$

where $\widetilde{X}_{11}$ and $\widetilde{Y}_{12}$ are $n \times n$.

If $\det Z \neq 0$, $L(\lambda)$ is a linearization of Q.

[Mackey, Mackey, Mehl, Mehrmann 2006]

As $v = [v_1 \quad \overline{v}_1]^T$ choose T as

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$$-Z = |v_1|^2 G + \overline{v}_1 W_1 + v_1 P \overline{W}_1 P.$$

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As $Q(\lambda) = \lambda^2 M + \lambda G + K$ is regular, we have for $W_1 = v_1 M$

$$-Z = |v_1|^2 (G + M + P\overline{M}P) = |v_1|^2 (G + M + K) = |v_1|^2 Q(1),$$

and $\det Z \neq 0$ iff 1 is not an eigenvalue of Q.

Other possible choice of W_1:

1. $W_1 = v_1 M$ yields $\det Z \neq 0$ if 1 is not an eigenvalue of Q.
2. $W_1 = -v_1 M$ yields $\det Z \neq 0$ if -1 is not an eigenvalue of Q.
3. $W_1 = \overline{v}_1 M$ yields $\det Z \neq 0$ if $\frac{\overline{v}_1}{v_1}$ is not an eigenvalue of Q.
4. $W_1 = -\overline{v}_1 M$ yields $\det Z \neq 0$ if $-\frac{\overline{v}_1}{v_1}$ is not an eigenvalue of Q.
5. $W_1 = v_1 G$ yields $\det Z \neq 0$ if $\det G \neq 0$.
6. ...

For $W_1 = -\overline{v}_1 M$ we have

$$\lambda \begin{bmatrix} v_1 M & \overline{v}_1 M \\ \overline{v}_1 M & \overline{v}_1 G - v_1 P\overline{M}P \end{bmatrix} + \begin{bmatrix} v_1 G - \overline{v}_1 M & v_1 P\overline{M}P \\ v_1 P\overline{M}P & \overline{v}_1 P\overline{M}P \end{bmatrix} \in \mathbb{L}_1(Q) \cap \mathbb{L}_2(Q$$

if $\frac{\overline{v}_1}{v_1}$ is not an eigenvalue of Q.

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Similar construction for general (anti-)PCP-polynomial possible.

Structured PCP-even/odd-Linearization

We have Q which satisfies

$$\pm P \cdot \overline{Q}(-\lambda) \cdot P = Q(\lambda)$$

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It is not immediately obvious what to use for $\widehat{P}$.

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Construction for general PCP-even/odd polynomial possible.

Conclusions

- New Structured Polynomial Eigenvalue Problem
- Spectral Symmetry
- Cayley Transformation
- Structured Linearization for (anti-)PCP and PCP-even/odd polynomials

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Open Problems:

- Choice of the Ansatz Vector v
- Structure-Preserving Transformation
- Structure-Preserving Eigenvalue Algorithm

GAMM Activity Group Meeting:

Today, 1:20 - 2:20 pm

Members as well as non-members are invited!