An application of the Bauer-Fike theorem to nonlinear eigenproblems

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joint work with Johan Karlsson, KTH

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Introduction:

- Nonlinear eigenvalue problems
- the Bauer-Fike Theorem
- Accuracy iteration

Quadratic eigenproblem:
Linearization[MMMM'06], SOAR[Bai,Su'05], JD[Slejpen et al'96]

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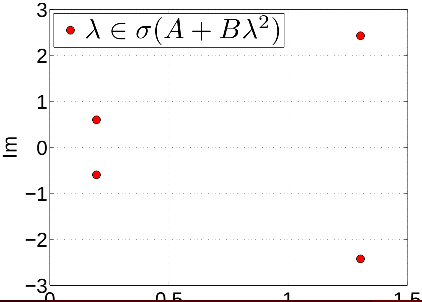
$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \; B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$



Nonlinear eigenproblems

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LMS[Engelborghs, et al'99], Pseudospectral differencing[Breda, et al'05]

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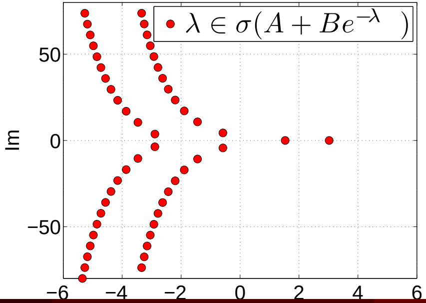

The Bauer-Fike Theorem [Bauer,Fike'60]

Classical formulation

Theorem (BF)

If $A_1 = V_1 D_1 V_1^{-1}$ *then for any* $\lambda_2 \in \sigma(A_2)$

$$\min_{\lambda_1 \in \sigma(A_1)} |\lambda_2 - \lambda_1| \leq \kappa(V_1) \|A_1 - A_2\|.$$

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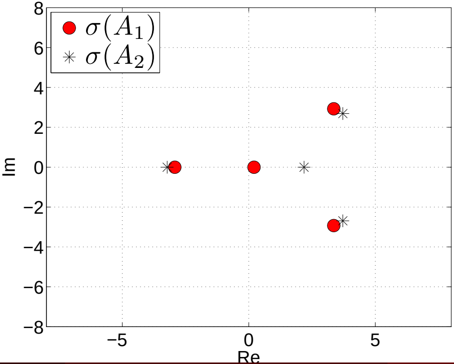

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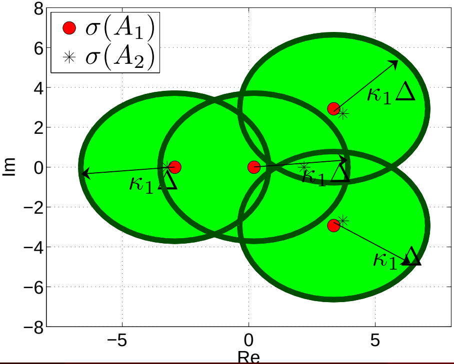

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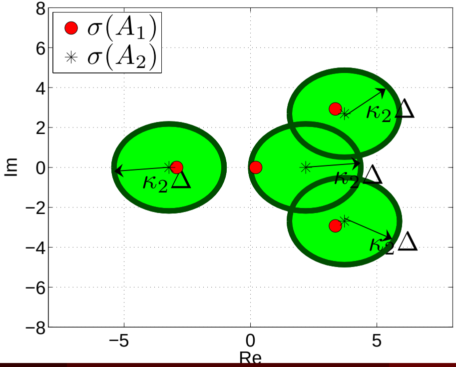

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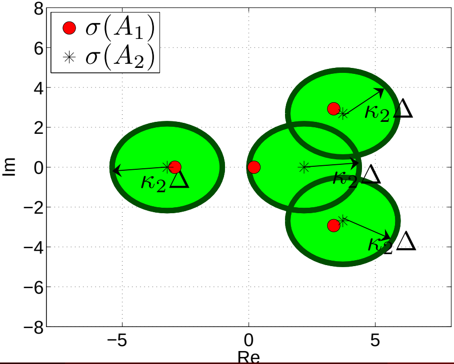

The Bauer-Fike Theorem

Theorem (BF - Dist)

If $\lambda_1 \in \sigma(A_1)$ *and* $\lambda_2 \in \sigma(A_2)$ *then*

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where $\operatorname{dist}(\lambda, S) = \min_{s \in S} |\lambda - s|$.

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For the Hausdorff metric

$$d_H(S_1, S_2) := \max\left(\max_{s_1 \in S_1} \operatorname{dist}(s_1, S_2), \max_{s_2 \in S_2} \operatorname{dist}(s_2, S_1)\right)$$

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Theorem (BF - Hausdorff metric)

$$d_H(\sigma(A_1), \sigma(A_2)) \leq \max(\kappa(V_1), \kappa(V_2))\|A_1 - A_2\|$$

Literature (non-exhaustive)

- Perturbation of polynomial eigenvalue problems:
  - [Tisseur'00] (backward error)
  - [Higham, Tisseur'03]
  - [Chu,Lin'04] (Bauer-Fike)
  - [Dedieu, Tisseur'03]
  - [Karow,Kressner,Tisseur'06] (condition number)
- Nonlinear eigenvalue problems:
  - [Hadeler '69] (generalized Rayleigh-functionals)
  - [Ehrmann '65]
  - [Cullum, Ruehli '01] (pseudospectra)
  - [Wagenknecht,Michiels,Green'07] (pseudospectra)

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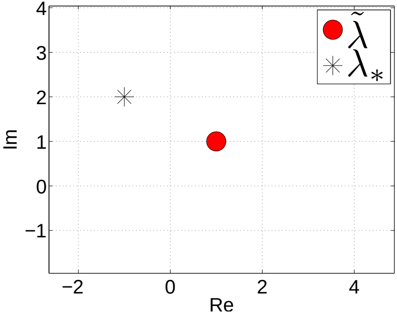

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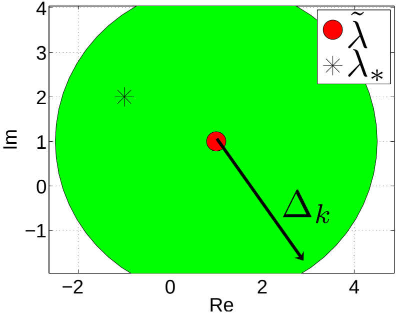

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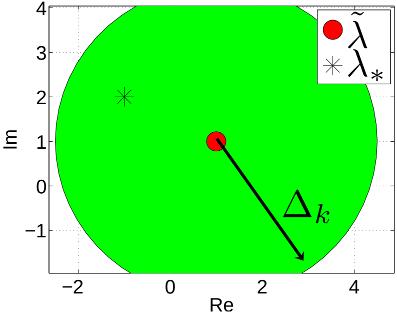

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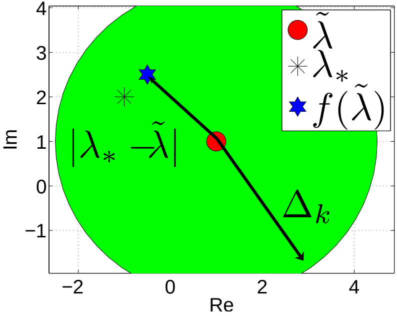

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&\leq |\tilde\lambda - f(\tilde\lambda)| + f'_{max}|\tilde\lambda - \lambda_*| \Rightarrow \\
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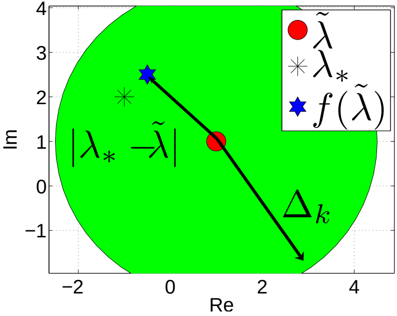

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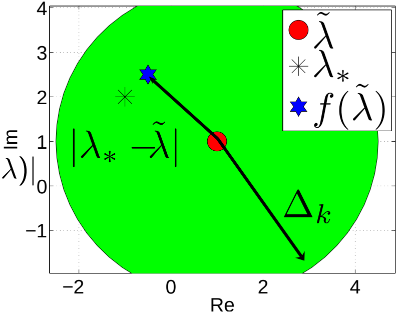

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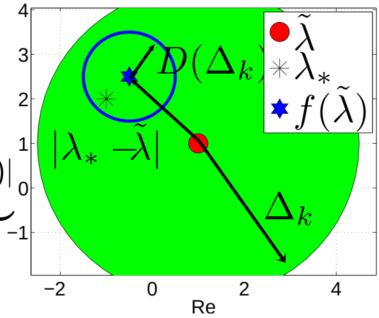

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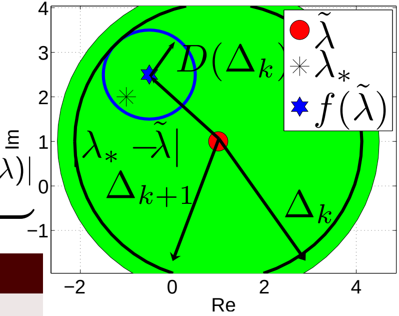

Accuracy Fixpoint Iteration

$$\lambda_* \in V(\Delta_k) \;\Rightarrow\; \lambda_* \in V(\Delta_{k+1}) \text{ where}$$

$$\Delta_{k+1} = \varphi(\Delta_k) := |\tilde\lambda - f(\tilde\lambda)| + D(\Delta_k)$$

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$$D(\Delta_k) = \max_{\lambda \in V(\Delta_k)} |f(\lambda) - f(\tilde{\lambda}))| = 0.2\Delta_k + 0.1\Delta_k^2$$

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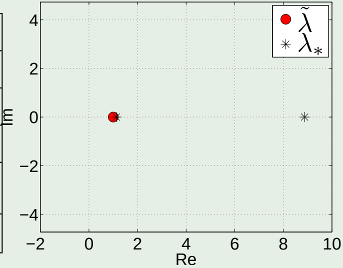
Δ_0	2
$\Delta_1 = \phi(\Delta_0)$	0.9
$\Delta_2 = \phi(\Delta_1)$	0.361
$\Delta_3 = \phi(\Delta_2)$	0.185
$\vdots$	$\vdots$
Δ_*	$4 - \sqrt{4^2 - 1} \approx 0.127$



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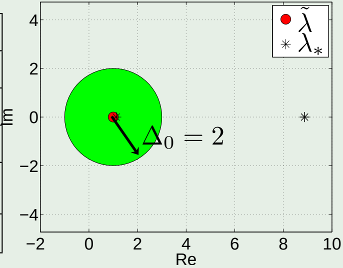

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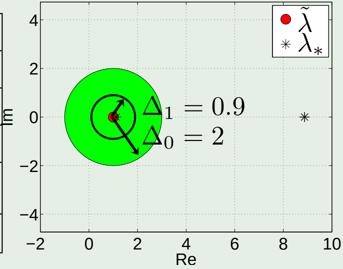

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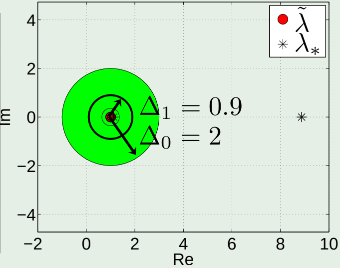

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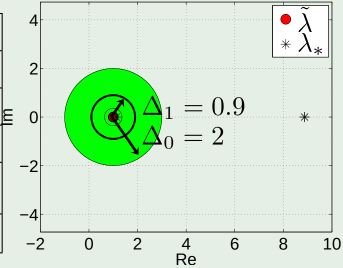

Error of $\tilde{\lambda}$ less than $\Delta_0 = 2 \Rightarrow$ error less than $\Delta_* = 0.127$.

Main results:

- A BF-NLVP comparison theorem
- Accuracy iteration with **constant comparison**
- Accuracy iteration with **linear comparison**
- Accuracy iteration with **linear normalized comparison**

Theorem (BF-NLEVP - General comparison)

Let $\lambda_1 \in \sigma(G_1(\lambda_1))$, $\lambda_2 \in \sigma(G_2(\lambda_2))$ *and*

$$\|G_1(\lambda) - G_2(\lambda)\| \leq \delta,\ \|G_1'(\lambda)\| \leq \varepsilon_1,\ \|G_2'(\lambda)\| \leq \varepsilon_2.$$

for all $\lambda \in V$. *Then,*

a) $$|\lambda_1 - \lambda_2| \leq \frac{\kappa_2}{1 - \kappa_2 \varepsilon_2} \delta$$

b) $$|\lambda_1 - \lambda_2| \leq \frac{\kappa_1}{1 - \kappa_1 \varepsilon_1} \delta$$

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$$\|G_1(\lambda) - G_2(\lambda)\| \leq \delta,\ \|G_1'(\lambda)\| \leq \varepsilon_1,\ \|G_2'(\lambda)\| \leq \varepsilon_2.$$

for all $\lambda \in V$. *Then,*

$$a) \qquad \lambda_2 = \underset{\lambda \in \sigma(G_2(\lambda_2))}{\operatorname{argmin}} |\lambda - \lambda_1| \Rightarrow |\lambda_1 - \lambda_2| \leq \frac{\kappa_2}{1 - \kappa_2 \varepsilon_2}\delta$$

$$b) \qquad \lambda_1 = \underset{\lambda \in \sigma(G_1(\lambda_1))}{\operatorname{argmin}} |\lambda - \lambda_2| \Rightarrow |\lambda_1 - \lambda_2| \leq \frac{\kappa_1}{1 - \kappa_1 \varepsilon_1}\delta$$

Theorem (BF-NLEVP - General comparison)

Let $\lambda_1 \in \sigma(G_1(\lambda_1))$, $\lambda_2 \in \sigma(G_2(\lambda_2))$ *and*

$$\|G_1(\lambda) - G_2(\lambda)\| \leq \delta,\ \|G_1'(\lambda)\| \leq \varepsilon_1,\ \|G_2'(\lambda)\| \leq \varepsilon_2.$$

for all $\lambda \in V$. *Then,*

$$a) \qquad \lambda_2 = \underset{\lambda \in \sigma(G_2(\lambda_2))}{\operatorname{argmin}} |\lambda - \lambda_1| \Rightarrow |\lambda_1 - \lambda_2| \leq \frac{\kappa_2}{1 - \kappa_2 \varepsilon_2}\delta$$

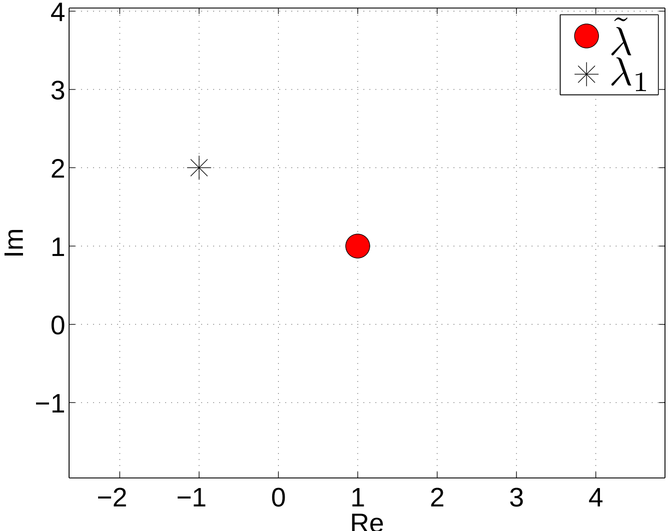
$$b) \qquad \lambda_1 = \underset{\lambda \in \sigma(G_1(\lambda_1))}{\operatorname{argmin}} |\lambda - \lambda_2| \Rightarrow |\lambda_1 - \lambda_2| \leq \frac{\kappa_1}{1 - \kappa_1 \varepsilon_1}\delta$$

Pf. Sketch.

$$\begin{aligned}
|\lambda_1 - \lambda_2| &\overset{BF}{\leq} \kappa_1 \|G_1(\lambda_1) - G_2(\lambda_2)\| \\
&\leq \kappa_1 \left(\|G_1(\lambda_1) - G_1(\lambda_2)\| + \|G_2(\lambda_2) - G_1(\lambda_2)\| \right) \\
&\leq \kappa_1 \left(\varepsilon_1 |\lambda_1 - \lambda_2| + \delta \right) \Rightarrow |\lambda_1 - \lambda_2| \leq \frac{\kappa_1}{1 - \kappa_1 \varepsilon_1}\delta
\end{aligned}$$

Accuracy iteration (constant comparison)

Given $\tilde{\lambda}$, how well does it approximate $\lambda_1 \in \sigma(G(\lambda_1))$?



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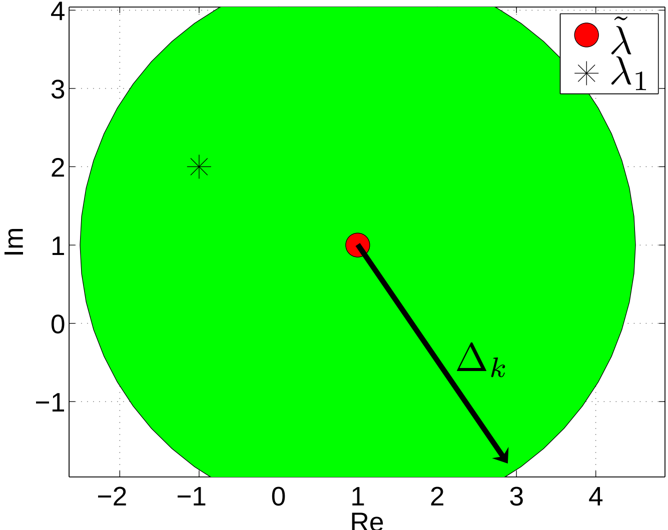
Given $\tilde{\lambda}$, how well does it approximate $\lambda_1 \in \sigma(G(\lambda_1))$?

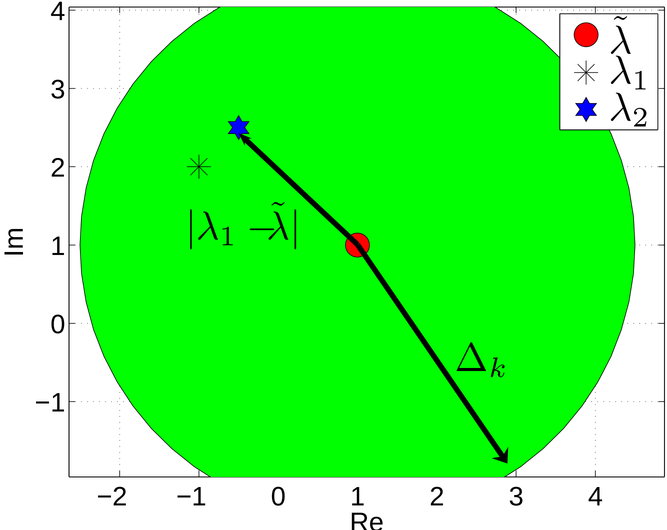
Assume $\lambda_1 \in V(\Delta_k)$.



Accuracy iteration (constant comparison)

Given $\tilde{\lambda}$, how well does it approximate $\lambda_1 \in \sigma(G(\lambda_1))$?

Assume $\lambda_1 \in V(\Delta_k)$. Define $\lambda_2 \in \sigma(G(\tilde{\lambda}))$.



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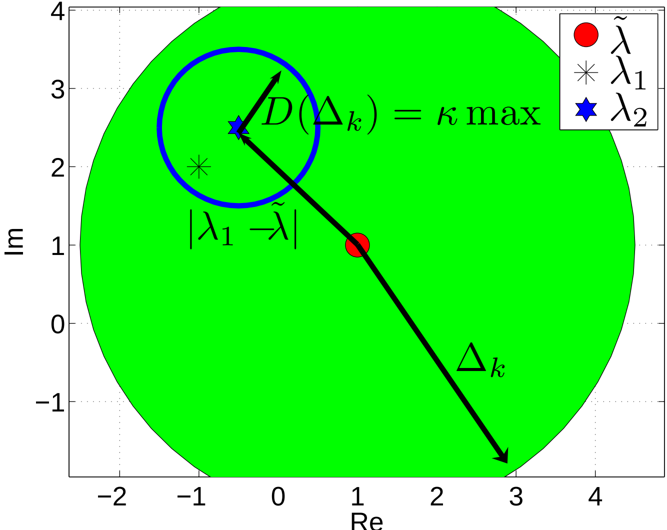
Given $\tilde{\lambda}$, how well does it approximate $\lambda_1 \in \sigma(G(\lambda_1))$?

Assume $\lambda_1 \in V(\Delta_k)$. Define $\lambda_2 \in \sigma(G(\tilde{\lambda}))$. Use BF-NLEVP to bound $|\lambda_1 - \lambda_2| < D(\Delta_k)$.



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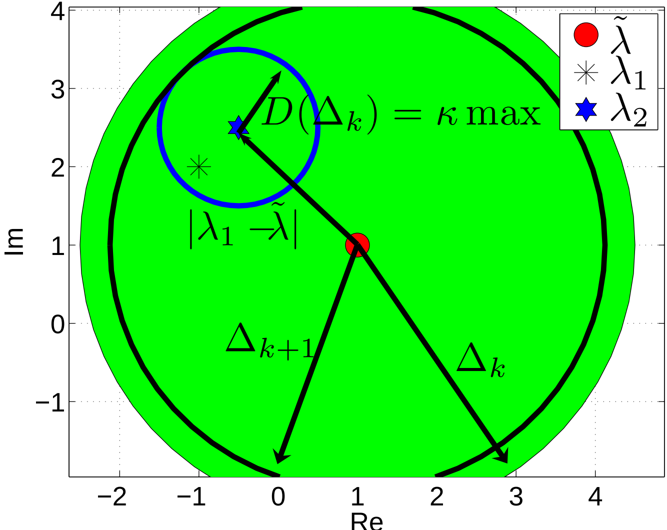
Assume $\lambda_1 \in V(\Delta_k)$. Define $\lambda_2 \in \sigma(G(\tilde{\lambda}))$. Use BF-NLEVP to bound $|\lambda_1 - \lambda_2| < D(\Delta_k)$. Triangle inequality $|\lambda_1 - \tilde{\lambda}| \le |\tilde{\lambda} - \lambda_2| + D(\Delta_k)$



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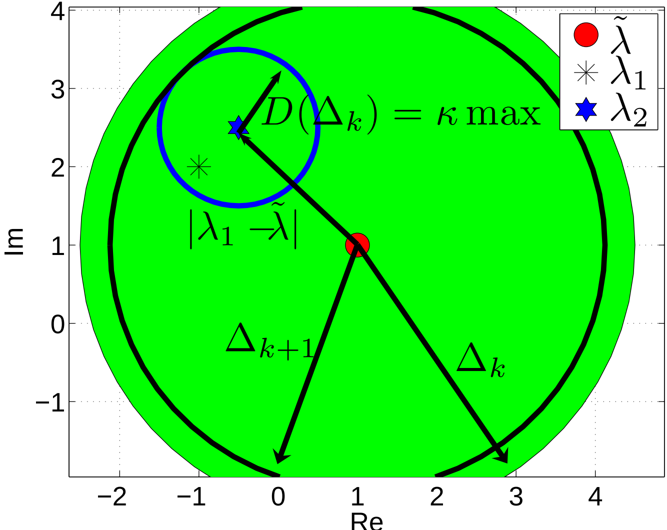
Assume $\lambda_1 \in V(\Delta_k)$. Define $\lambda_2 \in \sigma(G(\tilde{\lambda}))$. Use BF-NLEVP to bound $|\lambda_1 - \lambda_2| < D(\Delta_k)$. Triangle inequality $|\lambda_1 - \tilde{\lambda}| \leq |\tilde{\lambda} - \lambda_2| + D(\Delta_k) =: \Delta_{k+1}$



Accuracy iteration (constant comparison)

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Assume $\lambda_1 \in V(\Delta_k)$. Define $\lambda_2 \in \sigma(G(\tilde{\lambda}))$. Use BF-NLEVP to bound $|\lambda_1 - \lambda_2| < D(\Delta_k)$. Triangle inequality $|\lambda_1 - \tilde{\lambda}| \leq |\tilde{\lambda} - \lambda_2| + D(\Delta_k) =: \Delta_{k+1}$

Iteration

If $\lambda_1 \in V(\Delta_k) \Rightarrow \lambda_1 \in V(\Delta_{k+1})$ where

$$\Delta_{k+1} = |\tilde{\lambda} - \lambda_2| + \kappa_2 \max_{\lambda \in V(\Delta_k)} \|G(\lambda) - G(\tilde{\lambda})\|$$

Cubic toy example

$$A_0 = \begin{pmatrix} 1 & -1 & & & & & & \\ 1 & & -1 & & & & & \\ & 1 & & -1 & & & & \\ & & 1 & & -1 & & & \\ & & & 1 & & -1 & & \\ & & & & 1 & & -1 & \\ & & & & & 1 & & -1 \\ & & & & & & 1 & \end{pmatrix},\ A_1 = \begin{pmatrix} 0.02 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_* \in \sigma(A_0 + A_1 \lambda_*^3)$$

Cubic toy example

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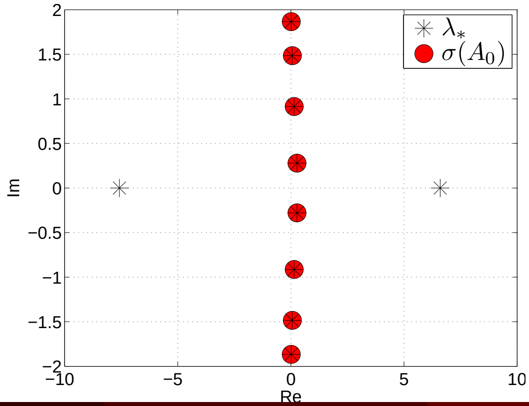
$$\lambda_* \in \sigma(A_0 + A_1\lambda_*^3)$$



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$$\lambda_* \in \sigma(A_0 + A_1\lambda_*^3)$$

Given approximation $\tilde{\lambda}$ we compute $\lambda_2 \in \sigma(A_0 + A_1\tilde{\lambda}^3)$

$$\Delta_{k+1} = |\lambda_2 - \tilde{\lambda}| + \kappa_2\|A_1\| \max_{\lambda \in V(\Delta_k)} |\lambda^3 - \tilde{\lambda}^3| =$$

$$= |\lambda_2 - \tilde{\lambda}| + \kappa_2\|A_1\| \left((|\tilde{\lambda}| + \Delta_k)^3 - |\tilde{\lambda}|^3 \right)$$

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$$= |\lambda_2 - \tilde{\lambda}| + \kappa_2\|A_1\| \left((|\tilde{\lambda}| + \Delta_k)^3 - |\tilde{\lambda}|^3 \right)$$

λ_*	$\tilde{\lambda}$	$\lvert\lambda_* - \tilde{\lambda}\rvert$	$\Delta_*^{(0)}$
$0.2721 \pm 0.2792i$	$0.2722 \pm 0.2787i$	5.82e-04	**6.02e-04**
$0.1447 \pm 0.9138i$	$0.1478 \pm 0.9135i$	3.09e-03	**3.89e-03**
$0.0596 \pm 1.4842i$	$0.0640 \pm 1.4848i$	4.49e-03	**9.44e-03**
$0.0140 \pm 1.8660i$	$0.0160 \pm 1.8664i$	1.99e-03	**1.20e-02**
6.5989			
-7.5798			

Accuracy iteration (linear comparison)

Compute one step of "Newton" (MSLP): $\lambda_2 \in \sigma(G(\tilde{\lambda}) + (\lambda_2 - \tilde{\lambda})G'(\tilde{\lambda}))$.

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Iteration (linear)

$$\Delta_{k+1} = |\lambda_2 - \tilde{\lambda}| + \frac{\kappa_2}{1 - \kappa_2 \|G'(\tilde{\lambda})\|} \max_{\lambda \in V(\Delta_k)} \|G(\lambda) - G(\tilde{\lambda}) - (\lambda - \tilde{\lambda})G'(\tilde{\lambda})\|$$

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For cubic toy example:

$$\Delta_{k+1} = |\lambda_2 - \tilde{\lambda}| + \frac{\kappa_2}{1 - 3\kappa_2 |\tilde{\lambda}|^2 \|A_1\|} \|A_1\| \max_{\lambda \in V(\Delta_k)} |\lambda^3 - \tilde{\lambda}^3 - 3(\lambda - \tilde{\lambda})\tilde{\lambda}^2|$$

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For cubic toy example:

$$\Delta_{k+1} = |\lambda_2 - \tilde{\lambda}| + \frac{\kappa_2}{1 - 3\kappa_2 |\tilde{\lambda}|^2 \|A_1\|} \|A_1\| \max_{\lambda \in V(\Delta_k)} |\lambda^3 - \tilde{\lambda}^3 - 3(\lambda - \tilde{\lambda})\tilde{\lambda}^2|$$

λ_*	$\tilde{\lambda}$	$\lvert\lambda_* - \tilde{\lambda}\rvert$	$\Delta_*^{(0)}$	$\boldsymbol{\Delta}_*^{(1)}$
$0.2721 \pm 0.2792i$	$0.2722 \pm 0.2787i$	5.82e-04	6.02e-04	**5.82e-04**
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$0.0596 \pm 1.4842i$	$0.0640 \pm 1.4848i$	4.49e-03	9.44e-03	**4.50e-03**
$0.0140 \pm 1.8660i$	$0.0160 \pm 1.8664i$	1.99e-03	1.20e-02	**1.20e-03**
6.5989				
-7.5798				

Example (DDE)

$$\lambda_* \in \sigma\left(\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} + \begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix} e^{-\lambda_*}\right)$$

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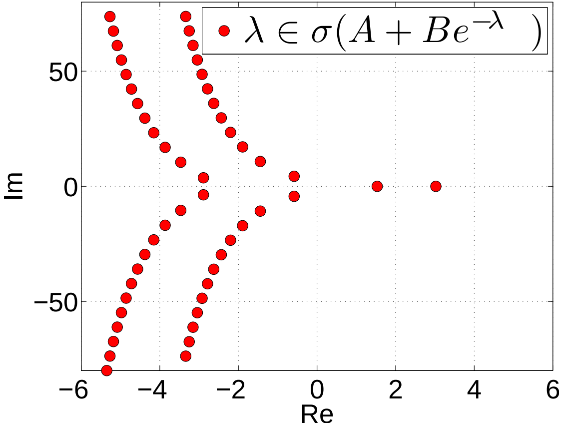

Example (DDE)

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Pick $\tilde{\lambda}$ as first LMS-approximation in DDE-BIFTOOL

λ_*	?	?	?	?	?
$\tilde{\lambda}$	1.5326	3.0246	-0.58±4.35i	-1.44±10.76i	-1.89±17.11i
$\lvert\lambda_* - \tilde{\lambda}\rvert$	?	?	?	?	?

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$\lvert\lambda_* - \tilde{\lambda}\rvert$	?	?	?	?	?
$\Delta_*^{(0)}$	5.04e-05	3.50e-06	Inf	Inf	Inf
$\Delta_*^{(1)}$	8.94e-06	2.06e-07	-Inf	-Inf	-Inf

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$$\Delta_*^{(0)} = |\lambda_2 - \tilde{\lambda}| - \kappa \|A_1\| e^{-\operatorname{Re}\tilde{\lambda}} - W_0\left(-\kappa \|A_1\| e^{-\operatorname{Re}\tilde{\lambda} + |\lambda_2 - \tilde{\lambda}| - \kappa \|A_1\| e^{-\operatorname{Re}\tilde{\lambda}}}\right)$$

Example (DDE)

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$\Delta_*^{(0)}$	5.04e-05	3.50e-06	Inf	Inf	Inf
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$\Delta_*^{(1n)}$	8.94e-06	2.06e-07	1.40e-05	5.45e-04	3.49e-03

Example (DDE)

$$\lambda_* \in \sigma\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} e^{-\lambda_*}\right)$$

Pick $\tilde{\lambda}$ as first LMS-approximation in DDE-BIFTOOL

λ_*	?	?	?	?	?
$\tilde{\lambda}$	1.5326	3.0246	-0.58±4.35i	-1.44±10.76i	-1.89±17.11i
$\lvert\lambda_* - \tilde{\lambda}\rvert \sim$	**8.93e-06**	**2.06e-07**	**1.40e-05**	**5.44e-04**	**3.48e-03**
$\Delta_*^{(0)}$	5.04e-05	3.50e-06	Inf	Inf	Inf
$\Delta_*^{(1)}$	8.94e-06	2.06e-07	-Inf	-Inf	-Inf
$\Delta_*^{(1n)}$	8.94e-06	2.06e-07	1.40e-05	5.45e-04	3.49e-03

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λ_*	?	?	?	?	?
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Normalization

$$\lambda \in \sigma(G(\lambda)) \Leftrightarrow \lambda \in \sigma(\underbrace{(I-A)^{-1}(G(\lambda)-\lambda A)}_{=:H(\lambda)})$$

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Normalization

$$\lambda \in \sigma(G(\lambda)) \Leftrightarrow \lambda \in \sigma(\underbrace{(I - A)^{-1}(G(\lambda) - \lambda A)}_{=:H(\lambda)})$$

Let $A = G'(\tilde{\lambda})$ then $H'(\tilde{\lambda}) = 0 \Rightarrow \frac{\kappa_2}{1-\kappa_2\varepsilon_2} = \kappa_2$

Final remarks and future work

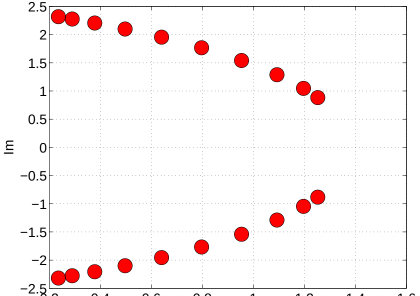
- A *Bauer-Fike comparison theorem* for nonlinear eigenvalue problems
- Accuracy of solutions *nonlinear eigenvalue problems* with a accuracy iterations
- Future: Relation to *method of linear/constant problems*

Example (DDE)

$$\lambda_* \in \sigma\left(\begin{pmatrix} & & & & & & & & & 1 \\ & & & & & & & & 1 & 1 \\ & & & & & & & 1 & 1 & \\ & & & & & & 1 & 1 & & \\ & & & & & 1 & 1 & & & \\ & & & & 1 & 1 & & & & \\ & & & 1 & 1 & & & & & \\ & & 1 & 1 & & & & & & \\ & 1 & 1 & & & & & & & \\ 1 & 1 & & & & & & & & \end{pmatrix} + \begin{pmatrix} -6 & 1 & & & & & & & & \\ 1 & -6 & 1 & & & & & & & \\ & 1 & -6 & 1 & & & & & & \\ & & 1 & -6 & 1 & & & & & \\ & & & 1 & -6 & 1 & & & & \\ & & & & 1 & -6 & 1 & & & \\ & & & & & 1 & -6 & 1 & & \\ & & & & & & 1 & -6 & 1 & \\ & & & & & & & 1 & -6 & 1 \\ & & & & & & & & 1 & -6 \end{pmatrix} e^{-\lambda_*}\right)$$

Example (DDE)

$$
\lambda_* \in \sigma\left(\begin{pmatrix}
 & & & & & & & & & 1\\
 & & & & & & & & 1 & 1\\
 & & & & & & & 1 & 1 & \\
 & & & & & & 1 & 1 & & \\
 & & & & & 1 & 1 & & & \\
 & & & & 1 & 1 & & & & \\
 & & & 1 & 1 & & & & & \\
 & & 1 & 1 & & & & & & \\
 & 1 & 1 & & & & & & & \\
1 & 1 & & & & & & & &
\end{pmatrix} + \begin{pmatrix}
-6 & 1 & & & & & & & & \\
1 & -6 & 1 & & & & & & & \\
 & 1 & -6 & 1 & & & & & & \\
 & & 1 & -6 & 1 & & & & & \\
 & & & 1 & -6 & 1 & & & & \\
 & & & & 1 & -6 & 1 & & & \\
 & & & & & 1 & -6 & 1 & & \\
 & & & & & & 1 & -6 & 1 & \\
 & & & & & & & 1 & -6 & 1\\
 & & & & & & & & 1 & -6
\end{pmatrix} e^{-\lambda_*}\right)
$$



Example (DDE)

$$\lambda_* \in \sigma\left(\begin{pmatrix} &&&&&&&&&1\\ &&&&&&&&1&1\\ &&&&&&&1&1&\\ &&&&&&1&1&&\\ &&&&&1&1&&&\\ &&&&1&1&&&&\\ &&&1&1&&&&&\\ &&1&1&&&&&&\\ &1&1&&&&&&&\\ 1&1&&&&&&&& \end{pmatrix} + \begin{pmatrix} -6&1&&&&&&&&\\ 1&-6&1&&&&&&&\\ &1&-6&1&&&&&&\\ &&1&-6&1&&&&&\\ &&&1&-6&1&&&&\\ &&&&1&-6&1&&&\\ &&&&&1&-6&1&&\\ &&&&&&1&-6&1&\\ &&&&&&&1&-6&1\\ &&&&&&&&1&-6 \end{pmatrix} e^{-\lambda_*}\right)$$

$\tilde{\lambda}$	$\Delta_*^{(0)}$	$\Delta_*^{(1)}$	$\Delta_*^{(1n)}$
1.2526±0.8837i	Inf	-Inf	2.62e-02
1.1967±1.0464i	Inf	-Inf	1.70e-01
1.0924±1.2885i	Inf	-Inf	2.80e-01
0.9536±1.5407i	NaN	-Inf	Inf
0.7973±1.7675i	NaN	-Inf	Inf
0.6404±1.9548i	NaN	-Inf	Inf
0.4973±2.0999i	NaN	-Inf	2.70e-01
0.3784±2.2058i	NaN	-Inf	1.73e-01
0.2900±2.2771i	Inf	-Inf	5.48e-02
0.2359±2.3179i	Inf	-Inf	6.46e-02