Iterative methods: Limits of performance via reachable set analysis

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1. Introduction
2. Reachable set analysis
3. Applications
   - Richardson Iteration
   - Inverse Iteration
4. Conclusion

Fact: Numeric is useful for control theory

(See talks of Embree, Mehrmann, Schröder, many more . . .)

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Many successful approaches:

Gustafsson et al. (1992), Batterson and Smillie (1990), Bhaya and Kaszkurewicz (2006), Grüne and Junge (2006)

Iterative methods with shifts

$$x_{t+1} = f(x_t, u_t)$$

Key Observation:

Iterative method with shift = Control system

Problems:

- How to find "good" or "optimal" $u_1, u_2, \ldots$
- How to find **feedback laws** Φ, s.t. $x_{t+1} = f(x_t, \Phi(x_t))$ converges
- Limits of performance

  $\longrightarrow$ Approach via reachable set analysis

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Reachable sets

Given $x_{t+1} = f(x_t, u_t)$, $u_t \in U$, $x_0 \in M$

Definition (Reachable set)

$$\mathcal{R}(x_0) = \{x \in M \,|\, x \text{ can be reached from } x_0 \text{ in finite many steps}\}$$

$$\overline{\mathcal{R}(x_0)} = \text{topological closure of } \mathcal{R}(x_0)$$

Observation: Let $\mathcal{E} \subset M$ set of desired states

If $u_1, u_2, \ldots$ with $x_t \to \mathcal{E}$ exists $\Rightarrow \overline{\mathcal{R}(x_0)} \cap \mathcal{E} \neq \emptyset$

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If $u_1, u_2, \ldots$ with $x_t \to \mathcal{E}$ exists $\Rightarrow$ $\overline{\mathcal{R}(x_0)} \cap \mathcal{E} \neq \emptyset$

If $\overline{\mathcal{R}(x_0)} \cap \mathcal{E}$ empty $\Rightarrow$ No convergent shift strategy

System semigroup

Definition (System semigroup)

$$S_\Sigma := \{f_{u_1} \circ \cdots \circ f_{u_T} \mid T \in \mathbb{N}, u_t \in U\};\ f_u := f(\cdot, u) : M \to M$$

Facts:

- $S_\Sigma \times M \to M$, $(s, x) \mapsto s(x)$ is a semigroup action
- Reachable set $=$ Semigroup orbit,
  i.e., $\mathcal{R}(x) = S_\Sigma \cdot x := \{s(x) \mid s \in S_\Sigma\}$

System group

Assumption: $f_u : M \to M, x \mapsto f(x, u)$ is invertible

Definition (System group)

$$G_\Sigma := \langle S_\Sigma \rangle := \{g_1 \circ \cdots \circ g_T \mid T \in N, g_t \in S_\Sigma \text{ or } g_t^{-1} \in S_\Sigma\}$$

Facts:

- Often: $S_\Sigma \neq G_\Sigma$
- $\mathcal{R}(x) \subset G_\Sigma \cdot x$
- Orbits of G_Σ form partition of M
- Often: $G_\Sigma \cdot x$ has geometric structure

Lemma

Let G_Σ be an abelian Lie group. **Then:**

$$\mathrm{Interior}_{G_\Sigma \cdot x}\, \mathcal{R}(x) \neq \emptyset$$

Lemma (J 2007)

Assume: (i) G_Σ abelian Lie group
(ii) $N := G_\Sigma \cdot x$ open and dense
(iii) $\mathcal{E} \subset \partial N$.

Then: *(1) $S_\Sigma = G_\Sigma$ implies $\overline{\mathcal{R}(x)} \cap \mathcal{E} \neq \emptyset$*

(2) $S_\Sigma \neq G_\Sigma$ implies

$$\overline{\mathcal{R}(x)} \cap \mathcal{E} = \emptyset \iff \overline{\mathcal{R}(y)} \cap \mathcal{E} = \emptyset \ \textit{ for all } \ y \in N$$

Apply reachable set analysis on numerical iteration schemes

(Following Fuhrmann and Helmke (2000), Helmke and Wirth (2001), Chu and Chu (2006), J (2007))

Richardson Iteration

Given $A \in \mathbb{R}^{n \times n}$ cyclic and invertible, $b \in \mathbb{R}^n$, $\mathcal{E} := \{A^{-1}b\}$

Richardson Iteration

$$x_{t+1} = x_t + u_t(b - Ax_t);\ \ x_0 \in \mathbb{R}^n$$

Facts:

- $G_{RI}(A)$ is an abelian Lie group
- $N_A = G_{RI}(A) \cdot x$ for almost all $x \in \mathbb{R}^n$.
- N_A is open and dense; $A^{-1}b \in \partial N_A$.

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Theorem (J 2007)

- interior $\mathcal{R}(x) \neq \emptyset$ *for all* $x \in N_A$.
- *If* $S_{RI}(A) = G_{RI}(A)$ *then* $A^{-1}b \in \overline{\mathcal{R}(x)}$ *for all* $x \in N_A$.
- $\exists \mathcal{F} \subset \mathbb{R}^{n \times n}$*, s.t.* $S_{RI}(A) \neq G_{RI}(A)$ *for all* $A \in \mathcal{F}$.
- *If* $A \in \mathcal{F}$ *then* $A^{-1}b \notin \overline{\mathcal{R}(x)}$ *for all* $x \in N_A$.

$\Rightarrow A \in \mathcal{F}$ **then** no convergence for GMRES(1), cyclic methods, etc.

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Examples for $A \in \mathcal{F}$:

$$A = T \begin{pmatrix} A_1 & 0 \\ 0 & \tilde{A} \end{pmatrix} T^{-1}, \; T \in \mathrm{GL}_n(\mathbb{R})$$

with

$$A_1 = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}, \; |\alpha| > 1;$$

Inverse Iteration

Given $A \in \mathbb{R}^{n \times n}$, cyclic

Inverse Iteration

$$x_{t+1} = (A - u_t I)^{-1} \cdot x_t; \; x_0 \in \mathbb{RP}^{n-1}$$

Set of desired points: $\mathcal{E} := \{$ eigenspaces $\}$

Facts:

- $G_{II}(A)$ is an abelian Lie group
- $N_A = G_{II}(A) \cdot x$ for almost all $x \in \mathbb{RP}^{n-1}$.
- N_A is open and dense; $\mathcal{E} \subset \partial N_A$

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Inverse Iteration

Theorem (Helmke, Wirth (2001),)

- *Generically* Inter $\mathcal{R}(x) \neq \emptyset$ *iff A cyclic.*
- *There exists a family* $\mathcal{F} \subset \mathbb{R}^{n \times n}$, *s.t.*

$$\mathcal{R}(x) \subsetneq G_{II}(A) \cdot x, \quad A \in \mathcal{F}$$

- *For all* $A \in \mathcal{F}$ *there exists an eigenspace* $\mathcal{E}$ *s.t.*

$$\mathcal{E} \cap \overline{\mathcal{R}(x)} = \emptyset$$

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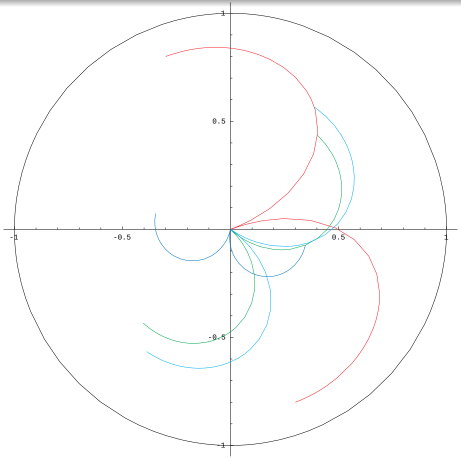
$$\text{with: } A_1 = \begin{pmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}, \beta \neq 0;$$

$$\text{or with: } A_2 = \begin{pmatrix} \alpha+\gamma & 0 & 0 & 0 \\ 0 & \alpha-\gamma & 0 & 0 \\ 0 & 0 & \gamma & \beta \\ 0 & 0 & -\beta & \gamma \end{pmatrix}, \; |\beta| > |\gamma|;$$

Example: $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{F}; \quad x_0 = \begin{pmatrix} -0.3 \\ 0.8 \\ 0.854 \end{pmatrix}$



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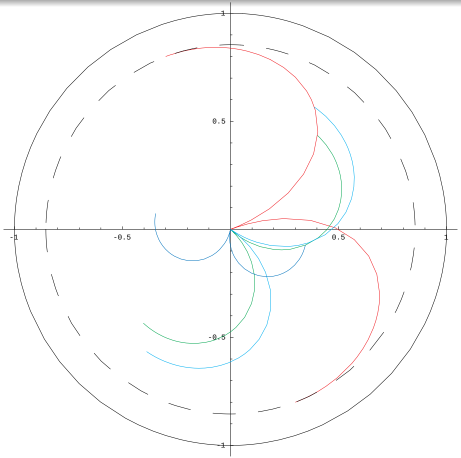
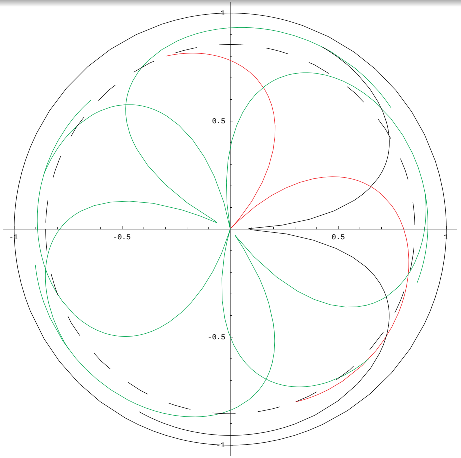


Example: $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0.1 \end{pmatrix} \notin \mathcal{F}; \quad x_0 = \begin{pmatrix} -0.3 \\ 0.8 \\ 0.854 \end{pmatrix}$



How to "repair" Inverse Iteration scheme?

Given $A \in \mathbb{R}^{n\times n}$, cyclic

Rational Iteration

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Set of desired points: $\mathcal{E} := \{$ eigenspaces $\}$

Theorem (J 07)

- $G_{RatI}(A) = S_{RatI}(A)$ *is an abelian Lie group*
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Conclusion and remarks

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- **Iterative algorithms** can be regarded as **control systems**
- The structure of **reachable sets** gives constraints on the **existence of shift strategies**
- In particular: Limitations for the convergence behavior of **Richardson Iteration** and **Inverse Iteration:**

Question:

How can we use this information to create new algorithms.

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