A technique for computing minors of orthogonal $(0, \pm 1)$ matrices

and an application to the Growth Problem

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Outline

1. Introduction
   - Definitions
   - Importance of this study
   - Preliminary Results
2. A technique for minors
3. Main Results
4. Application to the growth problem
   - Background
   - The proposed idea
5. Numerical experiments

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Definition. A is *orthogonal in a generalized sense* if

$$AA^T = A^T A = kI_n$$

or

$$AA^T = A^T A = k(I_n + J_n).$$

Examples.

1. A *Hadamard matrix* H of order n is an ± 1 matrix satisfying

$$HH^T = H^T H = nI_n.$$

2. A *weighing matrix of order* n *and weight* $n-k$ is a $(0, 1, -1)$ matrix $W = W(n, n-k)$, $k = 1, 2, \ldots,$ satisfying

$$WW^T = W^T W = (n-k)I_n.$$

$W(n, n)$, $n \equiv 0 \pmod 4$, is a Hadamard matrix.

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3. A *binary Hadamard matrix* or *S-matrix* is a $n \times n$ $(0,1)$ matrix S satisfying

$$SS^T = S^T S = \frac{1}{4}(n+1)(I_n + J_n).$$

Properties

1. $n \equiv 3 \pmod 4$.
2. $SJ_n = J_n S = \frac{1}{2}(n+1)J_n$
3. the inner product of every two rows and columns is $\frac{n+1}{4}$, if they are distinct, and $\frac{n+1}{2}$, otherwise.
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Construction. Take an $(n+1)\times(n+1)$ Hadamard matrix with first row and column all +1's, change +1's to 0's and -1's to +1's, and delete the first row and column.

Example.

$$H_4 = \left[\begin{array}{c|ccc} 1 & 1 & 1 & 1 \\ \hline 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right] \rightarrow S_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

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$$H_8 = \left[\begin{array}{c|ccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{array}\right]$$

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Why Hadamard, weighing and S-matrices?

1. Numerous Applications in various areas of Applied Mathematics:
   - Statistics-Theory of Experimental Designs
   - Coding Theory
   - Cryptography
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   - Image Processing
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(1) old and intensively studied mathematical object, but even nowadays of great research interest;

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(3) it is always useful to find analytical formulas of determinants of matrices with special structure and properties, e.g.

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- Hankel
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matrices.
Benefits:

- more efficient evaluation of determinants – avoidance of computational failure due to traditional expansion methods;
- more insight on some properties of a matrix.

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Preliminary Results.

Lemma

Let $A = (k-\lambda)I_v + \lambda J_v = \begin{bmatrix} k & \lambda & \cdots & \lambda \\ \lambda & k & \cdots & \lambda \\ \vdots & & \ddots & \\ \lambda & \lambda & \cdots & k \end{bmatrix}$*, where* $k,\ \lambda$ *are integers. Then,*

$$det\ A = [k + (v-1)\lambda](k-\lambda)^{v-1} \quad (1)$$

and for $k \neq \lambda, -(v-1)\lambda$*,* A *is nonsingular with* $A^{-1} =$

$$\frac{1}{k^2 + (v-2)k\lambda - (v-1)\lambda^2}\{[k + (v-2)\lambda + \lambda]I_v - \lambda J_v\}. \quad (2)$$

Lemma

Let $B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$, B_1 *nonsingular. Then*

$$det\ B = det\ B_1 \cdot det(B_4 - B_3 B_1^{-1} B_2). \tag{3}$$

Strategy for calculating all possible $(n-j)\times(n-j)$ minors of $(0,\pm 1)$ orthogonal matrices

Input: $A \in R^{n\times n}$, $AA^T = A^TA = kI_n$ for some k.

Write A in the form

$$A = \begin{bmatrix} B_{j\times j} & U_{j\times(n-j)} \\ V_{(n-j)\times j} & M_{(n-j)\times(n-j)} \end{bmatrix},$$

same columns clustered together in U.

Output: the appearing values for $det\ M$ for every possible upper left $j\times j$ corner B

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Main steps

1. Set up the linear system with unknowns the numbers of columns of U;

   (it results from the properties of A)

2. Figure out $M^T M$ taking into account $A^T A = kI_n$ and write the result in block form;

   (known block sizes $\leftrightarrow$ solution of the system)

3. Derive $det\ M^T M$ by consecutive applications of formula (3), with help of (1) and (2).

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Remarks.

1. Orthogonality of $A \Rightarrow$ all diagonal blocks of $M^T M$ will be of the form $(a-b)I + bJ$ and the others of the form cJ;
2. $M^T M$ is always symmetric and so is every principal submatrix of it;
3. Computations carried out effectively by exploiting structure;
4. All possible $(n-j) \times (n-j)$ minors are calculated;
5. Same columns are clustered together in $U_{j\times(n-j)}$, e.g.

$$U_3 = \begin{array}{cccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{array}$$

   $\Rightarrow$ computations are facilitated by the appearing block forms and derivation of formulas is possible;
6. The technique is demonstrated through the comprehensive example of S-matrices.

Main Results

Proposition

Let S be an S-matrix of order n. Then all possible $(n-1) \times (n-1)$ minors of S are of magnitude $2^{1-n}(n+1)^{\frac{n-1}{2}}$, and for $n > 2$ all possible $(n-2) \times (n-2)$ minors of S are of magnitude 0 or $2^{3-n}(n+1)^{\frac{n-3}{2}}$.

For $j > 2 \rightarrow$ the solution of the linear system has parameters. Bounds can be found with:

Lemma

For all possible columns $\underline{u}_1, \ldots, \underline{u}_{2^j}$ of an S-matrix S comprising the first j rows, $j \geq 3$, it holds

$$0 \leq u_i \leq \frac{n-3}{4}, \textit{for} \ \ i \in \left\{1, \ldots, \frac{1}{8} \cdot 2^j\right\} \cup \left\{\frac{7}{8} \cdot 2^j + 1, \ldots, 2^j\right\}$$

and

$$0 \leq u_i \leq \frac{n+1}{4}, \ \textit{otherwise}.$$

For $j > 2$, using the previous Lemma we get only n-dependant results

Proposition

Let S be an S-matrix of order $n = 11$. Then all possible $(n-3) \times (n-3)$ minors of S are of magnitude 0 or $2^{5-n}(n+1)^{\frac{n-5}{2}}$, and all possible $(n-4) \times (n-4)$ minors of S are of magnitude 0, $2^{7-n}(n+1)^{\frac{n-7}{2}}$ or $2^{8-n}(n+1)^{\frac{n-7}{2}}$.

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Application to the growth problem

Definition. For a *completely pivoted* (CP, no row and column exchanges are needed during GE with complete pivoting) matrix A the *growth factor* is given by

$$g(n, A) = \frac{max\{p_1, p_2, \ldots, p_n\}}{|a_{11}|},$$

where $p_1, p_2, \ldots, p_n$ are the pivots of A.

The Growth Problem: Determining $g(n, A)$ for CP $A \in R^{n \times n}$ and for various values of n.

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The Growth Problem: Determining $g(n, A)$ for CP $A \in R^{n \times n}$ and for various values of n.

Open conjecture (Cryer,1968):
For a CP Hadamard matrix H, $g(n, H) = n$.

New conjecture :
For a CP S-matrix S, $g(n, S) = \frac{n+1}{2}$.

First approach: $g(11, S_{11}) = 6$.
In other words, every possible S_{11} has growth 6.

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Difficulty of the problem

Pivot pattern invariant under equivalence operations,
i.e. equivalent matrices may have different pivot patterns.

A naive computer exhaustive search finding all possible S_{11} matrices by performing all possible row and/or column interchanges requires $(11!)^2 \approx 10^{15}$ trials.

In addition, the pivot pattern of each one of these matrices should be computed.

$\rightarrow$ many years of computations!

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Solution

Main idea 1: Calculation of pivots from the beginning and from the end *with different techniques*

$$p_1 \quad p_2 \quad \dots \quad p_6 \ \vdots \ p_7 \ \vdots \ p_8 \quad \dots \quad p_{11}$$

$$\longrightarrow \qquad\qquad\qquad\qquad \longleftarrow$$

and

$$p_7 = \frac{det\ S}{\prod_{i=1, i \neq 7}^{11} p_i}$$

Solution

Main idea 2: Calculate pivots with:

Lemma

Let A be a CP matrix and $A(j)$ denote the $j \times j$ principal minor of A.

(i) [Gantmacher 1959] *The magnitude of the pivots appearing after application of GE operations on A is given by*

$$p_j = \frac{A(j)}{A(j-1)}, \quad j = 1, 2, \ldots, n, \quad A(0) = 1. \tag{4}$$

(ii) [Cryer 1968] *The maximum $j \times j$ minor of A is $A(j)$.*

Main result

Theorem

If GE with complete pivoting is performed on an S-matrix of order 11 the pivot pattern is

$$(1, 1, 2, \frac{3}{2}, \frac{5}{3}, \frac{9}{5}, 2, \frac{3}{2}, 3, 3, 6).$$

So, the growth factor is 6.

Numerical experiments

class	pivot patterns (n=15)	number
I	$(1, 1, 2, 1, \frac{4}{3}, 1, 2, 1, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	12
	$(1, 1, 2, 1, \frac{4}{3}, 2, 3, 1, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	
	$(1, 1, 2, \frac{3}{2}, \frac{4}{3}, 1, 2, 2, 2, 2, 4, 4, 4, 4, 8)$	
II	$(1, 1, 2, 1, \frac{5}{3}, \frac{6}{5}, 2, 1, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	15
	$(1, 1, 2, 1, \frac{5}{3}, \frac{6}{5}, 2, \frac{4}{3}, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	
	$(1, 1, 2, 1, \frac{5}{3}, \frac{6}{5}, 2, \frac{8}{5}, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	
III	$(1, 1, 2, 1, \frac{4}{3}, \frac{9}{5}, 2, 1, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	18
	$(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, 1, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	
	$(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, \frac{4}{3}, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	
IV	$(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, 1, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	16
	$(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, \frac{4}{3}, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	
	$(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, \frac{8}{5}, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	
V	$(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, 2, 2, \frac{12}{5}, \frac{8}{3}, 2, 4, 4, 8)$	16
	$(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, 2, \frac{20}{9}, \frac{12}{5}, \frac{8}{3}, 2, 4, 4, 8)$	
	$(1, 1, 2, \frac{3}{2}, \frac{5}{3}, \frac{9}{5}, 2, 2, \frac{20}{9}, \frac{12}{5}, \frac{8}{3}, 2, 4, 4, 8)$	

Numerical experiments

n	pivot patterns	number
19	$(1, 1, 2, \frac{3}{2}, \frac{5}{3}, \frac{9}{5}, 2, \ldots, \frac{5}{2}, \frac{5}{2}, \frac{10}{3}, \frac{5}{2}, 5, 5, 10)$ $(1, 1, 2, \frac{3}{2}, \frac{5}{3}, \frac{9}{5}, \frac{9}{4}, \ldots, \frac{25}{9}, 3, 5, \frac{5}{2}, 5, 5, 10)$ $(1, 1, 2, \frac{3}{2}, \frac{5}{3}, \frac{9}{5}, \frac{5}{2}, \ldots, \frac{25}{8}, \frac{15}{4}, 5, \frac{5}{2}, 5, 5, 10)$	187
23	$(1, 1, 2, 1, \frac{5}{3}, \frac{8}{5}, 2, \ldots, 3, 3, 4, 3, 6, 6, 12)$ $(1, 1, 2, \frac{3}{2}, \frac{5}{2}, \frac{9}{5}, 3, \ldots, \frac{10}{3}, \frac{18}{5}, 6, 3, 6, 6, 12)$ $(1, 1, 2, \frac{3}{2}, 2, 2, 4, \ldots, \frac{15}{4}, \frac{9}{2}, 6, 3, 6, 6, 12)$	228

These results lead to the following conjecture.

The growth conjecture for S-matrices

Let S be an S-matrix of order n. Reduce S by GE with complete pivoting. Then, for large enough n,

(i) $g(n, S) = \frac{n+1}{2}$;

(ii) The three last pivots are (in backward order)

$$\frac{n+1}{2}, \frac{n+1}{4}, \frac{n+1}{4};$$

(iii) The fourth pivot from the end can be $\frac{n+1}{8}$ or $\frac{n+1}{4}$;

(iv) Every pivot before the last has magnitude at most $\frac{n+1}{2}$;

(v) The first three pivots are equal to $1, 2, 2$. The fourth pivot can take the values 1 or $3/2$.

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Conclusions-Discussions-Open Problems

- We proposed a technique for calculating all possible $(n-j)\times(n-j)$ minors of various $(0,\pm1)$ orthogonal matrices and demonstrated it with S-matrices;
- All possible pivots of the $S_{11} \rightarrow g(11, S_{11}) = 11$.
- Methods presented here can be used as basis for calculating the pivot pattern of S-matrices of higher orders, such as 15, 19 etc.
- High complexity of such problems $\rightarrow$ more effective implementation of the ideas introduced here, or other, more elaborate ideas.
- Reliable (i.e. non-skipping values) criterion for reducing the total amount of all possible upper left corners B?
- More precise upper bound than Lemma 3?

- Parallel implementation of the two main independent tasks.
- Statistical approach of the growth problem for Hadamard and S-matrices by examining the distribution of the pivots, according to:
  L. N. Trefethen and R. S. Schreiber, *Average-case stability of Gaussian elimination*, SIAM J. Matrix Anal. Appl. **11**, 335–360 (1990)
- Generalization for OD's: An *orthogonal design* (OD) of order n and type $(u_1, u_2, \ldots, u_t)$, u_i positive integers, is an $n \times n$ matrix D with entries from the set $\{0, \pm x_1, \pm x_2, \ldots, \pm x_t\}$ that satisfies

$$DD^T = D^T D = \left(\sum_{i=1}^{t} u_i x_i^2 \right) I_n.$$

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