A technique for computing minors of orthogonal $(0, \pm 1)$ matrices and an application to the Growth Problem

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joint work with Marilena Mitrouli - Harrachov 2007

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Outline

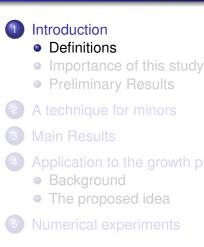


- Definitions
- Importance of this study
- Preliminary Results
- A technique for minors
- 3 Main Results
- Application to the growth problem
 - Background
 - The proposed idea
- 5 Numerical experiments

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Outline



Definitions Importance Preliminaries

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Definitions Importance Preliminaries

Definition. *A* is orthogonal in a generalized sense if $AA^T = A^T A = kI_n$

or

$$AA^{T} = A^{T}A = k(I_{n} + J_{n}).$$

Examples.

1. A Hadamard matrix H of order n is an ±1 matrix satisfying $HH^T = H^T H = nI_n.$

2. A weighing matrix of order n and weight n - k is a (0, 1, -1) matrix W = W(n, n - k), k = 1, 2, ..., satisfying

$$WW^T = W^T W = (n-k)I_n.$$

 $W(n, n), n \equiv 0 \pmod{4}$, is a Hadamard matrix.

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3. A *binary Hadamard matrix* or *S*-matrix is a $n \times n(0, 1)$ matrix *S* satisfying

$$SS^T = S^T S = \frac{1}{4}(n+1)(I_n+J_n).$$

Properties

$$1 n \equiv 3 \pmod{4}.$$

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$$SJ_n = J_n S = \frac{1}{2}(n+1)J_n$$

- It in the product of every two rows and columns is $\frac{n+1}{4}$, if they are distinct, and $\frac{n+1}{2}$, otherwise.
- (4) the sum of the entries of every row and column is $\frac{n+1}{2}$.

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Properties

$$n \equiv 3 \,(\text{mod } 4).$$

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$$SJ_n = J_n S = \frac{1}{2}(n+1)J_n$$

- So the inner product of every two rows and columns is $\frac{n+1}{4}$, if they are distinct, and $\frac{n+1}{2}$, otherwise.
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Construction. Take an $(n + 1) \times (n + 1)$ Hadamard matrix with first row and column all +1's, change +1's to 0's and -1's to +1's, and delete the first row and column.

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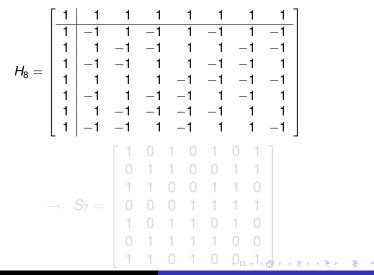
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A technique for minors Main Results Application to the growth problem Numerical experiments Summary-References

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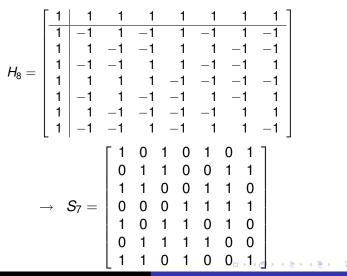
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Minors of $(0, \pm 1)$ orthogonal matrices

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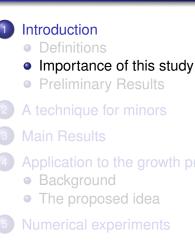
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Why Hadamard, weighing and S-matrices?

- Numerous Applications in various areas of Applied Mathematics:
 - Statistics-Theory of Experimental Designs
 - Coding Theory
 - Cryptography
 - Combinatorics
 - Image Processing
 - Signal Processing
 - Analytical Chemistry
- Interesting properties regarding the size of the pivots appearing after application of Gaussian Elimination (GE)

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Why computations of determinants?

(1) old and intensively studied mathematical object, but even nowadays of great research interest;

C. Krattenthaler, *Advanced determinant calculus: A complement*, Linear Algebra Appl., **411**, 68–166 (2005)

(2) contain their own intrinsic beauty;

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(3) it is always useful to find analytical formulas of determinants of matrices with special structure and properties, e.g.

- Vandermonde
- Hankel
- Cauchy
- integer

matrices.

Benefits:

- more efficient evaluation of determinants avoidance of computational failure due to traditional expansion methods;
- more insight on some properties of a matrix.
- (4) knowledge of determinants may lead to solution of interesting problems, e.g.
 - the growth problem;
 - evaluation of compound matrices.

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Generally: difficult and interesting problem to obtain analytical formulas for minors of various orders for a given arbitrary matrix but

possible for $(0, \pm 1)$ orthogonal matrices due to their special structure and properties.

First known effort for calculating the n - 1, n - 2 and n - 3 minors of Hadamard matrices:

F. R. Sharpe, *The maximum value of a determinant*, Bull. Amer. Math. Soc. **14**, 121–123 (1907)

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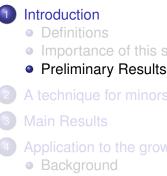
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Outline



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Preliminaries

Definitions Importance Preliminaries

Preliminary Results.

Lemma

Let
$$\mathbf{A} = (\mathbf{k} - \lambda)\mathbf{I}_{\mathbf{v}} + \lambda\mathbf{J}_{\mathbf{v}} = \begin{bmatrix} \mathbf{k} & \lambda & \cdots & \lambda \\ \lambda & \mathbf{k} & \cdots & \lambda \\ \vdots & \ddots & \vdots \\ \lambda & \lambda & \cdots & \mathbf{k} \end{bmatrix}$$
, where \mathbf{k}, λ are

integers. Then,

$$det A = [k + (v - 1)\lambda](k - \lambda)^{v - 1}$$
(1)

and for $k \neq \lambda, -(v-1)\lambda$, A is nonsingular with $A^{-1} =$

$$\frac{1}{k^2+(\nu-2)k\lambda-(\nu-1)\lambda^2}\{[k+(\nu-2)\lambda+\lambda]I_{\nu}-\lambda J_{\nu}\}.$$
 (2)

Definitions Importance Preliminaries

Lemma Let $B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$, B_1 nonsingular. Then $det B = det B_1 \cdot det(B_4 - B_3 B_1^{-1} B_2).$ (3)

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Strategy for calculating all possible $(n - j) \times (n - j)$ minors of $(0, \pm 1)$ orthogonal matrices

Input: $A \in \mathbb{R}^{n \times n}$, $AA^T = A^T A = kI_n$ for some k. Write A in the form

$$A = \begin{bmatrix} B_{j \times j} & U_{j \times (n-j)} \\ V_{(n-j) \times j} & M_{(n-j) \times (n-j)} \end{bmatrix}$$

same columns clustered together in U.

Output: the appearing values for det M for every possible upper left $j \times j$ corner B

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Strategy for calculating all possible $(n - j) \times (n - j)$ minors of $(0, \pm 1)$ orthogonal matrices

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Set up the linear system with unknowns the numbers of columns of U;

(it results from the properties of A)

Pigure out $M^T M$ taking into account $A^T A = kI_n$ and write the result in block form;

(known block sizes \leftrightarrow solution of the system)

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Main steps

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Oerive $det M^T M$ by consecutive applications of formula (3), with help of (1) and (2).

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Remarks.

- Orthogonality of $A \Rightarrow$ all diagonal blocks of $M^T M$ will be of the form (a b)I + bJ and the others of the form cJ;
- *M^TM* is always symmetric and so is every principal submatrix of it;
- Omputations carried out effectively by exploiting structure;
- All possible $(n j) \times (n j)$ minors are calculated;
- Same columns are clustered together in $U_{j\times(n-j)}$, e.g.

<i>U</i> ₃ =	U_1	U_2	U ₃	U_4	U_5	U ₆	U_7	U ₈
	1	1	1	1	0	0	0	0
	1	1	0	0	1	1	0	0
	1	0	1	0	1	0	1	0

 \Rightarrow computations are facilitated by the appearing block forms and derivation of formulas is possible;

The technique is demonstrated through the comprehensive example of S-matrices.

Main Results

Proposition

Let *S* be an *S*-matrix of order *n*. Then all possible $(n-1) \times (n-1)$ minors of *S* are of magnitude $2^{1-n}(n+1)^{\frac{n-1}{2}}$, and for n > 2 all possible $(n-2) \times (n-2)$ minors of *S* are of magnitude 0 or $2^{3-n}(n+1)^{\frac{n-3}{2}}$.

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For $j > 2 \rightarrow$ the solution of the linear system has parameters. Bounds can be found with:

Lemma

For all possible columns $\underline{u}_1, \ldots, \underline{u}_{2^j}$ of an S-matrix S comprising the first j rows, $j \ge 3$, it holds

$$0 \leq u_i \leq rac{n-3}{4}$$
 , for $i \in \left\{1, \ldots, rac{1}{8} \cdot 2^j
ight\} \cup \left\{rac{7}{8} \cdot 2^j + 1, \ldots, 2^j
ight\}$

and

$$0 \le u_i \le \frac{n+1}{4} \ , \ \text{otherwise}.$$

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For j > 2, using the previous Lemma we get only *n*-dependent results

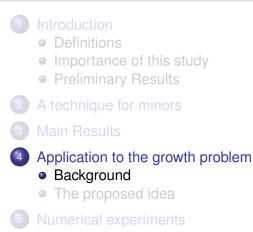
Proposition

Let *S* be an *S*-matrix of order n = 11. Then all possible $(n-3) \times (n-3)$ minors of *S* are of magnitude 0 or $2^{5-n}(n+1)^{\frac{n-5}{2}}$, and all possible $(n-4) \times (n-4)$ minors of *S* are of magnitude 0, $2^{7-n}(n+1)^{\frac{n-7}{2}}$ or $2^{8-n}(n+1)^{\frac{n-7}{2}}$.

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Application to the growth problem

Definition. For a *completely pivoted* (CP, no row and column exchanges are needed during GE with complete pivoting) matrix *A* the *growth factor* is given by

$$g(n, \mathbf{A}) = \frac{max\{p_1, p_2, \ldots, p_n\}}{|a_{11}|},$$

where p_1, p_2, \ldots, p_n are the pivots of *A*.

<u>The Growth Problem</u>: Determining g(n, A) for CP $A \in \mathbb{R}^{n \times n}$ and for various values of *n*.

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Open conjecture (Cryer, 1968): For a CP Hadamard matrix H, g(n, H) = n.

New conjecture : For a CP S-matrix S, $g(n, S) = \frac{n+1}{2}$.

First approach: $g(11, S_{11}) = 6$.

In other words, every possible S_{11} has growth 6.

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Difficulty of the problem

Pivot pattern invariant under equivalence operations, i.e. equivalent matrices may have different pivot patterns.

A naive computer exhaustive search finding all possible S_{11} matrices by performing all possible row and/or column interchanges requires $(11!)^2 \approx 10^{15}$ trials.

In addition, the pivot pattern of each one of these matrices should be computed.

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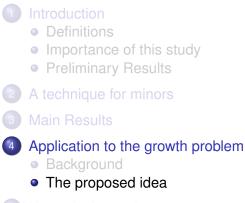
A naive computer exhaustive search finding all possible S_{11} matrices by performing all possible row and/or column interchanges requires $(11!)^2 \approx 10^{15}$ trials.

In addition, the pivot pattern of each one of these matrices should be computed.

→ many years of computations!

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Outline



Numerical experiments

The proposed idea

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Background The proposed idea

Solution

Main idea 1: Calculation of pivots from the beginning and from the end *with different techniques*

 $p_1 p_2 \ldots p_6 : p_7 : p_8 \ldots p_{11}$

and

$$p_7 = rac{\det S}{\prod_{i=1, i
eq 7}^{11} p_i}$$

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Background The proposed idea

Solution

Main idea 2: Calculate pivots with:

Lemma

Let A be a CP matrix and A(j) denote the $j \times j$ principal minor of A.

(i) [Gantmacher 1959] The magnitude of the pivots appearing after application of GE operations on A is given by

$$p_j = \frac{A(j)}{A(j-1)}, \quad j = 1, 2, \dots, n, \quad A(0) = 1.$$
 (4)

(ii) [Cryer 1968] The maximum $j \times j$ minor of A is A(j).

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Main result

Background The proposed idea

Theorem

If GE with complete pivoting is performed on an S-matrix of order 11 the pivot pattern is

$$(1, 1, 2, \frac{3}{2}, \frac{5}{3}, \frac{9}{5}, 2, \frac{3}{2}, 3, 3, 6).$$

So, the growth factor is 6.

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Numerical experiments

class	pivot patterns (n=15)	number
I	$(1, 1, 2, 1, \frac{4}{3}, 1, 2, 1, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	12
	$(1, 1, 2, 1, \frac{4}{3}, 2, 3, 1, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	
	$(1, 1, 2, \frac{3}{2}, \frac{4}{3}, 1, 2, 2, 2, 2, 4, 4, 4, 4, 8)$	
II	$\begin{array}{c}(1,1,2,1,\frac{5}{3},\frac{5}{5},2,1,2,2,\frac{3}{3},2,4,4,8)\\(1,1,2,1,\frac{5}{3},\frac{6}{5},2,\frac{4}{3},2,2,\frac{8}{3},2,4,4,8)\\(1,1,2,1,\frac{5}{3},\frac{6}{5},2,\frac{4}{3},2,2,\frac{8}{3},2,4,4,8)\\(1,1,2,1,\frac{5}{3},\frac{6}{5},2,\frac{8}{5},2,2,\frac{8}{3},2,4,4,8)\end{array}$	15
	$(1, 1, 2, 1, \frac{5}{3}, \frac{6}{5}, 2, \frac{4}{3}, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	
	$(1, 1, 2, 1, \frac{5}{3}, \frac{6}{5}, 2, \frac{8}{5}, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	
III	$(1, 1, 2, 1, \frac{4}{3}, \frac{9}{5}, 2, 1, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	18
	$(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, 1, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	
	$(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, \frac{4}{3}, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	
IV	$(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, 1, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	16
	$(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, \frac{4}{3}, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	
	$(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, \frac{8}{5}, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$	
V	$(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, 2, 2, \frac{12}{5}, \frac{8}{3}, 2, 4, 4, 8)$	16
	$(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, 2, \frac{20}{9}, \frac{12}{5}, \frac{8}{3}, 2, 4, 4, 8)$	
	$\begin{array}{c} (1,1,2,1,\frac{1}{3},\frac{1}{5},2,\frac{1}{5},2,\frac{1}{5},2,2,\frac{1}{3},2,4,4,8)\\ (1,1,2,1,\frac{4}{3},\frac{9}{5},2,1,2,2,\frac{8}{3},2,4,4,8)\\ (1,1,2,1,\frac{5}{3},\frac{9}{5},2,1,2,2,\frac{8}{3},2,4,4,8)\\ (1,1,2,1,\frac{5}{3},\frac{9}{5},2,\frac{4}{3},2,2,\frac{8}{3},2,4,4,8)\\ (1,1,2,1,\frac{5}{3},\frac{9}{5},2,\frac{4}{3},2,2,\frac{8}{3},2,4,4,8)\\ (1,1,2,1,\frac{5}{3},\frac{9}{5},2,\frac{4}{3},2,2,\frac{8}{3},2,4,4,8)\\ (1,1,2,1,\frac{5}{3},\frac{9}{5},2,\frac{8}{5},2,2,\frac{8}{3},2,4,4,8)\\ (1,1,2,1,\frac{5}{3},\frac{9}{5},2,\frac{8}{5},2,2,\frac{8}{3},2,4,4,8)\\ (1,1,2,1,\frac{5}{3},\frac{9}{5},2,2,2,2,\frac{12}{5},\frac{8}{3},2,4,4,8)\\ (1,1,2,1,\frac{5}{3},\frac{9}{5},2,2,2,2,\frac{12}{5},\frac{8}{3},2,4,4,8)\\ (1,1,2,1,\frac{5}{3},\frac{9}{5},2,2,2,2,\frac{12}{9},\frac{8}{3},2,4,4,8)\\ (1,1,2,1,\frac{5}{3},\frac{9}{5},2,2,2,2,\frac{12}{9},\frac{8}{3},2,4,4,8)\\ (1,1,2,1,\frac{5}{3},\frac{9}{5},2,2,2,2,\frac{12}{9},\frac{8}{3},2,4,4,8)\\ (1,1,2,1,\frac{5}{3},\frac{9}{5},2,2,2,2,\frac{12}{9},\frac{8}{3},2,4,4,8)\\ (1,1,2,1,\frac{5}{3},\frac{9}{5},2,2,2,2,\frac{12}{9},\frac{8}{3},2,4,4,8)\\ (1,1,2,1,\frac{5}{3},\frac{9}{5},2,2,2,2,\frac{12}{9},\frac{8}{3},2,4,4,8)\\ (1,1,2,\frac{1}{3},\frac{5}{3},\frac{5}{3},2,2,2,2,\frac{12}{9},\frac{8}{3},2,4,4,8)\\ (1,1,2,\frac{1}{3},\frac{5}{3},\frac{5}{3},2,2,2,2,\frac{12}{9},\frac{8}{3},2,4,4,8)\\ (1,1,2,\frac{1}{3},\frac{5}{3},\frac{5}{3},2,2,2,2,\frac{12}{9},\frac{8}{3},2,4,4,8)\\ (1,1,2,\frac{1}{3},\frac{5}{3},\frac{5}{3},2,2,2,2,\frac{12}{9},\frac{8}{3},2,4,4,8)\\ (1,1,2,\frac{1}{3},\frac{5}{3},\frac{5}{3},2,2,2,2,\frac{12}{9},\frac{8}{3},2,4,4,8)\\ (1,1,2,\frac{1}{3},\frac{5}{3},\frac{5}{3},2,2,2,2,\frac{12}{9},\frac{8}{3},2,4,4,8)\\ (1,1,2,\frac{1}{3},\frac{5}{3},\frac{5}{3},2,2,2,2,\frac{12}{9},\frac{8}{3},2,4,4,8)\\ (1,1,2,\frac{1}{3},\frac{5}{3},\frac{5}{3},\frac{5}{3},2,2,2,2,\frac{12}{9},\frac{8}{3},2,4,4,8)\\ (1,1,2,\frac{1}{3},\frac{5}{3},\frac{5}{3},2,2,2,2,\frac{12}{9},\frac{8}{3},2,2,4,4,8)\\ (1,1,2,\frac{1}{3},\frac{1}{3},\frac{5}{3},\frac{5}{3},2,2,2,\frac{12}{9},\frac{8}{3},2,4,4,8)\\ (1,1,2,\frac{1}{3},\frac{1}{3},\frac{5}{3},\frac{5}{3},2,2,2,\frac{12}{9},\frac{8}{3},2,4,4,8)\\ (1,1,2,\frac{1}{3},\frac{1}{3},\frac{5}{3},\frac{5}{3},2,2,2,\frac{12}{9},\frac{8}{3},2,4,4,8)\\ (1,1,2,\frac{1}{3},\frac{1}{3},\frac{5}{3},\frac{5}{3},2,2,2,\frac{12}{9},\frac{8}{3},2,4,4,8)\\ (1,1,2,\frac{1}{3},\frac{1}{3},\frac{5}{3},\frac{5}{3},2,2,2,\frac{12}{9},\frac{12}{3},\frac{8}{3},2,4,4,8)\\ (1,1,2,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{$	

C. Kravvaritis

Minors of $(0, \pm 1)$ orthogonal matrices

Numerical experiments

n	pivot patterns	number
19	$(1, 1, 2, \frac{3}{2}, \frac{5}{3}, \frac{9}{5}, 2, \dots, \frac{5}{2}, \frac{5}{2}, \frac{10}{3}, \frac{5}{2}, 5, 5, 10)$	187
	$(1, 1, 2, \frac{3}{2}, \frac{5}{3}, \frac{9}{5}, \frac{9}{4}, \dots, \frac{25}{9}, 3, 5, \frac{5}{2}, 5, 5, 10)$	
	$(1, 1, 2, \frac{3}{2}, \frac{5}{3}, \frac{9}{5}, \frac{5}{2}, \ldots, \frac{25}{8}, \frac{15}{4}, 5, \frac{5}{2}, 5, 5, 10)$	
23	$(1, 1, 2, 1, \frac{5}{3}, \frac{8}{5}, 2, \dots, 3, 3, 4, 3, 6, 6, 12)$	228
	$(1, 1, 2, \frac{3}{2}, \frac{5}{2}, \frac{9}{5}, 3, \ldots, \frac{10}{3}, \frac{18}{5}, 6, 3, 6, 6, 12)$	
	$(1, 1, 2, \frac{3}{2}, 2, 2, 4, \dots, \frac{15}{4}, \frac{9}{2}, 6, 3, 6, 6, 12)$	

These results lead to the following conjecture.

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The growth conjecture for S-matrices

Let S be an S-matrix of order n. Reduce S by GE with complete pivoting. Then, for large enough n,

(i)
$$g(n, S) = \frac{n+1}{2};$$

(ii) The three last pivots are (in backward order)

$$\frac{n+1}{2}, \frac{n+1}{4}, \frac{n+1}{4};$$

(iii) The fourth pivot from the end can be ⁿ⁺¹/₈ or ⁿ⁺¹/₄;
(iv) Every pivot before the last has magnitude at most ⁿ⁺¹/₂;
(v) The first three pivots are equal to 1, 2, 2. The fourth pivot can take the values 1 or 3/2.

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The growth conjecture for S-matrices

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(ii) The three last pivots are (in backward order)

$$\frac{n+1}{2}, \ \frac{n+1}{4}, \ \frac{n+1}{4};$$

(iii) The fourth pivot from the end can be $\frac{n+1}{8}$ or $\frac{n+1}{4}$;

- (iv) Every pivot before the last has magnitude at most $\frac{n+1}{2}$;
- (v) The first three pivots are equal to 1, 2, 2. The fourth pivot can take the values 1 or 3/2.

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Conclusions-Discussions-Open Problems

- We proposed a technique for calculating all possible (n − j) × (n − j) minors of various (0, ±1) orthogonal matrices and demonstrated it with S-matrices;
- All possible pivots of the $S_{11} \rightarrow g(11, S_{11}) = 11$.
- Methods presented here can be used as basis for calculating the pivot pattern of S-matrices of higher orders, such as 15, 19 etc.
- High complexity of such problems → more effective implementation of the ideas introduced here, or other, more elaborate ideas.
- Reliable (i.e. non-skipping values) criterion for reducing the total amount of all possible upper left corners *B*?
- More precise upper bound than Lemma 3?

- Parallel implementation of the two main independent tasks.
- Statistical approach of the growth problem for Hadamard and S-matrices by examining the distribution of the pivots, according to:

L. N. Trefethen and R. S. Schreiber, *Average-case stability of Gaussian elimination*, SIAM J. Matrix Anal. Appl. **11**, 335–360 (1990)

Generalization for OD's: An *orthogonal design* (OD) of order *n* and type (*u*₁, *u*₂,..., *u*_t), *u_i* positive integers, is an *n* × *n* matrix *D* with entries from the set {0, ±*x*₁, ±*x*₂,..., ±*x*_t} that satisfies

$$DD^{T} = D^{T}D = \left(\sum_{i=1}^{t} u_{i}x_{i}^{2}\right)I_{n}.$$

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Progress in the growth problem:

L. Tornheim, *Pivot size in Gauss reduction*, Tech. Report, Calif. Res. Corp., Richmond, Calif., February 1964.

C. W. Cryer, *Pivot size in Gaussian elimination*, Numer. Math. **12**, 335–345 (1968)

J. Day and B. Peterson, *Growth in Gaussian Elimination*, Amer. Math. Monthly **95**, 489–513 (1988)

N. Gould, *On growth in Gaussian elimination with pivoting*, SIAM J. Matrix Anal. Appl. **12**, 354–361 (1991)

A. Edelman and D. Friedman, *A counterexample to a Hadamard matrix pivot conjecture*, Linear Multilinear Algebra **44**, 53–56 (1998)



Books on orthogonal matrices:

A. V. Geramita and J. Seberry, *Orthogonal Designs: Quadratic Forms and Hadamard Matrices*, Marcel Dekker, New York-Basel (1979)

K. J. Horadam, *Hadamard matrices and their appplications*, Princeton University Press, Princeton (2007)

Existing work on S-matrices:

R. L. Graham and N. J. A. Sloane, *Anti-Hadamard Matrices*, Linear Algebra Appl., 62 (1984), pp. 113–137

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