The Discontinuous Galerkin Method for the Compressible Navier-Stokes Equations

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1. Discontinuous Galerkin Method Space Semidiscretization
   - Continuous Problem
   - Space semidiscretization

2. Time Discretization
   - Semi-implicit Time Discretization
   - Examples

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Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary $\partial\Omega = \Gamma_I \cup \Gamma_O \cup \Gamma_W$.

Continuous Problem

Find $\mathbf{w} : Q_T = \Omega \times (0,T) \to \mathbb{R}^4$ such that

$$\frac{\partial \mathbf{w}}{\partial t} + \sum_{s=1}^{2} \frac{\partial f_s(\mathbf{w})}{\partial x_s} = \sum_{s=1}^{2} \frac{\partial \mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w})}{\partial x_s} \quad \text{in } Q_T,$$

where

$$\mathbf{w} = (\rho, \rho v_1, \rho v_2, e)^{\mathrm{T}} \in \mathbb{R}^4,$$

$$f_i(w) = (\rho v_i, \rho v_1 v_i + \delta_{1i} p, \rho v_2 v_i + \delta_{2i} p, (e+p) v_i)^{\mathrm{T}},$$

$$\mathbf{R}_i(\mathbf{w}, \nabla \mathbf{w}) = (0, \tau_{i1}, \tau_{i2}, \tau_{i1} v_1 + \tau_{i2} v_2 + k \partial\theta/\partial x_i)^{\mathrm{T}},$$

$$\tau_{ij} = \lambda \delta_{ij} \mathrm{div}\mathbf{v} + 2\mu d_{ij}(\mathbf{v}),\ d_{ij}(\mathbf{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

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We add the thermodynamical relations

$$p = (\gamma - 1)(e - \rho|v|^2/2), \quad \theta = \left(\frac{e}{\rho} - \frac{1}{2}|v|^2\right)/c_v.$$

and the following set of boundary conditions:

Case Γ_I : a) $\rho|_{\Gamma_I\times(0,T)} = \rho_D$, b) $\mathbf{v}|_{\Gamma_I\times(0,T)} = \mathbf{v}_D = (v_{D1}, v_{D2})^{\mathrm{T}}$,

c) $\displaystyle\sum_{j=1}^{2}\left(\sum_{i=1}^{2}\tau_{ij}n_i\right)v_j + k\frac{\partial\theta}{\partial\mathbf{n}} = 0$ on $\Gamma_I \times (0,T)$;

Case Γ_W : a) $\mathbf{v}_{\Gamma_W\times(0,T)} = 0$, b) $\displaystyle\frac{\partial\theta}{\partial\mathbf{n}} = 0$ on $\Gamma_W \times (0,T)$;

Case Γ_O : a) $\displaystyle\sum_{i=1}^{2}\tau_{ij}n_i = 0, j = 1,2$, b) $\displaystyle\frac{\partial\theta}{\partial\mathbf{n}} = 0$ on $\Gamma_O \times (0,T)$;

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Let $\mathscr{T}_h$ be a partition of the closure $\overline{\Omega}$ into a finite number of closed triangles: $\mathscr{T}_h = \{K_i\}_{i \in I}$.

- For two neighboring elements we set $\Gamma_{ij} = \partial K_i \cap \partial K_j$ and for $i \in I$ we define $s(i) = \{j \in I; K_j$ is a neighbour of $K_i\}$. By $\mathbf{n}_{ij}$ we denote the unit outer normal to ∂K_i on the face Γ_{ij}.
- Over $\mathscr{T}_h$ we define the *broken Sobolev space*

$$H^k(\Omega, \mathscr{T}_h) = \{v; v|_K \in H^k(K)\ \forall K \in \mathscr{T}_h\}$$

and for $v \in H^1(\Omega, \mathscr{T}_h)$ we set

$$v|_{\Gamma_{ij}} = \textit{trace of } v|_{K_i} \text{ on } \Gamma_{ij},$$

$$\langle v \rangle_{\Gamma_{ij}} = \frac{1}{2}\left(v|_{\Gamma_{ij}} + v|_{\Gamma_{ji}}\right), \text{ average of traces of } v \text{ on } \Gamma_{ij},$$

$$[v]_{\Gamma_{ij}} = v|_{\Gamma_{ij}} - v|_{\Gamma_{ji}}, \textit{ jump} \text{ of traces of } v \text{ on } \Gamma_{ij}.$$

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$v|_{\Gamma_{ij}}$ = *trace* of $v|_{K_i}$ on Γ_{ij},

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- We discretize the continuous problem in the space of discontinuous piecewise polynomial functions

$$S_h = \{v; v|_K \in P_p(K)\ \forall K \in \mathscr{T}_h\},$$

where $P_p(K)$ is the space of all polynomials on K of degree $\leq p$.

- In order to derive a variational formulation, we multiply the Navier-Stokes equations by a test function $\boldsymbol{\varphi} \in H^2(\Omega, \mathscr{T}_h)$, apply Green's theorem on individual elements and other manipulations which take into account the discontinuity of the discrete functions between elements.

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Convective terms

$$\frac{\partial \mathbf{w}}{\partial t} + \sum_{s=1}^{2} \frac{\partial f_s(\mathbf{w})}{\partial x_s} = \sum_{s=1}^{2} \frac{\partial \mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w})}{\partial x_s}$$

- We multiply the convective term by a test function $\boldsymbol{\varphi} \in H^2(\Omega, \mathscr{T}_h)$, apply Green's theorem:

$$-\sum_{K_i \in \mathscr{T}_h} \int_{K_i} \sum_{s=1}^{2} f_s(\mathbf{w}) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_s} \, dx + \sum_{K_i \in \mathscr{T}_h} \sum_{j \in S(i)} \int_{\Gamma_{ij}} \sum_{s=1}^{2} f_s(\mathbf{w}) n_{ij}^{(s)} \cdot \boldsymbol{\varphi} \, dS,$$

- In the second term, incorporate a numerical flux $\mathbf{H}$:

$$\int_{\Gamma_{ij}} \sum_{s=1}^{2} f_s(\mathbf{w}) n_s^{ij} \cdot \boldsymbol{\varphi} \, dS \approx \int_{\Gamma_{ij}} \mathbf{H}(\mathbf{w}|_{\Gamma_{ij}}, \mathbf{w}|_{\Gamma_{ji}}, \mathbf{n}_{ij}) \cdot \boldsymbol{\varphi} \, dS,$$

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Inviscid Boundary Conditions

$$\frac{\partial \mathbf{w}}{\partial t} + \sum_{s=1}^{2} \frac{\partial f_s(\mathbf{w})}{\partial x_s} = \sum_{s=1}^{2} \frac{\partial \mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w})}{\partial x_s}$$

- Inviscid BCs at Γ_I, Γ_O are imposed by choosing the "outside" boundary state $\mathbf{w}_{ji}$ in the numerical flux. This is done by local linearization of the Euler equations and prescribing $\mathbf{w}_{ji}$ so that the linear problem is well posed.

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial f_1(\mathbf{q})}{\partial \tilde{x}_1} = 0$$

$$\Downarrow \text{ Linearization}$$

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbb{A}_1(\mathbf{q}_{ij}) \frac{\partial \mathbf{q}}{\partial \tilde{x}_1} = 0, \text{ where } \mathbb{A}_1 = \frac{Df_1}{D\mathbf{w}}.$$

Diffusion terms

$$\frac{\partial \mathbf{w}}{\partial t} + \sum_{s=1}^{2} \frac{\partial f_s(\mathbf{w})}{\partial x_s} = \sum_{s=1}^{2} \frac{\partial \mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w})}{\partial x_s}$$

Question

How does one discretize second order terms using spaces of discontinuous functions?

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Answer

Treat the second order terms as a first order system and apply the discretization from the previous slide.

Model problem

$$-\sum_{s=1}^{2}\frac{\partial \mathbf{R}_s(\mathbf{w},\nabla\mathbf{w})}{\partial x_s}=g.$$

- Due to properties of $\mathbf{R}_s(\mathbf{w},\nabla\mathbf{w})$ we can write

$$-\sum_{s=1}^{2}\frac{\partial}{\partial x_s}\left(\sum_{k=1}^{2}\mathbb{K}_{sk}(\mathbf{w})\frac{\partial \mathbf{w}}{\partial x_k}\right)=g.$$

- We introduce an auxiliary variable σ_k and write

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- This first order system for unknowns $\mathbf{w}, \sigma_1, \sigma_2$ can be discretized using the discontinuous Galerkin method. Different choices of the numerical flux for this system give different numerical schemes.
- If the numerical flux is appropriately chosen, it is possible to eliminate σ from the resulting numerical scheme.

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Nonsymmetric variant of the diffusion form

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$$a_h^N(\mathbf{w}, \boldsymbol{\varphi}) = \sum_{i \in I} \int_{K_i} \sum_{s=1}^{2} \mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w}) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_s} \, dx$$

$$- \sum_{i \in I} \sum_{\substack{j \in s(i) \\ j < i}} \int_{\Gamma_{ij}} \sum_{s=1}^{2} \langle \mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w}) \rangle n_{ij}^{(s)} \cdot [\boldsymbol{\varphi}] \, dS - \sum_{i \in I} \sum_{j \in \gamma_D(i)} \int_{\Gamma_{ij}} \sum_{s=1}^{2} \mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w}) n_{ij}^{(s)} \cdot \boldsymbol{\varphi} \, dS$$

$$+ \sum_{i \in I} \sum_{\substack{j \in s(i) \\ j < i}} \int_{\Gamma_{ij}} \sum_{s=1}^{2} \langle \widetilde{\mathbf{R}}_s(\mathbf{w}, \nabla \boldsymbol{\varphi}) \rangle n_{ij}^{(s)} \cdot [\mathbf{w}] \, dS + \sum_{i \in I} \sum_{j \in \gamma_D(i)} \int_{\Gamma_{ij}} \sum_{s=1}^{2} \widetilde{\mathbf{R}}_s(\mathbf{w}, \nabla \boldsymbol{\varphi}) n_{ij}^{(s)} \cdot \mathbf{w} \, dS,$$

Here $\widetilde{\mathbf{R}}_k(\mathbf{w}, \nabla \boldsymbol{\varphi}) := \sum_{s=1}^{2} \mathbb{K}_{sk}^T(\mathbf{w}) \frac{\partial \boldsymbol{\varphi}}{\partial x_s}$ and $\mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w}) = \sum_{k=1}^{2} \mathbb{K}_{sk}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x_s}$.

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Nonsymmetric, coercive and suboptimal convergence rate in L^2-norm for even p.

Symmetric variant of the diffusion form

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Symmetric, not coercive and optimal convergence rate in L^2-norm.

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$$- \sum_{i \in I} \sum_{\substack{j \in s(i) \\ j < i}} \int_{\Gamma_{ij}} \sum_{s=1}^{2} \langle \mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w}) \rangle n_{ij}^{(s)} \cdot [\boldsymbol{\varphi}] \, dS - \sum_{i \in I} \sum_{j \in \gamma_D(i)} \int_{\Gamma_{ij}} \sum_{s=1}^{2} \mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w}) n_{ij}^{(s)} \cdot \boldsymbol{\varphi} \, dS$$
$$- \sum_{i \in I} \sum_{\substack{j \in s(i) \\ j < i}} \int_{\Gamma_{ij}} \sum_{s=1}^{2} \langle \widetilde{\mathbf{R}}_s(\mathbf{w}, \nabla \boldsymbol{\varphi}) \rangle n_{ij}^{(s)} \cdot [\mathbf{w}] \, dS - \sum_{i \in I} \sum_{j \in \gamma_D(i)} \int_{\Gamma_{ij}} \sum_{s=1}^{2} \widetilde{\mathbf{R}}_s(\mathbf{w}, \nabla \boldsymbol{\varphi}) n_{ij}^{(s)} \cdot \mathbf{w} \, dS,$$

Symmetric, not coercive and optimal convergence rate in L^2-norm.

Symmetric variant of the diffusion form

$$\frac{\partial \mathbf{w}}{\partial t} + \sum_{s=1}^{2} \frac{\partial f_s(\mathbf{w})}{\partial x_s} = \sum_{s=1}^{2} \frac{\partial \mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w})}{\partial x_s}$$

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$$- \sum_{i \in I} \sum_{\substack{j \in s(i) \\ j < i}} \int_{\Gamma_{ij}} \sum_{s=1}^{2} \langle \mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w}) \rangle n_{ij}^{(s)} \cdot [\boldsymbol{\varphi}] \, dS - \sum_{i \in I} \sum_{j \in \gamma_D(i)} \int_{\Gamma_{ij}} \sum_{s=1}^{2} \mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w}) n_{ij}^{(s)} \cdot \boldsymbol{\varphi} \, dS$$

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Red terms are a result of applying a numerical flux to the first order system.

Incomplete variant of the diffusion form

$$\frac{\partial \mathbf{w}}{\partial t} + \sum_{s=1}^{2} \frac{\partial f_s(\mathbf{w})}{\partial x_s} = \sum_{s=1}^{2} \frac{\partial \mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w})}{\partial x_s}$$

$$a_h^I(\mathbf{w}, \boldsymbol{\varphi}) = \sum_{i \in I} \int_{K_i} \sum_{s=1}^{2} \mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w}) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_s} \, dx$$

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Not symmetric, not coercive and suboptimal convergence rate in L^2-norm for even p.

Incomplete variant of the diffusion form

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Simplest DG discretization of second order terms.

Interior and boundary penalty

- In theory and in practice we need to add the interior and boundary penalty jump terms:

$$J_h(\mathbf{w},\varphi) = C_W \sum_{i\in I} \sum_{\substack{j\in s(i)\\ j<i}} \int_{\Gamma_{ij}} \frac{1}{h_{ij}} [\mathbf{w}][\varphi]\, dS + C_W \sum_{i\in I} \sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \frac{1}{h_{ij}} \mathbf{w}\varphi\, dS.$$

  This term ensures coercivity, when the constant C_W is chosen sufficiently large.

- The boundary term is balanced on the right-hand side by

$$C_W \sum_{i\in I} \sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \sum_{s=1}^{2} \frac{1}{h_{ij}} \mathbf{w}_B \cdot \boldsymbol{\varphi}\, dS$$

  thus enforcing Dirichlet boundary conditions.

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  thus enforcing Dirichlet boundary conditions.

Discrete Problem

Definition

We say that $\mathbf{w}_h$ is a DGFE solution of the compressible Navier-Stokes equations if

a) $\mathbf{w}_h \in C^1([0,T];\mathbf{S}_h)$,

b) $$\frac{d}{dt}(\mathbf{w}_h(t),\boldsymbol{\varphi}_h)+b_h(\mathbf{w}_h(t),\boldsymbol{\varphi}_h)+J_h(\mathbf{w}_h(t),\boldsymbol{\varphi}_h)+a_h(\mathbf{w}_h(t),\boldsymbol{\varphi}_h)$$
$$=l_h(\mathbf{w}_h,\boldsymbol{\varphi}_h)(t), \quad \forall\boldsymbol{\varphi}_h\in\mathbf{S}_h, \forall t\in(0,T),$$

c) $\mathbf{w}_h(0)=\mathbf{w}_h^0$,

where $\mathbf{w}_h^0$ is an $\mathbf{S}_h$ approximation of the initial condition $\mathbf{w}^0$.

1. Discontinuous Galerkin Method Space Semidiscretization
   - Continuous Problem
   - Space semidiscretization

2. Time Discretization
   - Semi-implicit Time Discretization
   - Examples

$$\frac{d}{dt}(\mathbf{w}_h, \boldsymbol{\varphi}) + b_h(\mathbf{w}_h, \boldsymbol{\varphi}) + J_h(\mathbf{w}_h, \boldsymbol{\varphi}) + a_h(\mathbf{w}_h, \boldsymbol{\varphi}) = l_h(\mathbf{w}_h, \boldsymbol{\varphi})$$

- A fully implicit scheme requires the solution of a nonlinear system. In the semi-implicit scheme we linearize the nonlinear terms using their specific properties.
- We solve only one linear system per time level. The scheme is practically unconditionally stable.

$$\frac{d}{dt}(\mathbf{w}_h, \boldsymbol{\varphi}) + b_h(\mathbf{w}_h, \boldsymbol{\varphi}) + J_h(\mathbf{w}_h, \boldsymbol{\varphi}) + a_h(\mathbf{w}_h, \boldsymbol{\varphi}) = l_h(\mathbf{w}_h, \boldsymbol{\varphi})$$

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Time derivative:

$$\frac{d}{dt}(\mathbf{w}_h(t_{n+1}), \boldsymbol{\varphi}) \approx \frac{\mathbf{w}_h^{n+1} - \mathbf{w}_h^n}{\tau_n}$$

$$\frac{d}{dt}(\mathbf{w}_h, \boldsymbol{\varphi}) + b_h(\mathbf{w}_h, \boldsymbol{\varphi}) + J_h(\mathbf{w}_h, \boldsymbol{\varphi}) + a_h(\mathbf{w}_h, \boldsymbol{\varphi}) = l_h(\mathbf{w}_h, \boldsymbol{\varphi})$$

Convective terms:

$$-\sum_{K_i \in \mathscr{T}_h} \int_{K_i} \sum_{s=1}^{2} f_s(\mathbf{w}^{n+1}) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_s}\, dx + \sum_{K_i \in \mathscr{T}_h} \sum_{j \in S(i)} \int_{\Gamma_{ij}} \mathbf{H}(\mathbf{w}_{ij}^{n+1}, \mathbf{w}_{ji}^{n+1}, \mathbf{n}_{ij}) \cdot \boldsymbol{\varphi}\, dS$$

$$\frac{d}{dt}(\mathbf{w}_h, \boldsymbol{\varphi}) + b_h(\mathbf{w}_h, \boldsymbol{\varphi}) + J_h(\mathbf{w}_h, \boldsymbol{\varphi}) + a_h(\mathbf{w}_h, \boldsymbol{\varphi}) = l_h(\mathbf{w}_h, \boldsymbol{\varphi})$$

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It holds that

$$f_s(\mathbf{w}) = \mathbb{A}_s(\mathbf{w})\mathbf{w}, \quad \text{where} \quad \mathbb{A}_s(\mathbf{w}) = \frac{Df_s(\mathbf{w})}{D\mathbf{w}}.$$

We therefore linearize

$$f_s(\mathbf{w}^{n+1}) \approx \mathbb{A}_s(\mathbf{w}^n)\mathbf{w}^{n+1}.$$

$$\frac{d}{dt}(\mathbf{w}_h, \boldsymbol{\varphi}) + b_h(\mathbf{w}_h, \boldsymbol{\varphi}) + J_h(\mathbf{w}_h, \boldsymbol{\varphi}) + a_h(\mathbf{w}_h, \boldsymbol{\varphi}) = l_h(\mathbf{w}_h, \boldsymbol{\varphi})$$

Convective terms:

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We choose the Vijayasundaram numerical flux

$$\mathbf{H}_{VS}(\mathbf{w}_{ij}, \mathbf{w}_{ji}, \mathbf{n}_{ij}) = \mathbb{P}^{+}\left(\langle \mathbf{w} \rangle, \mathbf{n}_{ij}\right) \mathbf{w}_{ij} + \mathbb{P}^{-}\left(\langle \mathbf{w} \rangle, \mathbf{n}_{ij}\right) \mathbf{w}_{ji}$$

and linearize

$$\mathbf{H}_{VS}(\mathbf{w}_{ij}^{n+1}, \mathbf{w}_{ji}^{n+1}, \mathbf{n}_{ij}) \approx \mathbb{P}^{+}\left(\langle \mathbf{w}^{n} \rangle, \mathbf{n}_{ij}\right) \mathbf{w}_{ij}^{n+1} + \mathbb{P}^{-}\left(\langle \mathbf{w}^{n} \rangle, \mathbf{n}_{ij}\right) \mathbf{w}_{ji}^{n+1}.$$

$$\frac{d}{dt}(\mathbf{w}_h, \boldsymbol{\varphi}) + b_h(\mathbf{w}_h, \boldsymbol{\varphi}) + J_h(\mathbf{w}_h, \boldsymbol{\varphi}) + a_h(\mathbf{w}_h, \boldsymbol{\varphi}) = l_h(\mathbf{w}_h, \boldsymbol{\varphi})$$

Interior and boundary penalty jump terms are linear

$$J_h(\mathbf{w}^{n+1}, \varphi) = C_W \sum_{i \in I} \sum_{\substack{j \in s(i) \\ j < i}} \int_{\Gamma_{ij}} \frac{1}{h_{ij}} [\mathbf{w}^{n+1}][\varphi]\, dS + C_W \sum_{i \in I} \sum_{j \in \gamma_D(i)} \int_{\Gamma_{ij}} \frac{1}{h_{ij}} \mathbf{w}^{n+1} \varphi\, dS.$$

$$\frac{d}{dt}(\mathbf{w}_h, \boldsymbol{\varphi}) + b_h(\mathbf{w}_h, \boldsymbol{\varphi}) + J_h(\mathbf{w}_h, \boldsymbol{\varphi}) + a_h(\mathbf{w}_h, \boldsymbol{\varphi}) = l_h(\mathbf{w}_h, \boldsymbol{\varphi})$$

Diffusion terms (for instance *incomplete* variant):

$$a_h^I(\mathbf{w}, \boldsymbol{\varphi}) = \sum_{i \in I} \int_{K_i} \sum_{s=1}^{2} \mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w}) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_s} \, dx$$

$$- \sum_{i \in I} \sum_{\substack{j \in s(i) \\ j < i}} \int_{\Gamma_{ij}} \sum_{s=1}^{2} \langle \mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w}) \rangle n_{ij}^{(s)} \cdot [\boldsymbol{\varphi}] \, dS - \sum_{i \in I} \sum_{j \in \gamma_D(i)} \int_{\Gamma_{ij}} \sum_{s=1}^{2} \mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w}) n_{ij}^{(s)} \cdot \boldsymbol{\varphi} \, dS.$$

It holds that $\mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w}) = \sum_{k=1}^{2} \mathbb{K}_{sk}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x_s}$.

We can linearize $\mathbf{R}_s(\mathbf{w}^{n+1}, \nabla \mathbf{w}^{n+1}) \approx \sum_{k=1}^{2} \mathbb{K}_{sk}(\mathbf{w}^n) \frac{\partial \mathbf{w}^{n+1}}{\partial x_s}$.

Shock Capturing

In transonic and supersonic flows it is common that solutions develop discontinuities. In these cases spurious under and overshoots occur on elements near the discontinuity. Especially in the semi-implicit case, it is desirable to avoid such phenomena. We therefore locally add artificial diffusion to suppress these effects.

Shock Capturing

To the scheme we add two artificial viscosity forms. Internal diffusion:

$$\Phi_h^1(\mathbf{w}_h^n, \mathbf{w}_h^{n+1}, \boldsymbol{\varphi}) = \nu_1 \sum_{i \in I} h_{K_i} G^n(i) \int_{K_i} \nabla \mathbf{w}_h^{n+1} \cdot \nabla \boldsymbol{\varphi} \, \mathrm{d}x$$

with $\nu_1 = O(1)$ a given constant. Here $G(i)$ is a discontinuity indicator which measures interelement jumps of the solution:

$$G^k(i) = \begin{cases} 1 & \text{if interelement jumps of } \mathbf{w}_h^n \text{ are large near } K_i, \\ 0 & \text{otherwise.} \end{cases}$$

Shock Capturing

Interelement diffusion:

$$\Phi_h^2(\mathbf{w}_h^n, \mathbf{w}_h^{n+1}, \boldsymbol{\varphi}) = \nu_2 \sum_{i \in I} \sum_{j \in s(i)} \langle G^n \rangle_{ij} \int_{\Gamma_{ij}} [\mathbf{w}_h^{n+1}] \cdot [\boldsymbol{\varphi}] \, \mathrm{d}S,$$

with $\nu_2 = O(1)$ a given constant. This term allows to strengthen the influence of neighbouring elements and improves the behavior of the method in the case, when strongly unstructured and/or anisotropic meshes are used.

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Corner eddies near cylinder, $M_\infty = 0.0001$

L.E. Fraenkel: On Corner Eddies in Plane Inviscid Shear Flow, 1961

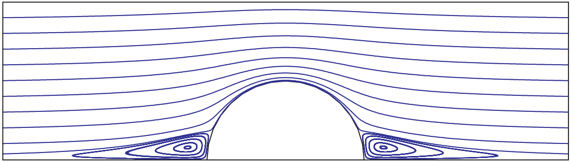

Figure: Exact solution streamlines.

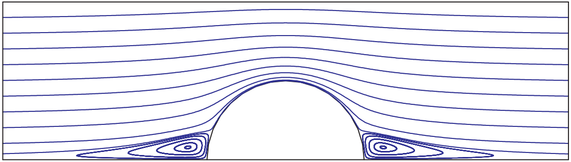

Figure: Numerical solution streamlines.

Corner eddies near cylinder, $M_\infty = 0.0001$

L.E. Fraenkel: On Corner Eddies in Plane Inviscid Shear Flow, 1961

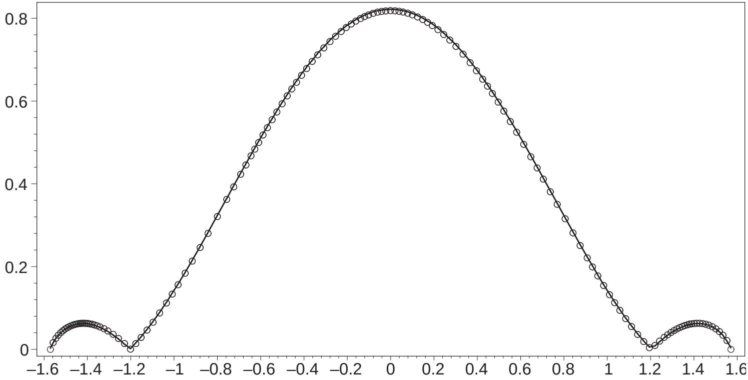

Figure: Velocity distribution on the surface of the half-cylinder: ◦ ◦ ◦ – exact solution of incompressible flow, —— – approximate solution of compressible flow.

Supersonic flow around Žukovský profile

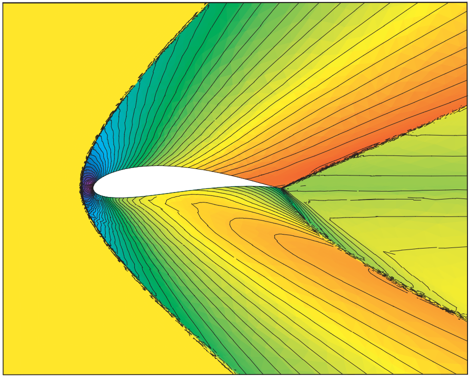

Figure: $M_\infty = 2.0$, Mach isolines.

NACA 0012 viscous flow

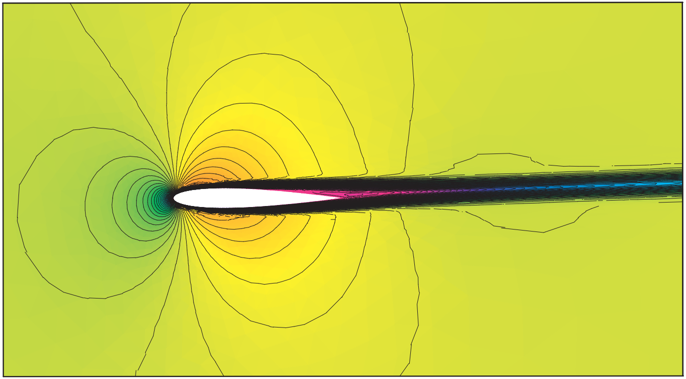

Figure: $M_\infty = 0.5$, $Re = 5000$, $\alpha = 2^\circ$, Mach isolines.

Thank you for your attention