Mathematical Analysis and Computational Simulations for flows of incompressible fluids

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- Mechanics of Incompressible Fluids **based on work by K.R. Rajagopal**
  - Framework (steady internal flows)
  - Constitutive equations (Expl, Impl)
  - Hierarchy of Power-law-like Fluids
  - Compressible vrs Incompressible fluids
  - Maximization of the rate of dissipation
  - Implicit constitutive theories
- Mathematical analysis of models (existence) *Steady flows*
  - Navier-Stokes fluid
  - Shear-rate dependent (power-law-like) fluids
  - Pressure dependent fluid
  - Pressure (and shear-rate) dependent fluid
  - Implicit Power-law-like Fluids
  - Concluding remarks

Incompressible Fluid Mechanics: Basic Framework/1

Goal: To describe flows of various *fluid-like-materials* exhibiting so different and so fascinating phenomena yet share one common feature: these materials are well approximated as *incompressible*.

SubGoal: To understand *theoretical foundations* and *mathematical properties* of these models.

$$\varrho_t + \operatorname{div} \varrho \boldsymbol{v} = 0$$
$$(\varrho \boldsymbol{v})_t + \operatorname{div}(\varrho \boldsymbol{v} \otimes \boldsymbol{v}) = \operatorname{div} \boldsymbol{T} + \varrho \boldsymbol{f}$$

$$\boldsymbol{T} \cdot \boldsymbol{D} - \rho \frac{d\psi}{dt} = \xi \qquad \text{with } \xi \geq 0$$

- ϱ density
- ψ Helmholtz free energy
- $\boldsymbol{T} \cdot \boldsymbol{D}$ stress power
- ξ rate of dissipation

- $\boldsymbol{v} = (v_1, v_2, v_3)$ velocity
- $\boldsymbol{f} = (f_1, f_2, f_3)$ external body forces
- $\boldsymbol{T} = (T_{ij})_{i,j=1}^3$ Cauchy stress
- $\boldsymbol{D} := \boldsymbol{D}(\boldsymbol{v}) := 1/2(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T)$

- Homogeneous fluids (the density is constant) $\Longrightarrow \operatorname{div} \boldsymbol{v} = \operatorname{tr} \boldsymbol{D} = 0$ $\quad \boxed{p := \frac{1}{3} \operatorname{tr} \boldsymbol{T}}$ pressure

Incompressible Fluid Mechanics: Basic Framework/2

Steady flows in $\Omega \subset \mathbb{R}^3$ (time discretizations lead to similar problems)

$$\begin{array}{llll} \operatorname{div} \boldsymbol{v} = 0 & \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \boldsymbol{S} = -\nabla p + \boldsymbol{f} & \text{in } \Omega \\ \boldsymbol{v} = \boldsymbol{0} \quad \text{on } \partial\Omega & \boldsymbol{T} \cdot \boldsymbol{D} = \xi & \text{with } \xi \geq 0 \end{array}$$

Constitutive equations: Relation between the Cauchy stress $\boldsymbol{T}$, of the form $\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{S}$, and the symmetric part of the velocity gradient $\boldsymbol{D} := \boldsymbol{D}(\boldsymbol{v})$

$$\boxed{\boldsymbol{g}(\boldsymbol{T}, \boldsymbol{D}) = \boldsymbol{0}} \qquad \text{Power-law-like Rheology (broad, accessible)}$$

- Explicit $\boldsymbol{S} = \mu(\dots)\boldsymbol{D}$
  - Navier-Stokes Fluids $\boldsymbol{S} = \mu^* \boldsymbol{D}$
  - Power-law Fluids $\boldsymbol{S} = \mu^* |\boldsymbol{D}|^{r-2} \boldsymbol{D}$
  - Power-law-like Fluids $\boldsymbol{S} = \mu(|\boldsymbol{D}|^2)\boldsymbol{D}$
  - Fluids with shear-rate dpt viscosity
  - Fluids with the yield stress
  - Fluids with activation criteria
- Implicit
  - Fluids with the yield stress (Bingham, Herschel-Bulkley)
  - Fluids with activation criteria
  - Implicit Power-law-like Fluids
  - Fluids with $\boldsymbol{S} = \mu(p)\boldsymbol{D}$
  - Fluids with $\boldsymbol{S} = \mu(p, |\boldsymbol{D}|^2)\boldsymbol{D}$

Framework is sufficiently robust to model behavior of various type of fluid-like materials. Models are frequently used in many areas of engineering and natural sciences: mechanics of colloids and suspensions, biological fluid mechanics (blood, synovial fluid), elastohydrodynamics, ice mechanics and glaciology, food processing

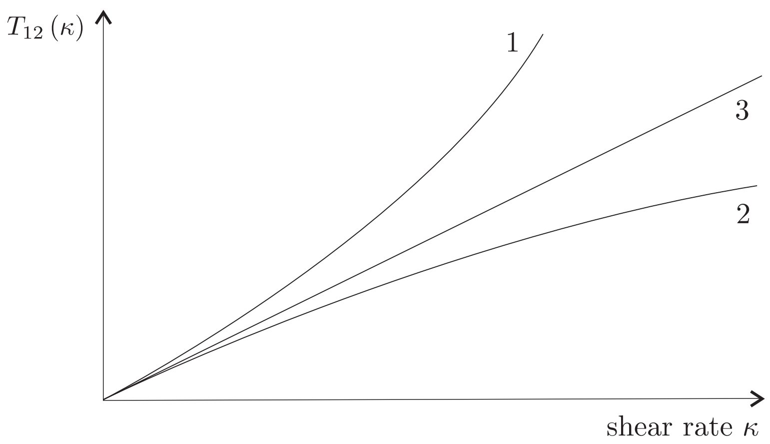
Fluids with shear-rate dependent viscosities

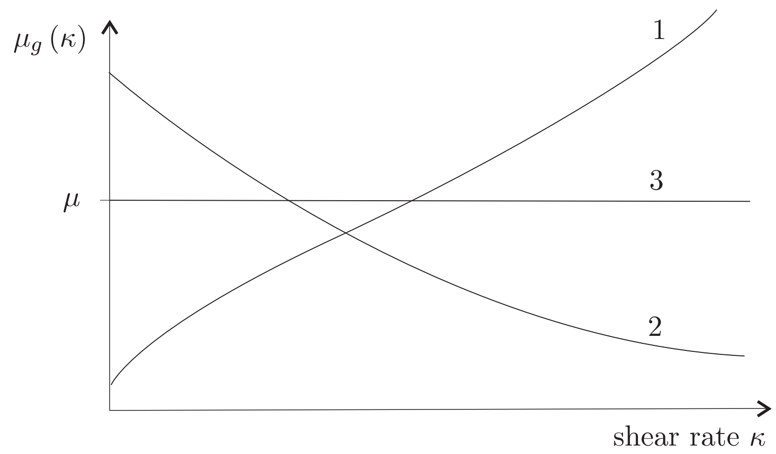
$\boxed{\nu(|\boldsymbol{D}|^2)}$

If $\boldsymbol{v} = (u(x_2), 0, 0)$, then $|\boldsymbol{D}(\boldsymbol{v})|^2 = 1/2|u'|^2$ ... shear rate.

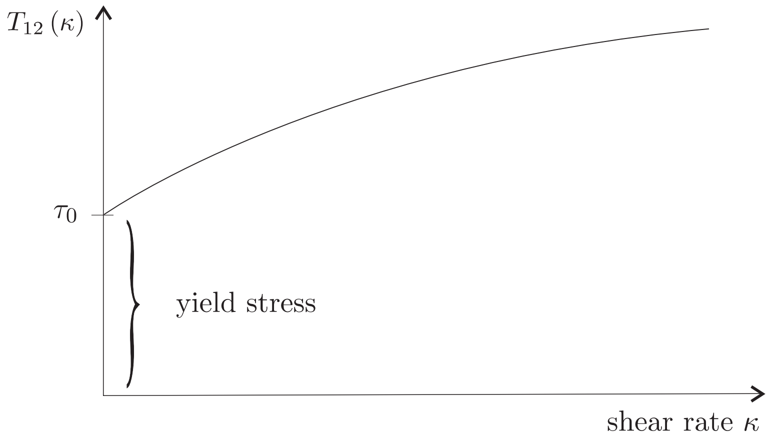
- $\nu(|\boldsymbol{D}|^2) = \mu^*|\boldsymbol{D}|^{r-2} \qquad 1 < r < \infty$
- power-law model
- $\nu(|\boldsymbol{D}|^2) \searrow$ as $|\boldsymbol{D}|^2 \nearrow$
- shear-rate thinning fluid ($r < 2$)

- $\nu(|\boldsymbol{D}|^2) = \mu_0^* + \mu_1^*|\boldsymbol{D}|^{r-2} \qquad r > 2$
- Ladyzhenskaya model (65)
- (Smagorinskii turbulence model: $r = 3$)

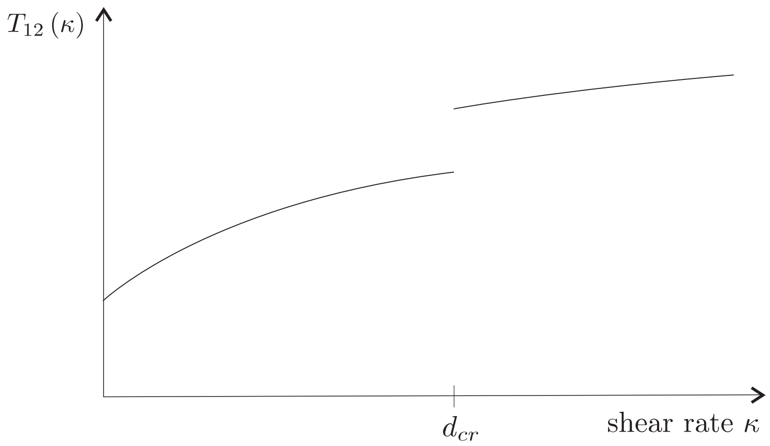




Fluids with the yield stress or the activation criteria/discontinuous stresses

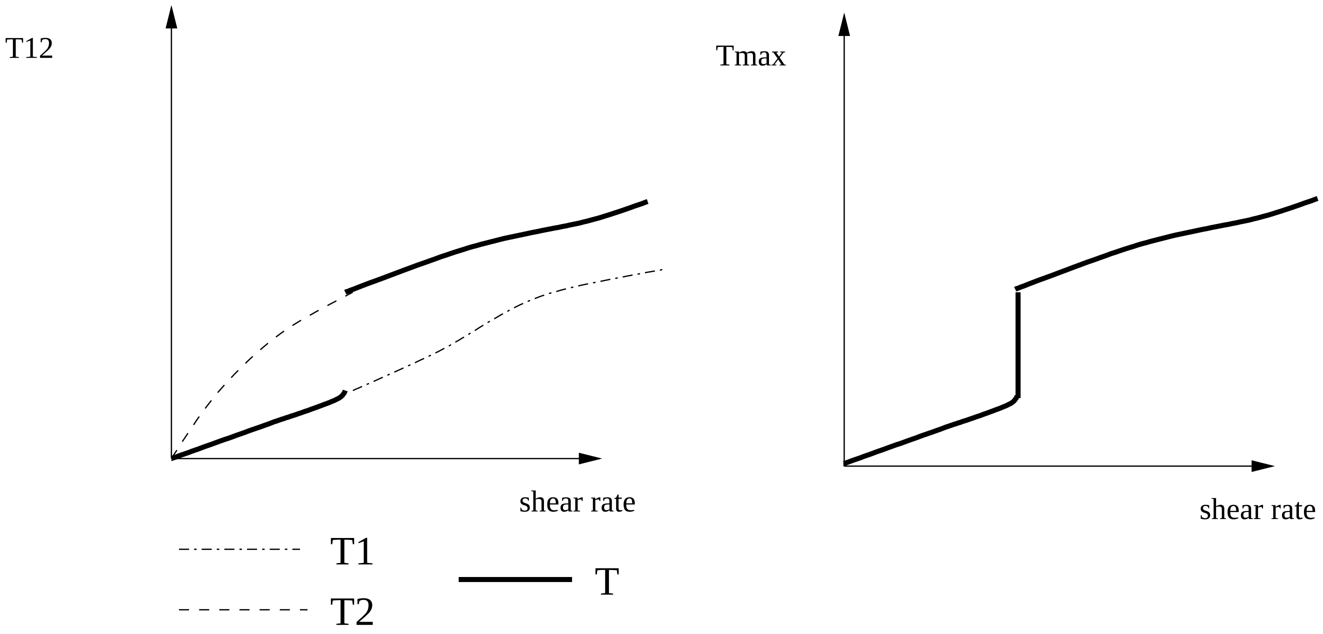


- threshold value for the stress to start flow
- Bingham fluid
- Herschel-Bingham fluid

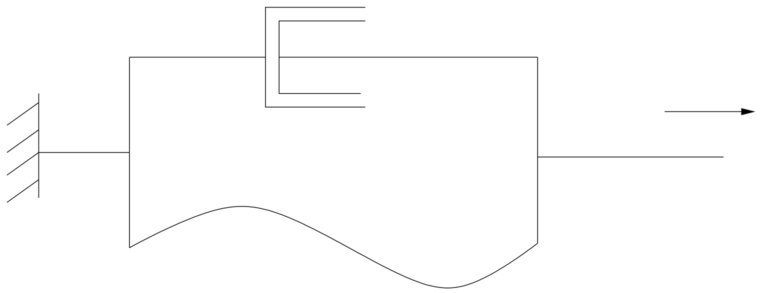


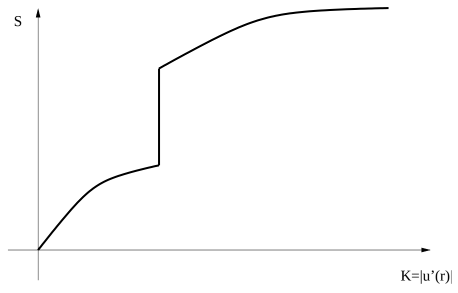
- drastic changes of the properties when certain criterion is met
- formation and dissolution of blood
- chemical reactions/time scale

Discontinuous stresses described by a maximal monotone graph



Implicit power-law-like fluids





Fluids with pressure-dependent viscosities

$\nu(p)$

$\nu(p) = \exp(\gamma p)$

Bridgman(31): "The physics of high pressure"

Cutler, McMickle, Webb and Schiessler(58)

Johnson, Cameron(67), Johnson, Greenwood(77), Johnson, Tewaarwerk(80)

elastohydrodynamics: Szeri(98) synovial fluids

No global existence result.

- Renardy(86), local, ($\frac{\nu(p)}{p} \longrightarrow 0$ as $p \longrightarrow \infty$)
- Gazzola(97), Gazzola, Secchi(98): local, severe restrictions

Fluids with shear- and pressure-dependent viscosities

$$\boxed{\nu(p, |\boldsymbol{D}|^2)}$$

$$\nu(p, |\boldsymbol{D}|^2) = (\eta_\infty + \frac{\eta_0 - \eta_\infty}{1 + \delta|\boldsymbol{D}|^{2-r}}) \exp(\gamma\, p)$$

Davies and Li(94), Gwynllyw, Davies and Phillips(96)

$$\nu(p, |\boldsymbol{D}|^2) = c_0 \frac{p}{|\boldsymbol{D}|} \qquad r = 1 \qquad \text{Schaeffer(87) - instabilities in granular materials}$$

$$\nu(p, |\boldsymbol{D}|^2) = (A + (1 + \exp(\alpha\, p))^{-q} + |\boldsymbol{D}|^2)^{\frac{r-2}{2}}$$

$$\alpha > 0, A > 0 \quad \boxed{1 \le r < 2} \quad 0 \le q \le \frac{1}{2\alpha} \frac{r-1}{2-r} A^{(2-r)/2}$$

Q.: Is the dependence of the viscosity on the pressure admissible in continuum mechanics and thermodynamics?

Compressible vrs Incompressible

Compressible fluid

$$\boldsymbol{T} = \boldsymbol{T}(\rho, \nabla \boldsymbol{v}) \implies \boldsymbol{T} = \boldsymbol{T}(\rho, \boldsymbol{D}(\boldsymbol{v}))$$

$$\boldsymbol{T} = \alpha_0(\rho, \mathrm{I}_{\boldsymbol{D}}, \mathrm{II}_{\boldsymbol{D}}, \mathrm{III}_{\boldsymbol{D}})\boldsymbol{I} + \alpha_1(\rho, \mathrm{I}_{\boldsymbol{D}}, \mathrm{II}_{\boldsymbol{D}}, \mathrm{III}_{\boldsymbol{D}})\boldsymbol{D} + \alpha_2(\rho, \mathrm{I}_{\boldsymbol{D}}, \mathrm{II}_{\boldsymbol{D}}, \mathrm{III}_{\boldsymbol{D}})\boldsymbol{D}^2$$

$$\mathrm{I}_{\boldsymbol{D}} := \mathrm{tr}\, \boldsymbol{D}, \quad \mathrm{II}_{\boldsymbol{D}} := \frac{1}{2}\left([\mathrm{tr}\, \boldsymbol{D}]^2 - \mathrm{tr}\, \boldsymbol{D}^2\right), \quad \mathrm{III}_{\boldsymbol{D}} := \det \boldsymbol{D}$$

Linearized model:

$$\boldsymbol{T} = \hat{\alpha}_0(\rho)\boldsymbol{I} + \lambda(\rho)[\mathrm{tr}\, \boldsymbol{D}]\boldsymbol{I} + 2\mu(\rho)\boldsymbol{D}, \qquad \text{Navier-Stokes fluid}$$

Usually: $\hat{\alpha}_0(\rho)$ is $-p(\rho)$ thermodynamic pressure constitutive relation for $p(\rho)$ needed to close the model $\implies \lambda, \mu$ depends on pressure

Incompressible fluid

$$\boldsymbol{T} = -p\boldsymbol{I} + \hat{\alpha}_1(\mathrm{II}_{\boldsymbol{D}}, \mathrm{III}_{\boldsymbol{D}})\boldsymbol{D} + \hat{\alpha}_2(\mathrm{II}_{\boldsymbol{D}}, \mathrm{III}_{\boldsymbol{D}})\boldsymbol{D}^2$$

Drawbacks:

- p in general is not the mean normal stress
- Can $\hat{\alpha}_i$ depend on p? No, as the derivation based on the principle: **Constraints do no work**

Maximization of the rate of dissipation subject to two constraints

Assume

$$\xi = 2\,\nu(\operatorname{tr}\boldsymbol{T}, |\boldsymbol{D}|^2)|\boldsymbol{D}|^2 \qquad \text{and} \qquad \psi = \psi(\theta, \rho) = const$$

Maximizing ξ with respect to $\boldsymbol{D}$ on the manifold described by the constraints

(1) $\boxed{\xi = \boldsymbol{T} \cdot \boldsymbol{D}}$ **(2)** $\boxed{\operatorname{div}\boldsymbol{v} = \operatorname{tr}\boldsymbol{D} = 0}$

results to

$$\frac{\partial \xi}{\partial \boldsymbol{D}} - \lambda_1 \boldsymbol{I} - \lambda_2 (\boldsymbol{T} - \frac{\partial \xi}{\partial \boldsymbol{D}}) = \boldsymbol{0}$$

$\Longrightarrow$

$$\boxed{\boldsymbol{T} = -p\boldsymbol{I} + 2\,\nu(p, |\boldsymbol{D}|^2)\boldsymbol{D}} \qquad \text{with} \qquad \boxed{p := \frac{1}{3}\operatorname{tr}\boldsymbol{T}}$$

Viscosity: resistance between two sliding surfaces of fluids

Implicit theories

Derivation of the model from the implicit constitutive eqn

$$\boxed{\boldsymbol{g}(\boldsymbol{T}, \boldsymbol{D}) = \boldsymbol{0}}$$

Isotropy of the material implies

$$\begin{aligned}\alpha_0 \boldsymbol{I} + \alpha_1 \boldsymbol{T} + \alpha_2 \boldsymbol{D} + \alpha_3 \boldsymbol{T}^2 + \alpha_4 \boldsymbol{D}^2 + \alpha_5 (\boldsymbol{T}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{T}) \\ + \alpha_6 (\boldsymbol{T}^2\boldsymbol{D} + \boldsymbol{D}\boldsymbol{T}^2) + \alpha_7 (\boldsymbol{T}\boldsymbol{D}^2 + \boldsymbol{D}^2\boldsymbol{T}) + \alpha_8 (\boldsymbol{T}^2\boldsymbol{D}^2 + \boldsymbol{D}^2\boldsymbol{T}^2) = \boldsymbol{0}\end{aligned}$$

α_i being a functions of

$$\rho, \operatorname{tr}\boldsymbol{T},\ \operatorname{tr}\boldsymbol{D},\ \operatorname{tr}\boldsymbol{T}^2,\ \operatorname{tr}\boldsymbol{D}^2,\ \operatorname{tr}\boldsymbol{T}^3,\ \operatorname{tr}\boldsymbol{D}^3,\ \operatorname{tr}(\boldsymbol{T}\boldsymbol{D}),\ \operatorname{tr}(\boldsymbol{T}^2\boldsymbol{D}),\ \operatorname{tr}(\boldsymbol{D}^2\boldsymbol{T}),\ \operatorname{tr}(\boldsymbol{D}^2\boldsymbol{T}^2)$$

For incompressible fluids

$$\boldsymbol{T} = \frac{1}{3}\operatorname{tr}\boldsymbol{T}\boldsymbol{I} + \nu(\operatorname{tr}\boldsymbol{T}, \operatorname{tr}\boldsymbol{D}^2)\boldsymbol{D}$$

Analysis of PDEs - Existence of (weak) solution - steady flows

- Mathematical consistency of the model (well-posedness), existence for any data in a reasonable function space
- Notion of the solution: balance equations for any subset of $\Omega \iff$ *weak* solution $\iff$ FEM
- Choice of the function spaces
- To know what is the object we approximate
- $\Omega \subset \mathbb{R}^3$ open, bounded with the Lipschitz boundary $\partial\Omega$

$$\boxed{\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0 \\ -\operatorname{div}(2\nu(p, |\boldsymbol{D}(\boldsymbol{v})|^2)\boldsymbol{D}(\boldsymbol{v})) + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) &= -\nabla p + \boldsymbol{f} \end{aligned} \quad \text{in } \Omega} \tag{1}$$

$$\boxed{\boldsymbol{v} = \boldsymbol{0} \qquad \text{on } \partial\Omega \qquad\qquad \frac{1}{|\Omega|}\int_\Omega p dx = p_0} \tag{2}$$

(0) $\nu(\ldots) = \nu_0$ NSEs
(1) $\nu(\ldots) = \nu(|\boldsymbol{D}|^2)$
(2) $\nu(\ldots) = \nu(p, |\boldsymbol{D}|^2)$
(3) discontinuous (implicit) power-law fluids

NSEs

$$\boxed{\operatorname{div} \boldsymbol{v} = 0 \qquad -\mu^* \Delta \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) = -\nabla p + \boldsymbol{f} \text{ in } \Omega}$$

Step 1. Finite-dimensional approximations: $(\boldsymbol{v}^N, p^N)$ - Fixed-point iterations

Step 2. Uniform estimates: $\sup_N \|\nabla \boldsymbol{v}^N)\|_2^2 + \sup_N \|p^N\|_2^2 \leq K$

Step 3. Weak compactness: $\nabla \boldsymbol{v}^N \rightarrow \boldsymbol{v}$ and $p^N \rightarrow p$ weakly in L^2

Step 4. Nonlinearity tretaed by compact embedding: $\boldsymbol{v}^N \rightarrow \boldsymbol{v}$ strongly in L^2

Step 5. Limit in approximations as $N \rightarrow \infty$

$$\mu^*(\nabla \boldsymbol{v}^N, \nabla \boldsymbol{\varphi}) - (\boldsymbol{v}^N \otimes \boldsymbol{v}^N, \nabla \boldsymbol{\varphi}) = (p^N, \operatorname{div} \boldsymbol{\varphi}) + \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle \qquad \forall \boldsymbol{\varphi}$$

$(\boldsymbol{v}, p)$ is a solution

$\boxed{\mu(|\boldsymbol{D}(\boldsymbol{v})|^2) = \mu^* |\boldsymbol{D}(\boldsymbol{v})|^{r-2} \boldsymbol{D}(\boldsymbol{v})}$ - Critical values for the power-law index

$$\boxed{\operatorname{div} \boldsymbol{v} = 0 \qquad -\mu^* \operatorname{div}(|\boldsymbol{D}(\boldsymbol{v})|^{r-2} \boldsymbol{D}(\boldsymbol{v})) + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) = -\nabla p + \boldsymbol{f} \text{ in } \Omega}$$

Energy estimates: $\sup_N \|\nabla \boldsymbol{v}^N\|_r^r \leq K$ - expect that $\boldsymbol{v} \in W^{1,r}_{0,\mathrm{div}}(\Omega)$

- Equation for the pressure: $p = (-\Delta)^{-1} \operatorname{div} \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v} - \mu^* |\boldsymbol{D}(\boldsymbol{v})|^{r-2} \boldsymbol{D}(\boldsymbol{v}))$

$$W^{1,r} \hookrightarrow L^{3r/(3-r)} \implies \boldsymbol{v} \otimes \boldsymbol{v} \in L^{\frac{3r}{2(3-r)}} \text{ and } |\boldsymbol{D}(\boldsymbol{v})|^{r-2} \boldsymbol{D}(\boldsymbol{v}) \in L^{r'} \text{ with } r' := r/(r-1)$$

$$\frac{r}{r-1} \leq \frac{3r}{2(3-r)} \iff r \geq \frac{9}{5}$$

- If $\boldsymbol{v}, \boldsymbol{\varphi} \in W^{1,r}$ then $\boldsymbol{v} \otimes \boldsymbol{v} \cdot \nabla \boldsymbol{\varphi} \in L^1$ only for $r \geq \frac{9}{5}$
- For $r \geq \frac{9}{5}$ the energy equality holds, higher differentiability accessible (useful tools)

– Compactness of the quadratic nonlinearity requires $W^{1,r} \hookrightarrow\hookrightarrow L^2$: holds for $r \geq \frac{6}{5}$

Analysis easier for $\boxed{r \geq 9/5}$ and more difficult for $\boxed{r \in (6/5, 9/5)}$, in both cases more difficult than NSEs

Results for $\boxed{\mu(|\boldsymbol{D}(\boldsymbol{v})|^2) = \mu^*|\boldsymbol{D}(\boldsymbol{v})|^{r-2}\boldsymbol{D}(\boldsymbol{v})}$

Th1 (Ladyzhenskaya, Lions 1967) *Let* $\boxed{r \geq 9/5}$ *and*

$$\boldsymbol{f} \in \left(W_0^{1,r}(\Omega)^3\right)^* \qquad and \qquad p_0 \in \mathbb{R} \tag{D}$$

Then there is a weak solution $(\boldsymbol{v}, p)$ *to* (1)-(2) *such that*

$$\boldsymbol{v} \in W_{0,div}^{1,r}(\Omega) \qquad \text{and } p \in L^{r'}(\Omega)$$

Tools: Monotone operator theory, Minty method (energy equality), compact embedding

Th2 (Frehse, Malek, Steinhauer 2003 and Diening, Malek, Steinhauer 2006) *Let* $\boxed{r \in (6/5, 9/5)}$ *and (D) hold.*

Then there is a weak solution $(\boldsymbol{v}, p)$ *to* (1)-(2) *such that*

$$\boldsymbol{v} \in W_{0,div}^{1,r}(\Omega) \qquad \text{and } p \in L^{3r/(2(r-1))}(\Omega)$$

Tools: Lipschitz approximations of Sobolev functions (strengthened version), Strictly monotone operator, Minty method, compact embedding

Assumptions on μ's for $\boxed{\mu(p, |\boldsymbol{D}(\boldsymbol{v})|^2)}$

(A1) given $\boxed{r \in (1,2)}$ there are $C_1 > 0$ and $C_2 > 0$ such that for all symmetric matrices $\boldsymbol{B}$, $\boldsymbol{D}$ and all p

$$C_1(1+|\boldsymbol{D}|^2)^{\frac{r-2}{2}}|\boldsymbol{B}|^2 \le \frac{\partial\left[(\mu(p,|\boldsymbol{D}|^2)\boldsymbol{D}\right]}{\partial \boldsymbol{D}} \cdot (\boldsymbol{B}\otimes\boldsymbol{B}) \le C_2(1+|\boldsymbol{D}|^2)^{\frac{r-2}{2}}|\boldsymbol{B}|^2$$

(A2) for all symmetric matrices $\boldsymbol{D}$ and all p

$$\left|\frac{\partial[\mu(p,|\boldsymbol{D}|^2)\boldsymbol{D}]}{\partial p}\right| \le \gamma_0(1+|\boldsymbol{D}|^2)^{\frac{r-2}{4}} \le \gamma_0 \qquad \gamma_0 < \frac{1}{C_{div,2}}\,\frac{C_1}{C_1+C_2}\,.$$

The constant $C_{div,q}$ occurs in the problem:

For a given $g \in L^q(\Omega)$ with zero mean value to find $\mathbf{z} \in W_0^{1,q}(\Omega)$ solving

$$\operatorname{div}\mathbf{z} = g \text{ in } \Omega,\ \mathbf{z} = \mathbf{0} \text{ on } \partial\Omega \text{ and } \|\mathbf{z}\|_{1,q} \le C_{div,q}\|g\|_q. \tag{3}$$

The solvability: Bogovskij (79) or Amrouche, Girault (94).

Results for $\boxed{\mu(p, |\boldsymbol{D}(\boldsymbol{v})|^2)}$

Th3 (Franta, Malek, Rajagopal 2005) *Let* $\boxed{r \in (9/5, 2)}$ *and (D) hold. Assume that* **(A1)**–**(A2)** *are fullfiled. Then there is a weak solution* $(\boldsymbol{v}, p)$ *to* (1)-(2) *such that*

$$\boldsymbol{v} \in W^{1,r}_{0,div}(\Omega) \qquad \text{and } p \in L^{r'}(\Omega)$$

Tools: Quasicompressible approximations, structure of the viscosities, solvability of equation $\operatorname{div} \boldsymbol{z} = g$, strictly monotone operator theory in $\boldsymbol{D}$-variable, compactness for the velocity gradient, compactness for the pressure, compact embedding

Th4 (Bulícek, Fišerová 2007) *Let* $\boxed{r \in (6/5, 9/5)}$ *and (D) holds. Assume that* **(A1)**–**(A2)** *are fullfiled. Then there is a weak solution* $(\boldsymbol{v}, p)$ *to* (1)-(2) *such that*

$$\boldsymbol{v} \in W^{1,r}_{0,div}(\Omega) \qquad \text{and } p \in L^{3r/(2(r-1))}(\Omega)$$

Tools: Quasicompressible approximations, Lipschitz approximations of Sobolev functions (strengthened version), structure of the viscosities, solvability of equation $\operatorname{div} \boldsymbol{z} = g$, strictly monotone operator theory in $\boldsymbol{D}$-variable, compactness for the velocity gradient, compactness for the pressure, decomposition of the pressure, compact embedding

Implicit Power-Law-like Fluids

Power-law index $r \in (1, \infty)$, its dual $r' := r/(r-1)$

$$\begin{aligned} &\boldsymbol{v} \in W^{1,r}_{0,\mathrm{div}}(\Omega), \;\; \boldsymbol{S} \in L^{r'}(\Omega)^{3\times 3}, \;, p \in L^{\tilde{r}}(\Omega) \,\mathrm{with}\, \tilde{r} = \min\{r', \tfrac{3r}{2(3-r)}\} \\ &\mathrm{div}\,(\boldsymbol{v} \otimes \boldsymbol{v} + p\boldsymbol{I} - \boldsymbol{S}) = \boldsymbol{f} \text{ in } \mathcal{D}'(\Omega), \\ &(\boldsymbol{D}\boldsymbol{v}(x), \boldsymbol{S}(x)) \in \mathcal{A}(x) \text{ for all } x \in \Omega_{a.e.} \end{aligned} \tag{4}$$

Properties of the maximal monotone graph $\mathcal{A}$ a.e.

(B1) $(0,0) \in \mathcal{A}(x)$;

(B2) If $(\boldsymbol{D}, \boldsymbol{S}) \in \mathbb{R}^{3\times 3}_{\mathrm{sym}} \times \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ fulfils

$$(\bar{\boldsymbol{S}} - \boldsymbol{S}) \cdot (\bar{\boldsymbol{D}} - \boldsymbol{D}) \geq 0 \quad \text{for all } (\bar{\boldsymbol{D}}, \bar{\boldsymbol{S}}) \in \mathcal{A}(x),$$

then $(\boldsymbol{D}, \boldsymbol{S}) \in \mathcal{A}(x)$ ($\mathcal{A}$ is maximal monotone graph);

(B3) There are a non-negative $m \in L^1(\Omega)$ and $c > 0$ such that for all $(\boldsymbol{D}, \boldsymbol{S}) \in \mathcal{A}(x)$

$$\boldsymbol{S} \cdot \boldsymbol{D} \geq -m(x) + c(|\boldsymbol{D}|^r + |\boldsymbol{S}|^{r'}) \qquad (\mathcal{A} \text{ is } r-\text{graph}); \tag{5}$$

(B4) At least one of the following two conditions (I) and (II) happens:

(I) for all $(\boldsymbol{D}_1, \boldsymbol{S}_1)$ and $(\boldsymbol{D}_2, \boldsymbol{S}_2) \in \mathcal{A}(x)$ fulfilling $\boldsymbol{D}_1 \neq \boldsymbol{D}_2$ we have

$$(\boldsymbol{S}_1 - \boldsymbol{S}_2) \cdot (\boldsymbol{D}_1 - \boldsymbol{D}_2) > 0,$$

(II) for all $(\boldsymbol{D}_1, \boldsymbol{S}_1)$ and $(\boldsymbol{D}_2, \boldsymbol{S}_2) \in \mathcal{A}(x)$ fulfilling $\boldsymbol{S}_1 \neq \boldsymbol{S}_2$ we have

$$(\boldsymbol{S}_1 - \boldsymbol{S}_2) \cdot (\boldsymbol{D}_1 - \boldsymbol{D}_2) > 0,$$

Results for Implicit Power-law-like Fluids

Th5 (Malek, Ruzicka, Shelukhin 2005) *Let* $\boxed{r \in (9/5, 2)}$ *and (D) hold. Consider Herschel-Bulkley fluids. Then there is a weak solution* $(\boldsymbol{v}, p, \boldsymbol{S})$ *to* (4).

Tools: Local regularity method free of involving the pressure, higher-differentiability, uniform monotone operator properties, compact embedding

Th6 (Gwiazda, Malek, Swierczewska 2007) *Let* $\boxed{r \geq 9/5}$ *and (D) holds. Assume that* **(B1)**–**(B4)** *are fullfiled. Then there is a weak solution* $(\boldsymbol{v}, p, \boldsymbol{S})$ *satisfying* (4).

Tools: Young measures (generalized version), energy equality, strictly monotone operator, compact embedding

Th7 (Gwiazda, Malek, Swierczewska 2007) *Let* $\boxed{r > 6/5}$ *and (D) holds. Assume that* **(B1)**–**(B4)** *are fullfiled. Then there is a weak solution* $(\boldsymbol{v}, p, \boldsymbol{S})$ *satisfying* (4).

Tools: Characterizatition of maximal monotone graphs in terms of 1-Lipschitz continuous mappings (Francfort, Murat, Tartar), Young measures, biting lemma, Lipschitz approximations of Sobolev functions (strengthened version), approximations of discontinuous functions, compact embedding

Concluding Remarks

- Consistent thermomechanical basis for incompressible fluids with power-law-like rheology
- 'Complete' set of results concerning mathematical analysis of these models (sophisticated methods, new tools)
- Computational tests (will be presented by M. Mádlík, J. Hron, M. Lanzendörfer) require numerical analysis of the models
- Hierarchy is incomplete 1: *unsteady flows, full thermodynamical setting*
- Hierarchy is incomplete 2: *rate type fluid models*
- Mutual interactions (includes more realistic boundary conditions - I/O)

1. J. Málek and **K.R. Rajagopal**: Mathematical Issues Concerning the Navier-Stokes Equations and Some of Its Generalizations, in: Handbook of Differential Equations, Evolutionary Equations, volume 2 371-459 2005
2. **J. Frehse**, J. Málek and **M. Steinhauer**: On analysis of steady flows of fluids with shear-dependent viscosity based on the Lipschitz truncation method, SIAM J. Math. Anal. 34, 1064-1083, 2003
3. **L. Diening**, J. Málek and **M. Steinhauer**: On Lipschitz truncations of Sobolev functions (with variable exponent) and their selected applications, accepted to ESAIM: COCV, 2006
4. **J. Hron**, J. Málek and **K.R. Rajagopal**: Simple Flows of Fluids with Pressure Dependent Viscosities, Proc. London Royal Soc.: Math. Phys. Engnr. Sci. 457, 1603–1622, 2001
5. **M. Franta**, J. Málek and **K.R. Rajagopal**: Existence of Weak Solutions for the Dirichlet Problem for the Steady Flows of Fluids with Shear Dependent Viscosities, Proc. London Royal Soc. A: Math. Phys. Engnr. Sci. 461, 651–670 2005
6. J. Málek, **M. Růžička** and **V.V. Shelukhin**: Herschel-Bulkley Fluids: Existence and regularity of steady flows, Mathematical Models and Methods in Applied Sciences, 15, 1845–1861, 2005
7. **P. Gwiazda**, J. Málek and **A. Świerczewska**: On flows of an incompressible fluid with a discontinuous power-law-like rheology, Computers & Mathematics with Applications, 53, 531–546, 2007
8. **M. Bulíček**, **P. Gwiazda**, J. Málek and **A. Świerczewska-Gwiazda**: On steady flows of an incompressible fluids with implicit power-law-like theology, to be submitted 2007