Multispace and Multilevel BDDC

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A Somewhat Biased Short Overview of Iterative Substructuring a.k.a. Nonoverlapping DD

- assemble elements into substructures, eliminate interiors $\Longrightarrow$ reduced problem (Schur complement) on interface (in Sobolev space $H^{1/2} \to H^{-1/2}$)

  $$\text{cond} \approx O\left(\frac{\text{number of substructures}^2 * \text{substructure size}}{\text{mesh step (=element size)}}\right) = O\left(N^2\right) O\left(\frac{H}{h}\right)$$

- for parallel solution; Schur complement matrix-vector multiply = solve substructure Dirichlet problem

- only matrix datastructures needed; condition number better in practice than the $O\left(1/h^2\right)$ for the original problem (small N)

Early Preconditioners

- diagonal preconditioning: Gropp and Keyes 1987, "probing" the diagonal of the Schur complement Chan and Mathew 1992 (because creating the diagonal of the Schur complement is expensive)...

- preconditioning by solving substructure Neumann problems ($H^{-1/2} \rightarrow H^{1/2}$) Glowinski and Wheeler 1988, Le Talleck and De Roeck 1991 (a.k.a. the Neumann-Neumann method)

- optimal substructuring methods: coarse problem: $O\left(N^2\right) \rightarrow$ const, asymptotically optimal preconditioners $H/h \rightarrow \log^2(1 + H/h)$ (Bramble, Pasciak, Schatz 1986+, Widlund 1987, Dryja 1988,... **but all these methods require access to mesh details and depend on details of the Finite Element code, which makes them hard to implement in a professional software framework**

FETI and BDD

- algebraic - need only substructure 1. solvers (Neumann, Dirichlet), 2. connectivity, 3. basis of nullspace (BDD - constant function for the Laplacian, FETI - actual nullspace)

- both involve singular substructure problems (Neumann) and build the coarse problem from local substructure nullspaces (in different ways) to assure that the singular systems are consistent

- Balancing Domain Decomposition (BDD, Mandel 1993): solve the system reduced to interfaces, interface degrees of freedom common (this is the Neumann-Neumann with a particular coarse space)

- Finite Element Tearing and Interconnecting (FETI, Farhat and Roux 1991): enforce continuity across interfaces by Lagrange multipliers, solve the dual system for the multipliers

FETI and BDD Developments

- Both methods require only matrix level information that is readily available in Finite Element software (no fussing with the meshes, coordinates, and individual elements...) and can be implemented easily outside of the FE engine. So they became very popular and widely used. The methods work well in 2D and 3D (solids).

- But the performance for plates/shells/biharmonic not so good. Reason: the condition numbers depend on the energy (trace norm) of functions with jumps across a substructure corner. In 2D, OK for $H^{1/2}$ traces of H^1 functions, not $H^{3/2}$ traces of H^2 functions (embedding theorem).

- Fix: avoid this by increasing the coarse space and so restricting the space where the method runs, to **make sure that nothing gets torn across the corners** (BDD: LeTallec Mandel Vidrascu 1998, FETI: Farhat Mandel 1998, Farhat Mandel Tezaur 1998). Drawback: complicated, expensive - a large coarse problem with custom basis functions

FETI-DP and BDDC

- To assure that nothing gets torn across the corners, enforce identical values at corners from all neighboring substructures a-priori $\Longrightarrow$ corner values are coarse degrees of freedom

- Continuity elsewhere at the interfaces by Lagrange multipliers as in FETI $\Longrightarrow$ FETI-DP (Farhat et al 2001)

- Continuity elsewhere by common values as in BDD $\Longrightarrow$ BDDC (Dohrmann 2003; independently Cros 2003, Frakagis and Papadrakakis 2003, with corner coarse degrees of freedom only)

- **Additional coarse degrees of freedom (side/face averages)** required in 3D for good conditioning: Farhat, Lesoinne, Pierson 2000 (algorithm only), Dryja Windlud 2002 (with proofs)

FETI-DP and BDDC Developments

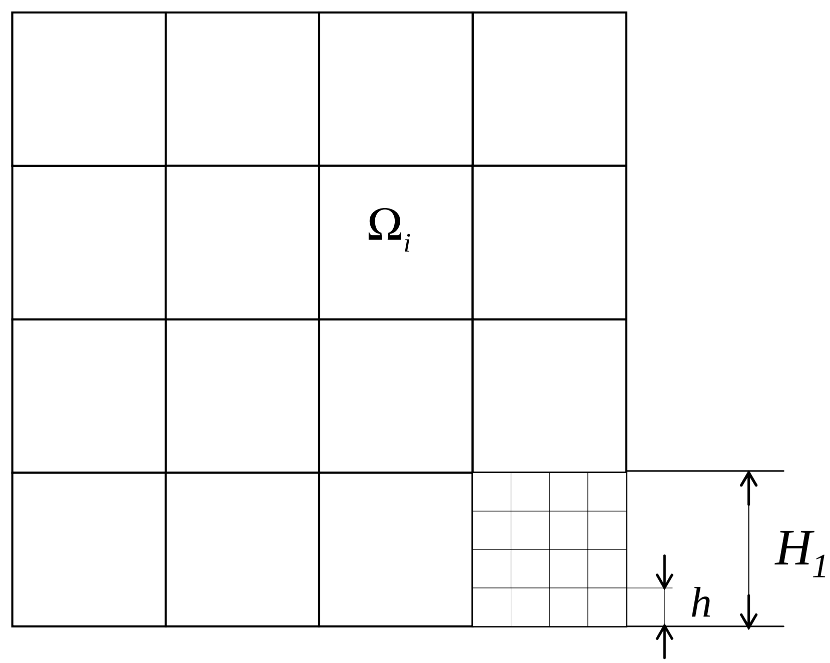
- Convergence estimate (energy norm of an averaging operator) Mandel Dohrmann 2003

- **The eigenvalues of the preconditioned FETI-DP and BDDC operators are the same** (Mandel, Dohrman, Tezaur 2005, simplified proofs: Li and Widlund 2006, Brenner and Sung 2007)

- **For large problems, coarse problem is a bottleneck**$\Longrightarrow$three-level BDDC (BDDC with two coarse levels) in two and three dimensions (Tu 2004, 2005).

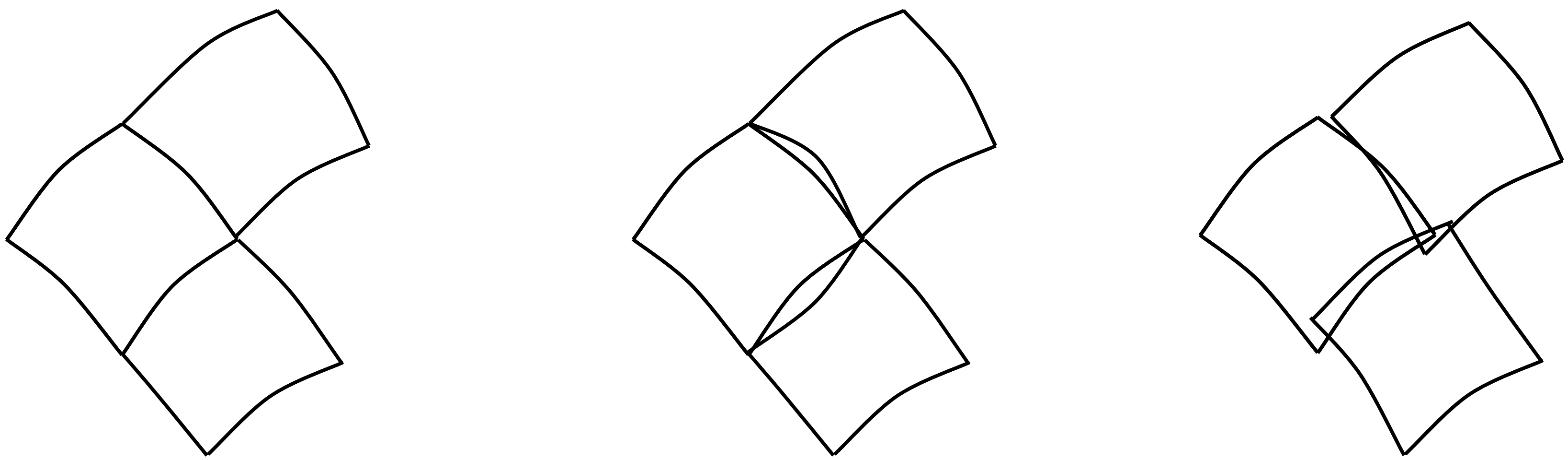
- **Here: multilevel BDDC. Theory by a new abstract multispace BDDC formulation.**

Substructuring for a Problem with H/h=4



BDDC Description - Example Spaces

$W = \bigotimes_{i=1}^{N} W_i :$ space of block vectors, one block per substructure



$U \subset \widetilde{W} \subset W$

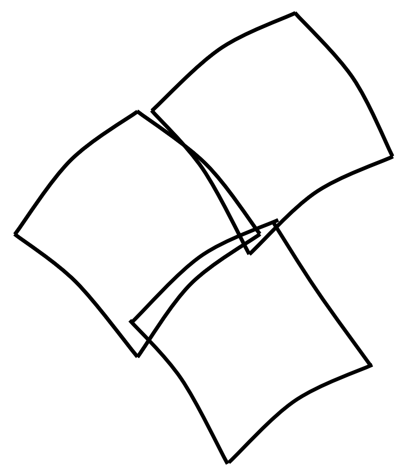
U	$\widetilde{W}$	W
continuous across whole substructure interfaces fully assembled	continuous across corners only partially assembled	no continuity required partially assembled

Want to solve: $u \in U : a(u,v) = \langle f, v\rangle \quad \forall v \in U$, $a(\cdot,\cdot)$ SPD on U

$a(\cdot,\cdot)$ defined on the bigger space W

BDDC Description - Example Form and RHS

$W = \bigotimes_{i=1}^{N} W_i$: space of block vectors, one block per substructure Ω_i,
$w = (w_i)$



The bilinear form a and the right-hand side f defined by integrals over substructures:

$$a(w,v) = \sum_{i=1}^{N} \int_{\Omega_i} \nabla w_i \nabla v_i, \quad \langle f, v\rangle = \sum_{i=1}^{N} \int_{\Omega_i} f_i v_i$$

Abstract BDDC (Two Levels): Variational Setting of the Problem and Algorithm Components

$$u \in U : a(u, v) = \langle f, v \rangle \,, \quad \forall v \in U$$

a SPD on U and **positive semidefinite** on $W \supset U$, $\langle , \rangle$ is inner product

Example:
$W = W_1 \times \cdots \times W_N$ (spaces on substructures)
$U =$ functions continuous across interfaces

Choose preconditioner components:

space $\widetilde{W}$, $U \subset \widetilde{W} \subset W$, such that a is positive definite on $\widetilde{W}$.
Example: functions with continuous coarse dofs, such as values at substructure corners

projection $E : \widetilde{W} \to U, \quad \text{range}\, E = U$.
Example: averaging across substructure interfaces

Abstract BDDC Preconditioner

Given a on $W \supset U$, define $A : U \to U$ by $a(v, w) = \langle Av, w \rangle \ \forall v, w \in U$
Choose $\widetilde{W}$ such that $U \subset \widetilde{W} \subset W$, and projection $E : \widetilde{W} \to U$ onto

Theorem 1 **The abstract BDDC preconditioner** $M : U \longrightarrow U$,

$$M : r \longmapsto u = Ew, \quad w \in \widetilde{W} : \quad a\left(w, z\right) = \left\langle r, Ez \right\rangle, \quad \forall z \in \widetilde{W},$$

satisfies

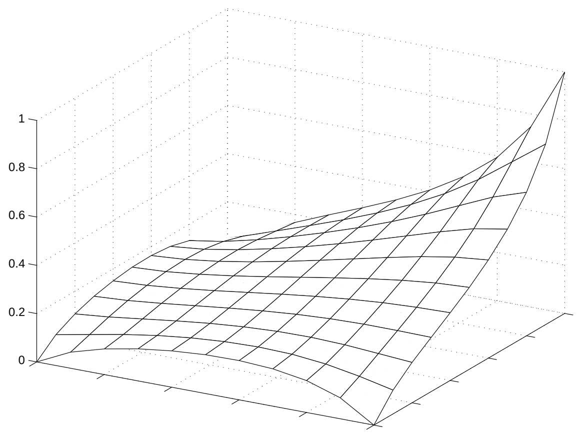
$$\kappa = \frac{\lambda_{\mathsf{max}}(MA)}{\lambda_{\mathsf{min}}(MA)} \leq \omega = \sup_{w \in \widetilde{W}} \frac{\|Ew\|_a^2}{\|w\|_a^2}.$$

In implementation, $\widetilde{W}$ is decomposed into

$$\widetilde{W} = \widetilde{W}_\Delta \oplus \widetilde{W}_\Pi$$

$\widetilde{W}_\Delta$ = functions with zero coarse dofs $\Rightarrow$ local problems on substructures
$\widetilde{W}_\Pi$ = functions given by coarse dofs & energy minimal
$\Rightarrow$ global coarse problem

The Coarse Problem

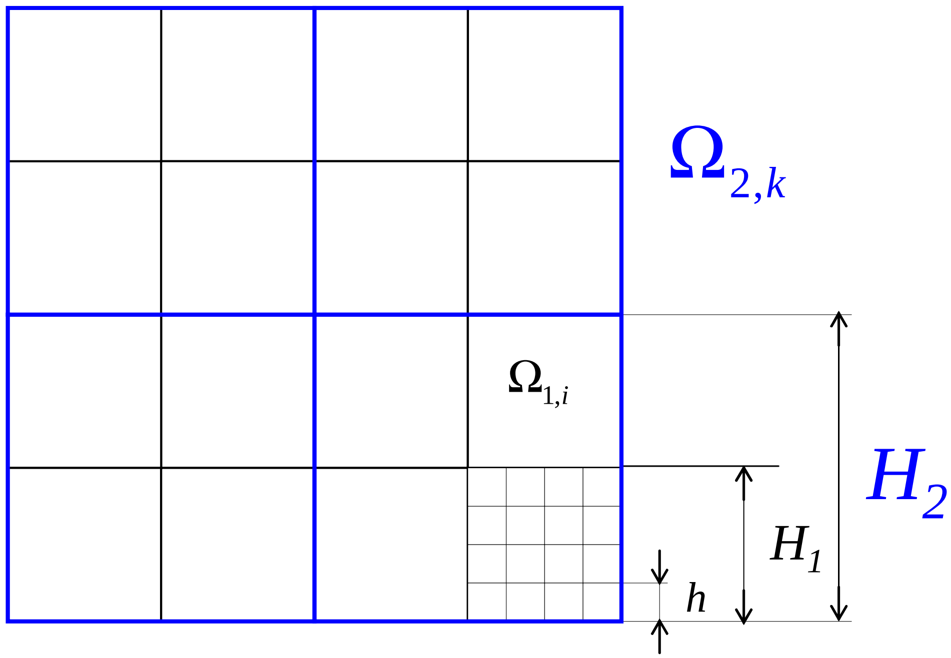


A basis function from $\widetilde{W}_{\Pi}$ is energy minimal subject to given values of coarse degrees of freedom on the substructure. The function is discontinuous across the interfaces between the substructures but the values of coarse degrees of freedom on the different substructures coincide.

The coarse problem has the same structure as the original FE problem $\Longrightarrow$ solve it approximately by one iteration of BDDC $\Longrightarrow$ three-level BDDC

Apply recursively $\Longrightarrow$ multi-level BDDC

Substructuring for a Three-level Model Problem



Abstract Multispace BDDC

Choose a space with **decomposition** $\sum_{k=1}^{N} V_k$ and **projections** Q_k as

$$\boxed{U \subset \sum_{k=1}^{N} V_k \subset W, \quad Q_k : V_k \to U}$$

V_k energy orthogonal: $V_k \perp_a V_\ell, k \neq \ell$,

assume $\forall u \in U : \left[u = \sum_{k=1}^{N} v_k, \; v_k \in V_k \right] \Longrightarrow u = \sum_{k=1}^{N} Q_k v_k$

Equivalently, assume $\Pi_k : \bigoplus_{j=1}^{M} V_j \to V_k$ are $a-$orthogonal projections, and

$$I = \sum_{k=1}^{N} Q_k \Pi_k \text{ on } U$$

Abstract Multispace BDDC

Theorem 2 **The abstract Multispace BDDC preconditioner** $M : U \longrightarrow U$ *defined by*

$$M : r \mapsto u, \quad u = \sum_{k=1}^{N} Q_k v_k, \quad v_k \in V_k : \quad a(v_k, z_k) = \langle r, Q_k z_k \rangle, \quad \forall z_k \in V_k,$$

satisfies

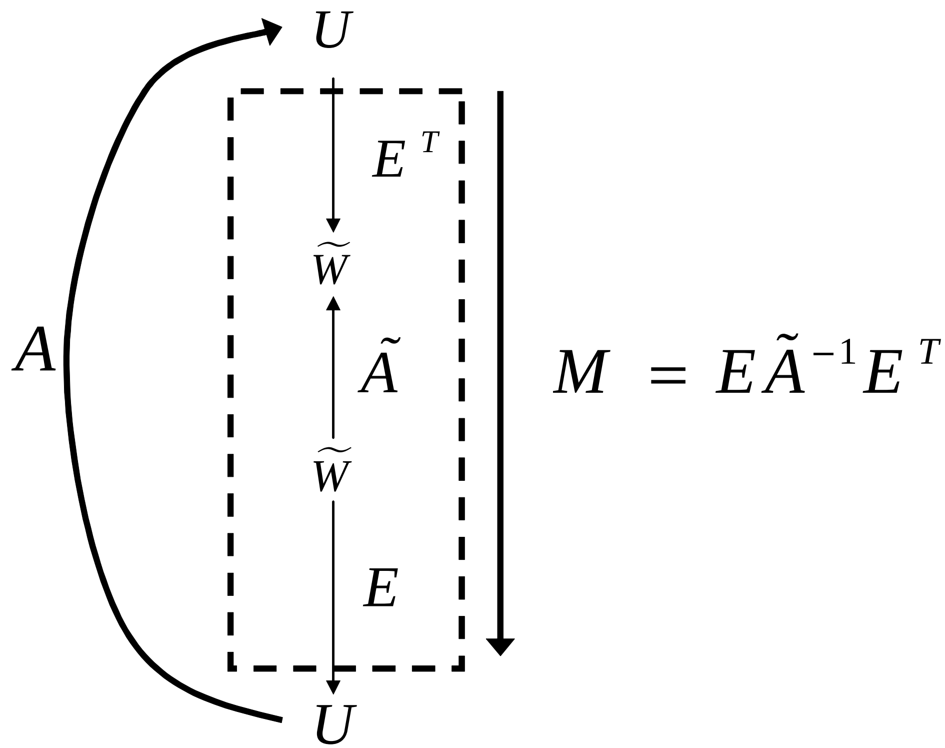
$$\kappa = \frac{\lambda_{\max}(MA)}{\lambda_{\min}(MA)} \leq \omega = \max_k \sup_{v_k \in V_k} \frac{\|Q_k v_k\|_a^2}{\|v_k\|_a^2}.$$

For $N = 1$ we recover the abstract BDDC algorithm and condition number bound. Proved from generalized Schwarz theory (Dryja and Widlund, 1995). Unlike in the Schwarz theory, we decompose some large space than U.

In a sense:

1. the spaces V_k decompose the space $\widetilde{W}$, and
2. the projections $Q_k : V_k \to U$ decompose the projection $E : \widetilde{W} \to U$.

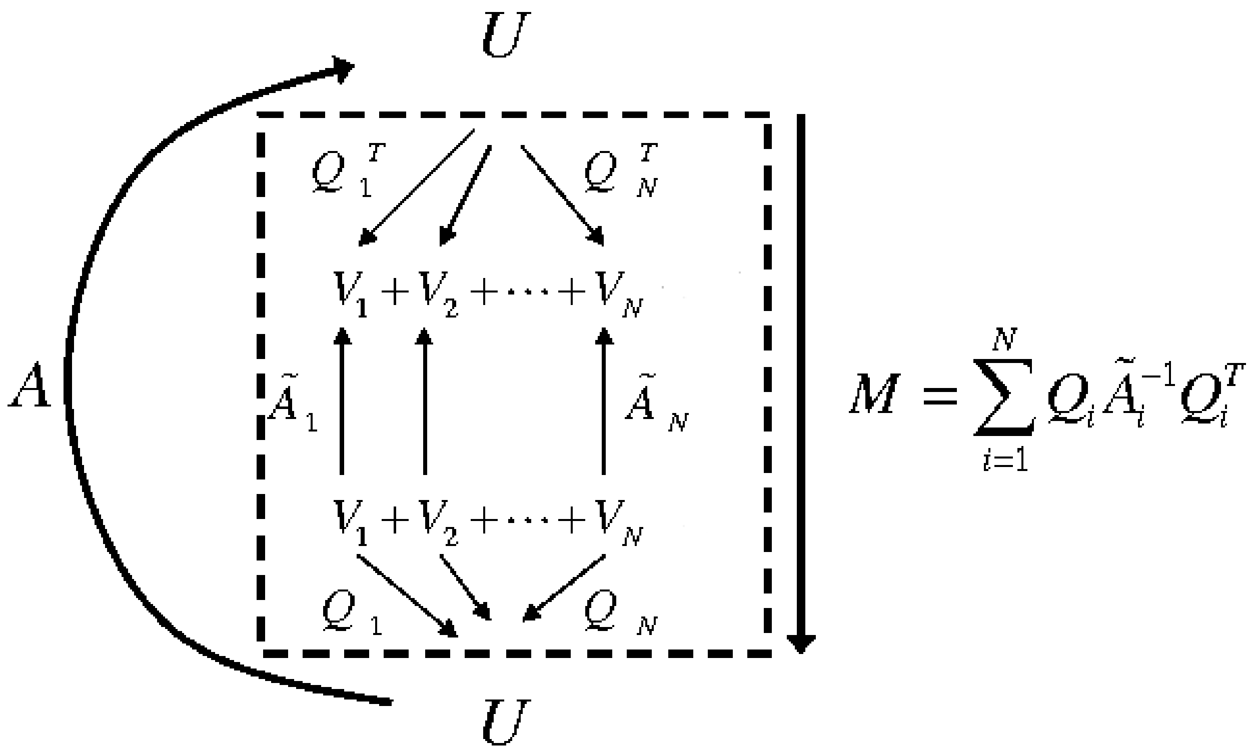
Algebraic View of the Abstract BDDC Preconditioner



The same bilinear form a defines $A : U \rightarrow U$ and $\widetilde{A} : \widetilde{W} \rightarrow \widetilde{W} \supset U$

The preconditioner M to A is obtained by solving a problem with the same bilinear form on the bigger space $\widetilde{W}$ and mapping back to U via the projection E and its transpose E^T.

Algebraic View of the Abstract Multispace BDDC Preconditioner



The same bilinear form a defines $A : U \to U$ and $\widetilde{A}_i : V_i \to V_i$, $\sum_{i=1}^{N} V_i \supset U$
The preconditioner M to A is obtained by solving problems with the same bilinear form on the bigger spaces V_i and mapping back to U via the projections Q_i and their transposes Q_i^T.

BDDC with Interiors as Multispace BDDC

Abstract BDDC often presented on the space of discrete harmonic functions. The original BDDC formulation had "interior correction":

$$\boxed{U_I \overset{P}{\underset{\subset}{\leftarrow}} U \overset{E}{\underset{\subset}{\leftarrow}} \widetilde{W}}$$

Lemma 3 *The original BDDC preconditioner is the abstract Multispace BDDC method with $N = 2$ and the spaces and operators*

$$V_1 = U_I, \quad V_2 = (I - P)\widetilde{W}, \quad Q_1 = I, \quad Q_2 = (I - P)\, E.$$

The space $\widetilde{W}$ has an a-orthogonal decomposition

$$\widetilde{W} = \widetilde{W}_{\Delta} \oplus \widetilde{W}_{\Pi}.$$

so the problem on $\widetilde{W}$ splits into independent problems on $\widetilde{W}_{\Delta}$ and $\widetilde{W}_{\Pi}$.

Example:
$\widetilde{W}_{\Delta}$ = functions zero on substructure corners
$\widetilde{W}_{\Pi}$ = given by values on substructure corners and energy minimal

BDDC with Interiors as Multispace BDDC

The same BDDC formulation with “interior correction” and splitting of $\widetilde{W}$:

$$\begin{array}{ccccc} U_I & \overset{P}{\underset{\subset}{\leftarrow}} & U & \overset{E}{\underset{\subset}{\leftarrow}} & \widetilde{W} \\ & & & & \parallel \\ & & \widetilde{W}_\Pi & \oplus & \widetilde{W}_\Delta \end{array}$$

Lemma 4 *The original BDDC preconditioner is the abstract multi-space BDDC method with $N = 3$ and the spaces and operators*

$$V_1 = U_I, \quad V_2 = \widetilde{W}_\Pi, \quad V_3 = (I-P)\widetilde{W}_\Delta, \quad Q_1 = I, \quad Q_2 = Q_3 = (I-P)\,E.$$

Solving on $U_I \Rightarrow$ independent Dirichet problems on substructures
Solving on $(I-P)\widetilde{W}_\Delta \Rightarrow$ independent constrained Neumann problems on substructures + correction in U_I
Solving on $\widetilde{W}_\Pi \Rightarrow$ **Global coarse problem with substructures as coarse elements and energy minimal function as coarse shape functions.**

Three-level BDDC

Coarse problem solved approximately by the BDDC preconditioner.

$$
\begin{array}{ccccccc}
 & & U & & & & \\
 & & \parallel & & & & \\
U_{I1} & \underset{\subset}{\overset{P_1}{\leftarrow}} & U_1 & \underset{\subset}{\overset{E_1}{\leftarrow}} & \widetilde{W}_1 & & \\
 & & & & \parallel & & \\
 & & & \widetilde{W}_{\Pi 1} & \oplus & \widetilde{W}_{\Delta 1} & \\
 & & & \parallel & & & \\
 & U_{I2} & \underset{\subset}{\overset{P_2}{\leftarrow}} & U_2 & \underset{\subset}{\overset{E_2}{\leftarrow}} & \widetilde{W}_2 & \\
 & & & & & \parallel & \\
 & & & & \widetilde{W}_{\Pi 2} & \oplus & \widetilde{W}_{\Delta 2}
\end{array}
$$

Lemma 5 *The three-level BDDC preconditioner is the abstract Multispace BDDC method with* $N = 5$ *and the spaces and operators*

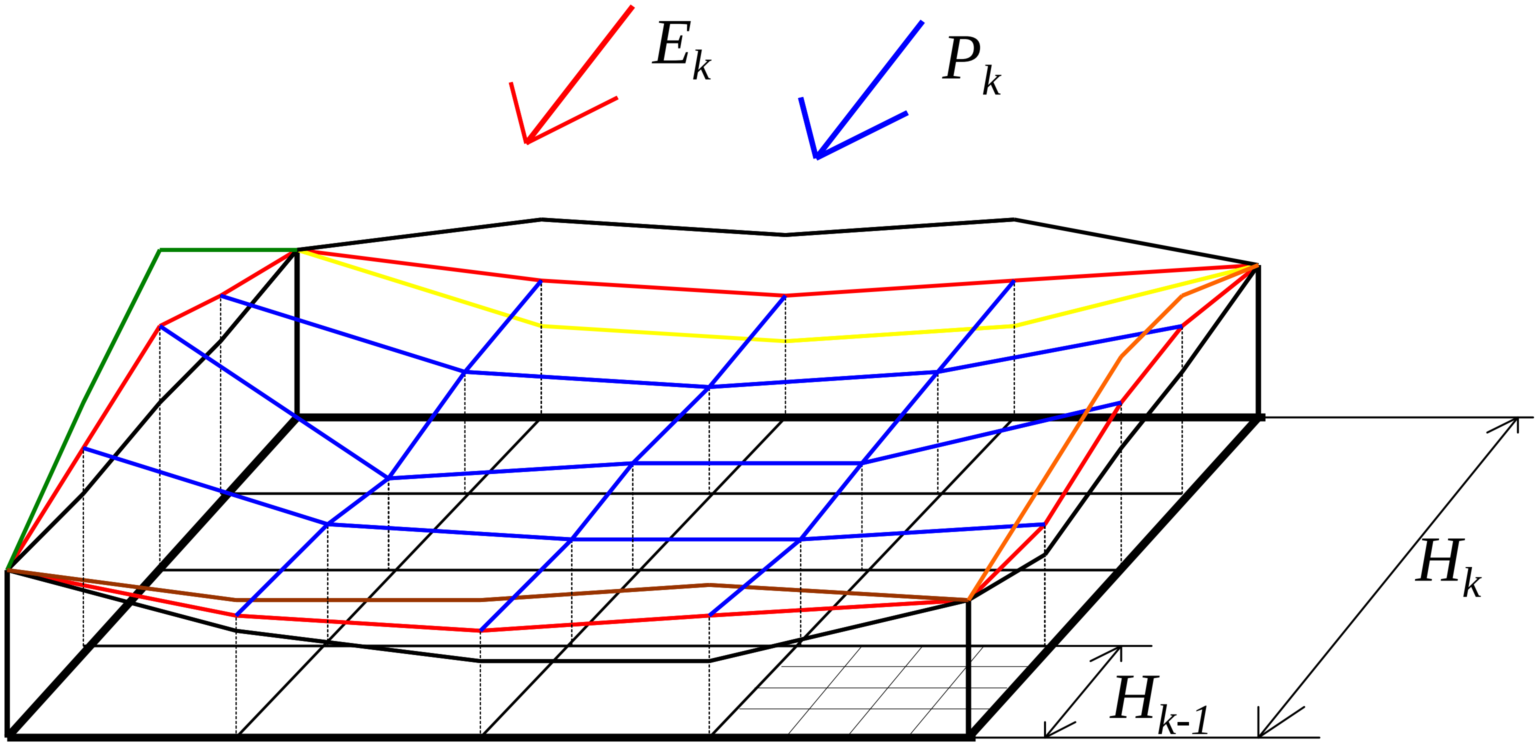
$$
V_1 = U_{I1}, \quad V_2 = (I - P_1)\widetilde{W}_{\Delta 1}, \quad V_3 = U_{I2}, \quad V_4 = (I - P_2)\widetilde{W}_{\Delta 2}, \quad V_5 = \widetilde{W}_{\Pi 2},
$$
$$
Q_1 = I, \quad Q_2 = Q_3 = (I - P_1)\, E_1, \quad Q_4 = Q_5 = (I - P_1)\, E_1 \,(I - P_2)\, E_2.
$$

Multilevel BDDC

Coarse problem solved by the BDDC preconditioner, recursive.

$$
\begin{array}{cccccccc}
 & & U & & & & & \\
 & & \parallel & & & & & \\
U_{I1} & \overset{P_1}{\underset{\subset}{\leftarrow}} & U_1 & \overset{E_1}{\underset{\subset}{\leftarrow}} & \widetilde{W}_1 & & & \\
 & & & & \parallel & & & \\
 & & & \widetilde{W}_{\Pi 1} & \oplus & \widetilde{W}_{\Delta 1} & & \\
 & & & \parallel & & & & \\
 & U_{I2} & \overset{P_2}{\underset{\subset}{\leftarrow}} & U_2 & \overset{E_2}{\underset{\subset}{\leftarrow}} & \widetilde{W}_2 & & \\
 & & & & & \parallel & & \\
 & & & & \widetilde{W}_{\Pi 2} & \oplus & \widetilde{W}_{\Delta 2} & \\
 & & & & \parallel & & & \\
 & & & & \vdots & & & \\
 & & & & \parallel & & & \\
 & & U_{IL-1} & \overset{P_{L-1}}{\underset{\subset}{\leftarrow}} & U_{L-1} & \overset{E_{L-1}}{\underset{\subset}{\leftarrow}} & \widetilde{W}_{L-1} & \\
 & & & & & & \parallel & \\
 & & & & & \widetilde{W}_{\Pi L-1} & \oplus & \widetilde{W}_{\Delta L-1}
\end{array}
$$

An Example of Action of Operators E_k and P_k



Values on this substructure and its neighbors are averaged by E_k, then extended as "discrete harmonic" by P_k.

Basis functions on level k are given by dofs on level k & **energy minimal** w.r.t. basis functions on level $k-1$. Discrete harmonics on level k are given by boundary values & **energy minimal** w.r.t. basis functions on level $k-1$.

Multilevel BDDC

Coarse problem solved by the BDDC preconditioner, recursive.

Lemma 6 *The Multilevel BDDC preconditioner is the abstract Multispace BDDC preconditioner with N=2L-2 and*

$$
\begin{aligned}
V_1 &= U_{I1}, \quad V_2 = (I - P_1)\widetilde{W}_{\Delta 1}, \quad V_3 = U_{I2}, \\
V_4 &= (I - P_2)\widetilde{W}_{\Delta 2}, \quad V_5 = U_{I3}, \quad \ldots \\
V_{2L-4} &= (I - P_{L-2})\widetilde{W}_{\Delta L-2}, \quad V_{2L-3} = U_{IL-1,}, \\
V_{2L-2} &= (I - P_{L-1})\widetilde{W}_{L-1} \\
Q_1 &= I, \quad Q_2 = Q_3 = (I - P_1)\, E_1, \\
Q_4 &= Q_5 = (I - P_1)\, E_1 \,(I - P_2)\, E_2, \quad \ldots \\
Q_{2L-4} &= Q_{2L-3} = (I - P_1)\, E_1 \cdots (I - P_{L-2})\, E_{L-2}. \\
Q_{2L-2} &= (I - P_1)\, E_1 \cdots (I - P_{L-1})\, E_{L-1}
\end{aligned}
$$

Recall condition number bound:

$$
\kappa = \frac{\lambda_{\mathsf{max}}(MA)}{\lambda_{\mathsf{min}}(MA)} \le \omega = \max_k \sup_{v_k \in V_k} \frac{\|Q_k v_k\|_a^2}{\|v_k\|_a^2}.
$$

Algebraic Condition Estimate of Multilevel BDDC

Lemma 7 *If*

$$\begin{aligned}
\|(I-P_1)E_1w_1\|_a^2 &\le \omega_1\,\|w_1\|_a^2 \quad \forall w_1\in\widetilde{W}_1,\\
\|(I-P_2)E_2w_2\|_a^2 &\le \omega_2\,\|w_2\|_a^2 \quad \forall w_2\in\widetilde{W}_2,\\
&\vdots\\
\|(I-P_{L-1})E_{L-1}w_{L-1}\|_a^2 &\le \omega_{L-1}\,\|w_{L-1}\|_a^2 \quad \forall w_{L-1}\in\widetilde{W}_{L-1}.
\end{aligned}$$

then the multilevel BDDC preconditioner satisfies $\kappa \le \prod_{i=1}^{L-1}\omega_i$.

- All spaces are subspaces of the single space W.

- The functions $(I-P_i)E_iw_i$ are discrete harmonic functions on level i with energy minimal extension into the interior after averaging on level i, such that w_i has continuous coarse dofs (such as values at corners) at the decomposition level $i-1$.

Condition Number Estimate with Corner Contraints

Theorem 8 *The Multilevel BDDC preconditioner in 2D with corner constraints only satisfies*

$$\kappa \leq C_1 \left(1 + \log \frac{H_1}{h}\right)^2 C_2 \left(1 + \log \frac{H_2}{H_1}\right)^2 \cdots C_{L-1} \left(1 + \log \frac{H_{L-1}}{H_{L-2}}\right)^2.$$

For $L = 3$ we recover the estimate by Tu (2004).
This bound implies at most polylogarithmic growth of the condition number in the ratios of mesh sizes for a fixed number of levels L
For fixed H_i/H_{i-1} the growth of the condition number can be exponential in L and this is indeed seen in numerical experiments
With additional constraints, such as side averages, the condition number will be less but the bound is still principally the same, though possibly with (much) smaller constants. For small enough constants, the exponential growth of the condition number may no longer be apparent.

Numerical Examples

Multilevel BDDC implemented for the 2D Laplace eq. on a square domain:

2D Laplace equation results for $H_i/H_{i-1} = 8$ at all levels.

L	corners only		corners & faces		n	n_Γ
	iter	cond	iter	cond		
2	10	2.99	7	1.33	1024	240
3	19	7.30	11	2.03	65,536	15,360
4	31	18.6	13	2.72	4,194,304	983,040
5	50	47.38	15	3.40	268,435,456	62,914,560

2D Laplace equation results for $H_i/H_{i-1} = 16$ at all levels.

L	corners only		corners & faces		n	n_Γ
	iter	cond	iter	cond		
2	19	6.90	10	1.93	65,536	7936
3	23	12.62	13	2.67	1,048,576	126,976
4	43	41.43	16	3.78	268,435,456	32,505,856

2D Laplace equation results for $H_i/H_{i-1} = 4$ at all levels.

L	corners only		corners & faces		n	n_Γ
	iter	cond	iter	cond		
2	9	2.20	6	1.14	256	112
3	15	4.02	8	1.51	4096	1792
4	21	7.77	10	1.88	65,536	28,672
5	30	15.2	12	2.24	1,048,576	458,752
6	42	29.7	13	2.64	16,777,216	7,340,032

2D Laplace eq. results for $H_1/H_0 = 4$, $H_2/H_1 = 4$, and varying H_3/H_2.

H_3/H_2	corners only		corners & faces		n	n_Γ
	iter	cond	iter	cond		
4	21	7.77	10	1.88	65,536	28,672
8	23	10.74	11	2.23	262,144	114,688
16	25	14.54	13	2.63	1,048,576	458,752
32	28	19.10	14	3.08	4,194,304	1,835,008
64	31	24.39	14	3.57	16,777,216	7,340,032

Conclusion

- Described an algorithm for Multilevel BDDC preconditioning and derived a condition number estimate for case of corner constraints.

- Method tested on Laplace equation in 2D. Numerical results confirm the theory.

- The new concept of Multispace BDDC and algebraic estimate of its condition number could be of independent interest.

Future Developments

- 3D condition number bounds + extension to linear elasticity.
- Other types of constraints - why does the condition number grow so much less when side averages are added in 2D?
- Lower bounds.
- Extensions of the adaptive approach (Mandel, Sousedík 2007) to the multilevel case $\Longrightarrow$ solution of problems that are both very large and numerically difficult.