Convergence Issues of Iterative Aggregation/Disaggregation

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Outline

1. Some motivation
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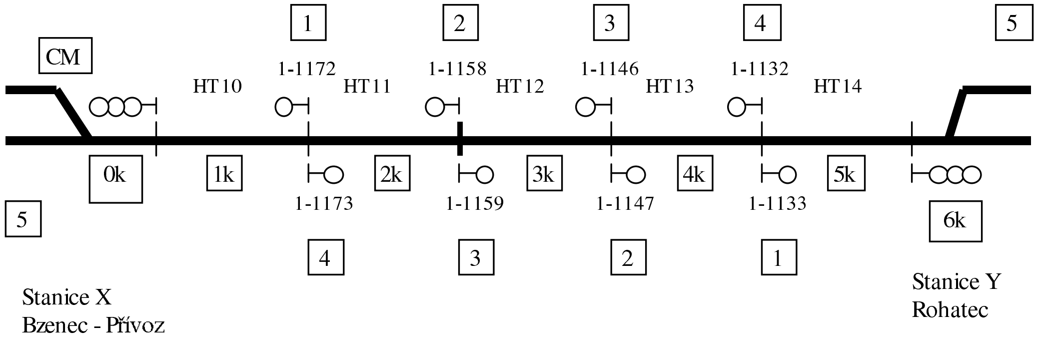
Some motivation

Definition 1. *Let elements of $T \in \Re^{n \times n}$ be non negative and $Te = e$, where $e = (1, ..., 1)^T \in \Re^n$. Then we call T the stochastic matrix.*

Definition 2. *A finite Markov chain is stochastic process, which moves through finite number of states, and for which the probability of entering a certain state depends only on the last state occupied.*

Definition 3. *A transient state has a non-zero probability that the chain will never return to this state.*
A reccurent(persistent) state has a zero probability that the chain will never return to this state.

E ... matrix of all ones
e ... vector of all ones
I ... identity matrix


1
2
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CM
1-1172
1-1158
1-1146
1-1132
HT10
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HT12
HT13
HT14
0k
1k
2k
3k
4k
5k
6k
1-1173
1-1159
1-1147
1-1133
5
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3
2
1
Stanice X
Bzenec - Přívoz
Stanice Y
Rohatec

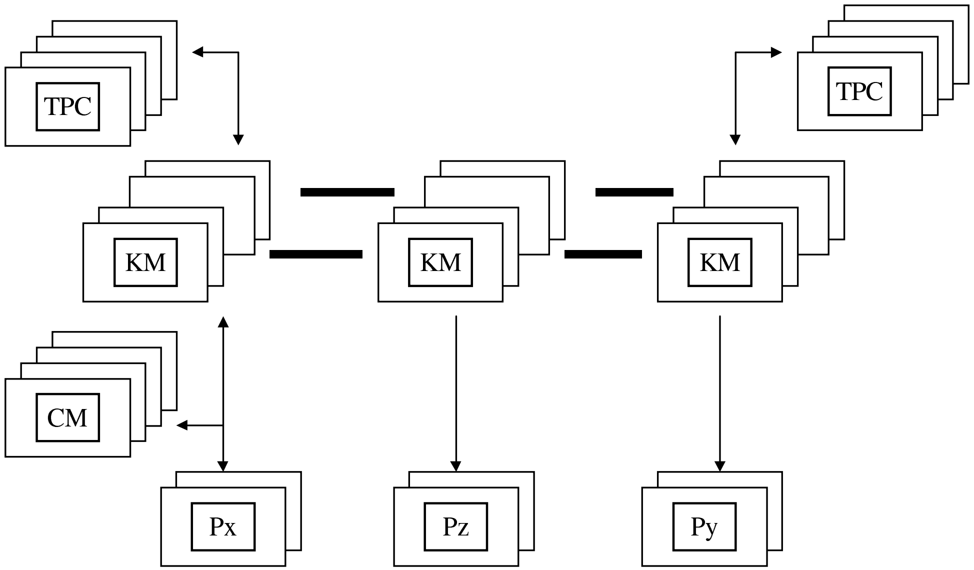
TPC
TPC
KM
KM
KM
CM
Px
Pz
Py

Figure 1.

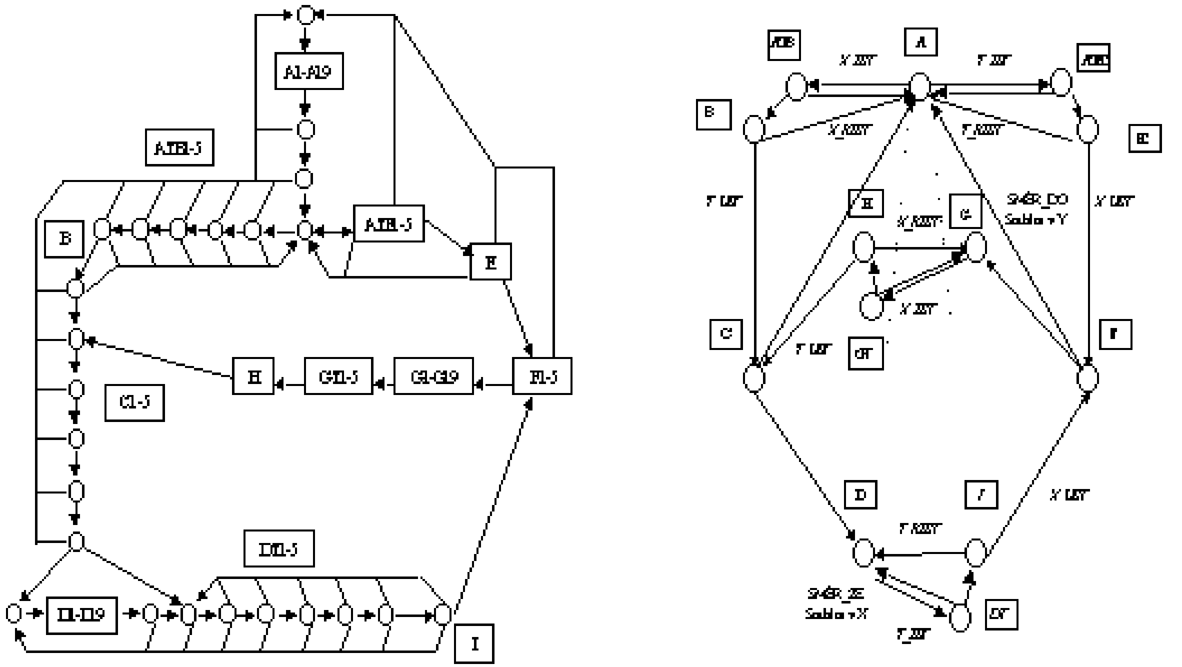

Figure 2.

To find long time behaviour of such system, we have to solve

Problem 1. We solve

$$Tx = x,\; e^T x = 1 \tag{1}$$

Let $Q \in \Re^{N \times N}$, such that $e^T Q = 0$, $\operatorname{diag} Q \leqslant 0$, $\operatorname{offdiag} Q \geqslant 0$, we try to compute

$$u(t) = e^{Qt} u(0)$$

using the Implicit Euler method, then we have to solve at every step system

$$u(t) = \tau Q\, u(t) + u(t - \tau).$$

After some rearangement ve finish with system of the type

$$x = Tx + b, \tag{2}$$

where T is nonnegative matrix with spectral radius less than one.

IAD method for stationary probability vector

$$g\colon \{1, \ldots, N\} \longrightarrow \{1, \ldots, n\}.$$

The restriction matrix $R \in \Re^{N \times n}$:

$$R_{g(i),i} = 1$$

$$(Rx)_j = \sum_{j=1,\, g(j)=i}^{N} x_j.$$

The prolongation matrix $S(x)$ is parametrised by vector $x \in \Re^N$, the nonzero elements of this matrix are

$$(S(x))_{i,g(i)} = \frac{x_i}{(Rx)_{g(i)}},$$

$$(S(x)z)_i = z_{g(i)} \frac{x_i}{(Rx)_{g(i)}} .$$

Aggregated matrix : $A(x) = RTS(x)$.

Lemma 4. *Let T be a column stochastic matrix, let g be an aggregation mapping and $x \in \Re^N$ such that $x \geqslant 0$ and $R\,x > 0$. Then aggregated matrix $A(x)$ is collumn stochastic. If the matrix T is irreducible and the vector x is strictly positive, then $A(x)$ is irreducible.*

Note 5. Let us note that the strict positivity of x is essential.

$$T = \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{2}{3} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \end{pmatrix}, \quad x = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}, \quad g\colon \begin{array}{c} 1 \mapsto 1 \\ 2 \mapsto 1 \\ 3 \mapsto 2 \\ 4 \mapsto 2 \end{array}.$$

We get the matrix $A(x) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, which is reducible.

Algorithm IAD(input: $T, M, W, x_{\text{init}}, \varepsilon, g, s$ output: x)

1. $k := 1, x_1 := x_{\text{init}}$
2. while $\|Tx_k - x_k\| > \varepsilon$ do
3. $\tilde{x} := (M^{-1}W)^s x_k$
4. $A(\tilde{x}) := RTS(\tilde{x})$
5. solve $A(\tilde{x})z = z$ and $e^T z = 1$
6. $k := k + 1$
7. $x_k = S(\tilde{x})z$
8. end while

Convergence theory for IAD can be found in [1].

Theorem 6. *Let T be a column stochastic matrix, let $\hat{x}$ be the solution of (1), then there exist s_0 and neighborhood of $\hat{x}$ such that for any x_{init} from this neighborhood and any $s > s_0$ the Algoritm IAD is convergent.*

IAD with right hand side

we solve problem (2), i.e.

$$x = Tx + b$$

Algorithm RHS(input: $T, M, W, x_{\text{init}}, \varepsilon, g, s$ output: x)

1. $k := 1, x_1 := x_{\text{init}}$
2. while $\|Tx_k - x_k\| > \varepsilon$ do
3. $\tilde{x}_0 = x_k$
4. for j=1,s do
5. $\tilde{x}_j := (M^{-1}W)\tilde{x}_{j-1} + M^{-1}b$
6. end do
7. $\tilde{x} = \tilde{x}_s$
8. $A(\tilde{x}) := RTS(\tilde{x})$
9. solve $z = A(\tilde{x})z + Rb$
10. $k := k + 1$
11. $x_k = S(\tilde{x})z$
12. end while

Error formula

For both previous processes we have same error formula

$$x_k - x^* = (M^{-1}W)^s (I - P(x_{k-1})T)^{-1}(I - P(x_{k-1}))$$

where

$$P(x) = S(x)\,R$$

Fast convergence

Theorem 7. *Let for splitting* M, W *be* $\mathrm{range}(M^{-1}W) \subseteq \mathrm{range}(S(\hat{x}))$. *Then Algorithm IAD terminates after the first iteration.*

Example 8. Let

$$T = \begin{pmatrix} 0.1 & 0.1 & 0.1 & 0.05 & 0.15 & 0.25 \\ 0.5 & 0.2 & 0 & 0.02 & 0.06 & 0.10 \\ 0.2 & 0.1 & 0.1 & 0.03 & 0.09 & 0.15 \\ 0.04 & 0.12 & 0.16 & 0.2 & 0.2 & 0.1 \\ 0.08 & 0.24 & 0.32 & 0.6 & 0.2 & 0.1 \\ 0.08 & 0.24 & 0.32 & 0.1 & 0.3 & 0.3 \end{pmatrix}$$

and splitting $I - T = M - W$ be

$$M = \begin{pmatrix} 0.9 & -0.1 & -0.1 & 0 & 0 & 0 \\ -0.5 & 0.8 & 0 & 0 & 0 & 0 \\ -0.2 & -0.1 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & -0.2 & -0.1 \\ 0 & 0 & 0 & -0.6 & 0.8 & -0.1 \\ 0 & 0 & 0 & -0.1 & -0.3 & 0.7 \end{pmatrix}$$

$$W = \begin{pmatrix} 0 & 0 & 0 & 0.05 & 0.15 & 0.25 \\ 0 & 0 & 0 & 0.02 & 0.06 & 0.10 \\ 0 & 0 & 0 & 0.03 & 0.09 & 0.15 \\ 0.04 & 0.12 & 0.16 & 0 & 0 & 0 \\ 0.08 & 0.24 & 0.32 & 0 & 0 & 0 \\ 0.08 & 0.24 & 0.32 & 0 & 0 & 0 \end{pmatrix}$$

Let aggregation mapping be

$$g\colon \begin{aligned} 1 &\mapsto 1 \\ 2 &\mapsto 1 \\ 3 &\mapsto 1 \\ 4 &\mapsto 2 \\ 5 &\mapsto 2 \\ 6 &\mapsto 2 \end{aligned}$$

$$x_0 = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})^T$$

$$\tilde{x} = (0.104124, 0.102577, 0.084536, 0.182906, 0.309402, 0.311111)^T$$

$$A(\tilde{x}) = \begin{pmatrix} 0.4849558 & 0.3319149 \\ 0.5150442 & 0.6680851 \end{pmatrix}$$

$$z = (0.3918901, 0.6081099)^T$$

$$x_1 = (0.140109, 0.138029, 0.113752, 0.138442, 0.234187, 0.235481)^T$$

It is not true for Algorithm RHS, we need to add condition

$$b \in \text{range}\,(S(\hat{x})).$$

Other possibility is to replace steps 8, 9, 11 by

Step 8 : $A(\tilde{x}_s - \tilde{x}_{s-1}) := RTS(\tilde{x}_s - \tilde{x}_{s-1})$
Step 9 : solve $z = A(\tilde{x})z + R(b - Tx_k)$
Step 11: $x_k = S(\tilde{x})z + x_{k-1}$

Algorithm RHS(input: $T, M, W, x_{\text{init}}, \varepsilon, g, s$ output: x)

1. $k := 1, x_1 := x_{\text{init}}$
2. while $\|Tx_k - x_k\| > \varepsilon$ do
3. $\tilde{x}_0 = x_k$
4. for j=1,s do
5. $\tilde{x}_j := (M^{-1}W)\tilde{x}_{j-1} + M^{-1}b$
6. end do
7. $\tilde{x} = \tilde{x}_s$
8. $A(\tilde{x}_s - \tilde{x}_{s-1}) := RTS(\tilde{x}_s - \tilde{x}_{s-1})$
9. solve $z = A(\tilde{x})z + R(b - Tx_k)$
10. $k := k + 1$
11. $x_k = S(\tilde{x})z + x_{k-1}$
12. end while

Remark

for every irreducible stochastic matrix T there exist

$$\lim_{k\to\infty} T^k e = x^* \text{ and } Tx^* = x^*$$

Nearly dyadic matrices: bl = 60 sz = 420

method	p=0 w=0.5	p=0 w=0.1	p=0 w=0.01	p=0.1 w=0.5	p=0.1 w=0.1	p=0.1 w=0.01	p=0.5 w=0.5	p=0.5 w=0.1	p=0.5 w=0.01
power	53	224	3130	44	142	1812	55	70	915
MM	16	25	27	12	23	27	12	19	27
Vant	1	1	1	11	10	9	12	12	10
KMS	1	1	1	9	9	7	9	10	8
Jacobi	55	71	76	54	71	77	59	75	82
G.S.	41	51	54	30	42	47	29	40	45

Conclusion

– IAD methods are best for computing of SPV

– for cyclic matrices the power method applied to all ones vector is a reasonable choice

– structure of solution is significant

– Algorithm RHS is applicable for computing moments of Markov chains

Bibliography

[1] I. Marek and P. Mayer. Iterative aggregation/disaggregation methods for computing some characteristic of of markov chains. In *Large Scale Scientific Computing*, pages 68–82, 2001. Third International Conference, LSSC 2001, Sozopol, Bulgaria.