On Stein-Rosenberg type theorems for nonnegative splittings

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Computational methods with Applications, Harrachov-2007

Czech Republic, August 19–25, 2007

The Stein Rosenberg Theorem:

Theorem [Stein, P. and Rosenberg R. L., 1948] Let the Jacobi matrix $B \equiv L + U$ be a nonnegative $n \times n$ matrix with zero diagonal entries, and let $\mathcal{L}_1$ be the Gauss-Seidel matrix. Then one and only one of the following mutually exclusive relations is valid:

(i) $\rho(B) = \rho(\mathcal{L}_1) = 0$.

(ii) $0 < \rho(\mathcal{L}_1) < \rho(B) < 1$.

(iii) $\rho(B) = \rho(\mathcal{L}_1) = 1$.

(iv) $1 < \rho(B) < \rho(\mathcal{L}_1)$. •

The Aim

The aim of this paper is:

- **to extend and generalize the Stein-Rosenberg Theorem for nonnegative splittings.**
- **to give an outline of the extension and generalization of the Stein-Rosenberg Theorem to the Perron-Frobenius splittings.**

Definitions

- $A \in \mathbb{R}^{n \times n}$ **has the** ***Perron-Frobenius (PF) property*** **if** $\rho(A) \in \sigma(A)$ **and there exists a nonnegative eigenvector corresponding to** $\rho(A)$**.**
- $A \in \mathbb{R}^{n \times n}$ **has the** ***strong Perron-Frobenius property*** **if, in addition,**
  $$\rho(A) > |\lambda| \quad \text{for all} \quad \lambda \in \sigma(A), \quad \lambda \neq \rho(A)$$
  and the corresponding eigenvector is positive.
- **D. Noutsos, On Perron-Frobenius property of matrices having some negative entries.** ***LAA*****,** ***LAA*****, 412 (2006) 132–153.**

Definitions of Splittings

A splitting $A = M - N$ is called:

- M-splitting if M is an M-matrix and $N \geq 0$,
- Regular splitting if $M^{-1} \geq 0$ and $N \geq 0$,
- Weak regular of 1st type if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$,
- Weak regular of 2nd type if $M^{-1} \geq 0$ and $NM^{-1} \geq 0$,
- Nonnegative of 1st type if $M^{-1}N \geq 0$,
- Nonnegative of 2nd type if $NM^{-1} \geq 0$,
- Perron-Frobenius of 1st type if $M^{-1}N$ has the PF property,
- Perron-Frobenius of 2nd type if NM^{-1} has the PF property,

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The Stein-Rosenberg Theorem on Nonnegative Splittings

Theorem 1 **Let** $A = M_1 - N_1 = M_2 - N_2$ **be both nonnegative splittings, (**$M_i^{-1}N_i \geq 0, \;\; i = 1, 2$**) and**

$$M_1^{-1}N_1 \geq M_1^{-1}N_2 \geq 0, \;\; M_1^{-1}N_1 \neq M_1^{-1}N_2, \;\; M_1^{-1}N_2 \neq 0.$$

Then exactly one of the statements holds:

(i) $0 \leq \rho(M_2^{-1}N_2) \leq \rho(M_1^{-1}N_1) < 1$

(ii) $\rho(M_2^{-1}N_2) = \rho(M_1^{-1}N_1) = 1$

(iii) $\rho(M_2^{-1}N_2) \geq \rho(M_1^{-1}N_1) > 1$**.** ●

Characterization of the inequalities

Theorem 2 Let $A = M_1 - N_1 = M_2 - N_2$ be both nonnegative splittings, ($M_i^{-1} N_i \geq 0, \;\; i = 1, 2$) and

$$M_1^{-1} N_1 \geq M_1^{-1} N_2 \geq 0, \;\; M_1^{-1} N_1 \neq M_1^{-1} N_2, \;\; M_1^{-1} N_2 \neq 0.$$

Assume that the matrices $M_1^{-1} N_1, \;\; T = M_1^{-1}(N_1 - N_2)$ and $F = M_1^{-1} N_2$ are up to a permutation of the form

$$M_1^{-1} N_1 = \begin{pmatrix} P_{11} & 0 \\ P_{21} & 0 \end{pmatrix}, \; T = \begin{pmatrix} T_{11} & 0 \\ T_{21} & 0 \end{pmatrix}, \; F = \begin{pmatrix} F_{11} & 0 \\ F_{21} & 0 \end{pmatrix}$$

with $P_{11}, \;\; T_{11}$ and F_{11} being $k \times k$ matrices ($k \leq n$), P_{11} irreducible and $T_{11}, F_{11} \neq 0$.

Then, exactly one of the statements holds:

(i) $0 < \rho(M_2^{-1}N_2) < \rho(M_1^{-1}N_1) < 1$
(ii) $\rho(M_2^{-1}N_2) = \rho(M_1^{-1}N_1) = 1$
(iii) $\rho(M_2^{-1}N_2) > \rho(M_1^{-1}N_1) > 1$.

If $T_{11} = 0$**, the second inequality of (i) and the first one of (iii) become equalities, while if** $F_{11} = 0$**, the first inequality of (i) becomes equality.** •

Analogous theorem holds for nonnegative splittings of the second type.

Example 1

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, M_1 = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 3 \end{pmatrix}, M_2 = \begin{pmatrix} 2 & -1 & 1 \\ -0.5 & 2 & -1 \\ 0.5 & -1 & 3 \end{pmatrix},$$

$$M_1^{-1} = \frac{1}{9}\begin{pmatrix} 5 & 2 & -1 \\ -1 & 5 & 2 \\ -2 & 1 & 4 \end{pmatrix}, M_1^{-1}N_1 = \frac{1}{9}\begin{pmatrix} 1 & 0 & 4 \\ 7 & 0 & 1 \\ 5 & 0 & 2 \end{pmatrix},$$

$$M_2^{-1} = \begin{pmatrix} 0.5882 & 0.2353 & -0.1176 \\ 0.1176 & 0.6471 & 0.1765 \\ -0.0588 & 0.1765 & 0.4118 \end{pmatrix}, M_2^{-1}N_2 = \begin{pmatrix} 0.0588 & 0 & 0.4706 \\ 0.4118 & 0 & 0.2941 \\ 0.2941 & 0 & 0.3529 \end{pmatrix}.$$

$$F = M_1^{-1}N_2 = \frac{1}{18}\begin{pmatrix} 1 & 0 & 8 \\ 7 & 0 & 2 \\ 5 & 0 & 4 \end{pmatrix}, T = M_1^{-1}(N_1 - N_2) = \frac{1}{18}\begin{pmatrix} 1 & 0 & 0 \\ 7 & 0 & 0 \\ 5 & 0 & 0 \end{pmatrix},$$

$$\rho(M_2^{-1}N_2) = 0.6059 < \rho(M_1^{-1}N_1) = \frac{2}{3} < 1.$$

Example 2

$$A = \begin{pmatrix} 0.5 & 4.5 & -2.5 \\ 0.5 & -4.5 & 2 \\ -0.5 & 0.5 & -0.5 \end{pmatrix}, M_1 = \begin{pmatrix} 1 & 5 & -3 \\ 1 & -4 & 3 \\ -0.5 & 1 & 0 \end{pmatrix}, M_2 =$$

$$\begin{pmatrix} 1 & 4.5 & -3 \\ 1 & -4 & 3 \\ -0.5 & 1 & 0 \end{pmatrix}, M_1^{-1} = \frac{1}{3} \begin{pmatrix} 1.2 & 1.2 & -1.2 \\ 0.6 & 0.6 & 2.4 \\ 0.4 & 1.4 & 3.6 \end{pmatrix}, M_1^{-1} N_1 =$$

$$\begin{pmatrix} 0.4 & 0.2 & 0 \\ 0.2 & 0.6 & 0.5 \\ 0.3 & 0.9 & 1 \end{pmatrix}, M_2^{-1} = \begin{pmatrix} 0.4444 & 0.4444 & -0.2222 \\ 0.2222 & 0.2222 & 0.8889 \\ 0.1481 & 0.4815 & 1.2593 \end{pmatrix}, M_2^{-1} N_2 =$$

$$\begin{pmatrix} 0.4444 & 0.1111 & 0.1111 \\ 0.2222 & 0.5556 & 0.5556 \\ 0.3148 & 0.8704 & 1.0370 \end{pmatrix}, F = \begin{pmatrix} 0.3 & 0 & 0 \\ 0.2 & 0.5 & 0.5 \\ 0.3 & 0.8333 & 1 \end{pmatrix}, T =$$

$$\frac{1}{3} \begin{pmatrix} 0 & 0.6 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0.2 & 0 \end{pmatrix}, \rho(M_2^{-1} N_2) = 1.5842 > \rho(M_1^{-1} N_1) = 1.5316 > 1.$$

Example 3

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \; M_1 = \begin{pmatrix} 1.4 & -1 \\ 0.7 & 2 \end{pmatrix}, \; M_2 = \begin{pmatrix} 1.5 & -1 \\ 0.5 & 2 \end{pmatrix},$$

$$M_1^{-1} = \frac{1}{7}\begin{pmatrix} 4 & 2 \\ -1.4 & 2.8 \end{pmatrix}, \; M_1^{-1}N_1 = \frac{1}{7}\begin{pmatrix} 1 & 0 \\ 5.6 & 0 \end{pmatrix}$$

$$M_2^{-1} = \frac{1}{7}\begin{pmatrix} 4 & 2 \\ -1 & 3 \end{pmatrix}, \; M_2^{-1}N_2 = \frac{1}{7}\begin{pmatrix} 1 & 0 \\ 5 & 0 \end{pmatrix},$$

$$F = M_1^{-1}N_2 = \frac{1}{7}\begin{pmatrix} 1 & 0 \\ 4.9 & 0 \end{pmatrix}, \; T = M_1^{-1}(N_1 - N_2) = \begin{pmatrix} 0 & 0 \\ 0.1 & 0 \end{pmatrix}.$$

Assumptions of Theorem 2 hold true except that $T_{11} \neq 0$. We have $T_{11} = 0$, while $P_{11} = 1$. So, equality of the spectral radii is confirmed

$$\rho(M_1^{-1}N_1) = \rho(M_2^{-1}N_2) = \frac{1}{7} < 1,$$

Theorem 3 Let $A = M_1 - N_1 = M_2 - N_2$ be both nonnegative splittings, ($M_i^{-1}N_i \geq 0, \;\; i = 1, 2$) and

$$M_2^{-1}N_1 \geq M_2^{-1}N_2 \geq 0, \;\; M_2^{-1}N_1 \neq M_2^{-1}N_2, \;\; M_2^{-1}N_2 \neq 0.$$

Assume that the matrices $M_2^{-1}N_2, \;\; T = M_2^{-1}(N_1 - N_2)$ and $F = M_2^{-1}N_1$ are up to a permutation of the form

$$M_2^{-1}N_2 = \begin{pmatrix} P_{11} & 0 \\ P_{21} & 0 \end{pmatrix}, \; T = \begin{pmatrix} T_{11} & 0 \\ T_{21} & 0 \end{pmatrix}, \; F = \begin{pmatrix} F_{11} & 0 \\ F_{21} & 0 \end{pmatrix}$$

with $P_{11}, \; T_{11}$ and F_{11} being $k \times k$ matrices ($k \leq n$), P_{11} irreducible and $T_{11}, F_{11} \neq 0$.

Then, exactly one of the statements holds:

(i) $0 < \rho(M_2^{-1}N_2) < \rho(M_1^{-1}N_1) < 1$
(ii) $\rho(M_2^{-1}N_2) = \rho(M_1^{-1}N_1) = 1$
(iii) $\rho(M_2^{-1}N_2) > \rho(M_1^{-1}N_1) > 1$.

If $T_{11} = 0$, the second inequality of (i) and the first one of (iii) become equalities, while if $P_{11} = 0$, the first inequality of (i) becomes equality. •

Analogous theorem holds for nonnegative splittings of the second type.

Example 1, 2, 3

1. $F = M_2^{-1}N_1 = \begin{pmatrix} 0.1176 & 0 & 0.4706 \\ 0.8235 & 0 & 0.2941 \\ 0.5882 & 0 & 0.3529 \end{pmatrix}$, $M_2^{-1}N_2 = \begin{pmatrix} 0.0588 & 0 & 0.4706 \\ 0.4118 & 0 & 0.2941 \\ 0.2941 & 0 & 0.3529 \end{pmatrix}$,

$T = M_2^{-1}(N_1 - N_2) = \begin{pmatrix} 0.0588 & 0 & 0 \\ 0.4118 & 0 & 0 \\ 0.2941 & 0 & 0 \end{pmatrix}$, $\rho(M_2^{-1}N_2) < \rho(M_1^{-1}N_1) < 1.$

2. $F = \begin{pmatrix} 0.4444 & 0.3333 & 0.1111 \\ 0.2222 & 0.6667 & 0.5556 \\ 0.3148 & 0.9444 & 1.0370 \end{pmatrix}$, $M_2^{-1}N_2 = \begin{pmatrix} 0.4444 & 0.1111 & 0.1111 \\ 0.2222 & 0.5556 & 0.5556 \\ 0.3148 & 0.8704 & 1.0370 \end{pmatrix}$,

$T = M_2^{-1}(N_1 - N_2) = \begin{pmatrix} 0 & 0.2222 & 0 \\ 0 & 0.1111 & 0 \\ 0 & 0.0741 & 0 \end{pmatrix}$, $\rho(M_2^{-1}N_2) > \rho(M_1^{-1}N_1) > 1.$

3. $F = \begin{pmatrix} 0.1429 & 0 \\ 0.8143 & 0 \end{pmatrix}$, $M_2^{-1}N_2 = \begin{pmatrix} 0.1429 & 0 \\ 0.7143 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 0 & 0 \\ 0.1 & 0 \end{pmatrix}$,

$\rho(M_2^{-1}N_2) = \rho(M_1^{-1}N_1) = 0.1429.$

Theorem 4 **If the assumptions of Theorem 2 hold, then the assumptions of Theorem 3 hold also.**

So, Theorem 3 is stronger than Theorem 2.

Proof:

$$\begin{aligned} T' &= M_2^{-1}(N_1 - N_2) = (M_1 - N_1 + N_2)^{-1}(N_1 - N_2) \\ &= \left(I - M_1^{-1}(N_1 - N_2)\right)^{-1} M_1^{-1}(N_1 - N_2) \\ &= (I - T)^{-1} T \geq 0, \end{aligned}$$

since $T \geq 0$ **and** $(I - T)^{-1}T = T + T^2 + T^3 + \cdots \geq 0$. ●

Example 4

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \ M_1 = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 2 & -0.7 \\ 1 & -1 & 2.9 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 2 & -1 & 1 \\ -0.5 & 2 & -1 \\ 0.5 & -1 & 3 \end{pmatrix}, \ M_1^{-1}N_1 = \begin{pmatrix} 0.0674 & 0 & 0.5056 \\ 0.6966 & 0 & 0.2247 \\ 0.5618 & 0 & 0.2135 \end{pmatrix},$$

$$M_2^{-1}N_2 = \begin{pmatrix} 0.0588 & 0 & 0.4706 \\ 0.4118 & 0 & 0.2941 \\ 0.2941 & 0 & 0.3529 \end{pmatrix}, \ T = M_1^{-1}(N_1 - N_2) =$$

$$\begin{pmatrix} 0.0337 & 0 & 0.0787 \\ 0.3483 & 0 & 0.1461 \\ 0.2809 & 0 & -0.0112 \end{pmatrix}, \ T' = M_2^{-1}(N_1 - N_2) = \begin{pmatrix} 0.0588 & 0 & 0.0824 \\ 0.4118 & 0 & 0.1765 \\ 0.2941 & 0 & 0.0118 \end{pmatrix},$$

$$\rho(M_2^{-1}N_2) = 0.6059 < \rho(M_1^{-1}N_1) = 0.6784 < 1.$$

Further Extension

Theorem 5 Let $A = M_1 - N_1 = M_2 - N_2$ be both nonnegative splittings, ($M_i^{-1}N_i \geq 0,\ \ i = 1, 2$) with $x_1,\ x_2$ being the right Perron eigenvectors, respectively, and

$$M_1^{-1}N_1x_2 \geq M_1^{-1}N_2x_2 \geq 0,\ \ M_1^{-1}N_1x_2 \neq M_1^{-1}N_2x_2 \neq 0.$$

Assume that the matrices $M_1^{-1}N_1,\ \ T = M_1^{-1}(N_1 - N_2)$ and $F = M_1^{-1}N_2$ are up to a permutation of the form

$$M_1^{-1}N_1 = \begin{pmatrix} P_{11} & 0 \\ P_{21} & 0 \end{pmatrix},\ T = \begin{pmatrix} T_{11} & 0 \\ T_{21} & 0 \end{pmatrix},\ F = \begin{pmatrix} F_{11} & 0 \\ F_{21} & 0 \end{pmatrix}$$

with $P_{11},\ T_{11}$ and F_{11} being $k \times k$ matrices ($k \leq n$), P_{11} irreducible and $T_{11}, F_{11} \neq 0$.

Then, exactly one of the statements holds:

(i) $0 < \rho(M_2^{-1}N_2) < \rho(M_1^{-1}N_1) < 1$

(ii) $\rho(M_2^{-1}N_2) = \rho(M_1^{-1}N_1) = 1$

(iii) $\rho(M_2^{-1}N_2) > \rho(M_1^{-1}N_1) > 1$**.**

If $T_{11} = 0$**, the second inequality of (i) and the first one of (iii) become equalities, while if** $F_{11} = 0$**, the first inequality of (i) becomes equality. •**

Observe that Theorem 5 works on Example 4. Although $T = M^{-1}(N_1 - N_2)$ **is not a nonnegative matrix,**

$$Tx_2 = \begin{pmatrix} 0.0337 & 0 & 0.0787 \\ 0.3483 & 0 & 0.1461 \\ 0.2809 & 0 & -0.0112 \end{pmatrix} \begin{pmatrix} 0.5064 \\ 0.6300 \\ 0.5888 \end{pmatrix} = \begin{pmatrix} 0.0634 \\ 0.2624 \\ 0.1356 \end{pmatrix} > 0$$

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Work in Progress: Extension to the Perron-Frobenius Splittings

Theorem 6 Let $A = M_1 - N_1 = M_2 - N_2$ be both Perron-Frobenius splittings of the first kind, with $(\rho_1, x_1), (\rho_2, x_2)$ being the Perron-Frobenius eigenpairs ($M_i^{-1} N_i x_i = \rho_i x_i \geq 0, \ \ i = 1, 2$) and $M_1^{-1} N_1 x_2 \geq M_1^{-1} N_2 x_2 \geq 0, \ \ M_1^{-1} N_1 x_2 \neq M_1^{-1} N_2 x_2 \neq 0.$

Then exactly one of the statements holds:

(i) $0 \leq \rho(M_2^{-1} N_2) \leq \rho(M_1^{-1} N_1) < 1$

(ii) $\rho(M_2^{-1} N_2) = \rho(M_1^{-1} N_1) = 1$

(iii) $\rho(M_2^{-1} N_2) \geq \rho(M_1^{-1} N_1) > 1$. ●

Thank You

for your

Attention!