Ritz Values of Hermitian Matrices

A nice majorization result.

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Our main result

Theorem: Let $\mathcal{X}$, $\mathcal{Y}$ be subspaces of $\mathbb{C}^n$ having the same dimension k, with orthonormal bases given by the columns of the matrices X and Y respectively. Let $A \in \mathbb{C}^{n \times n}$ be Hermitian, $\mathcal{X}$ be A-invariant and let $\theta \equiv \theta(\mathcal{X}, \mathcal{Y})$ denote the vector of principal angles between the subspaces $\mathcal{X}$ and $\mathcal{Y}$. Then

$$|\lambda(X^H A X) - \lambda(Y^H A Y)| \prec_w \operatorname{spr}(A)(\sin^2\theta + \frac{\sin^4\theta}{2}).$$

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$$|\lambda(X^H AX) - \lambda(Y^H AY)| \prec_w \operatorname{spr}(A)(\sin^2\theta + \frac{\sin^4\theta}{2}).$$

Moreover, if the A-invariant subspace $\mathcal{X}$ corresponds to the set of k largest or smallest eigenvalues of A then

$$|\lambda(X^H AX) - \lambda(Y^H AY)| \prec_w \operatorname{spr}(A)\sin^2\theta.$$

Basic notation

Hermitian matrix $A = A^H \in \mathbb{C}^{n \times n}$

Eigenvalues $\lambda_1(A) \geq \ldots \geq \lambda_n(A)$

$$\mathrm{spr}(A) \equiv \lambda_1(A) - \lambda_n(A)$$

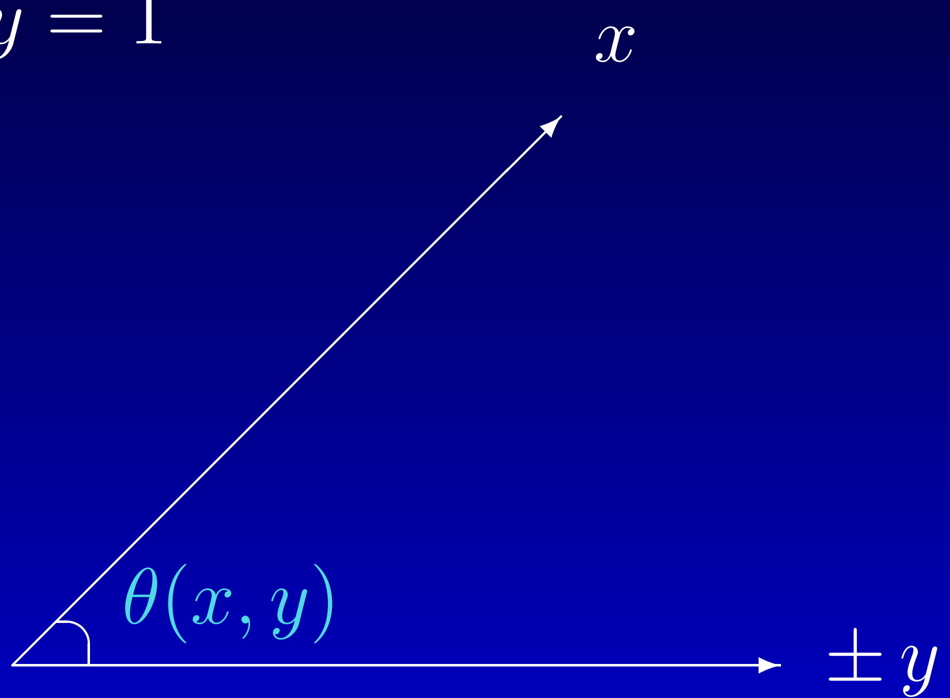
Singular values $\sigma_1(B) \geq \sigma_2(B) \geq \ldots \geq 0$

$$\|B\| \equiv \sigma_1(B)$$

Angles

Acute angle between vectors x and y:

$$x^H x = y^H y = 1$$



$$\cos\theta(x,y) = |x^H y| = \sigma(x^H y)$$

Eigenvalue approximation

Assume $x^H x = y^H y = 1$

Eigenvector x, i.e. $Ax = x.x^H Ax$

y an approximation to x

Rayleigh quotient $y^H Ay \approx x^H Ax$ accuracy?

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Classical *a priori* result: ?Axel Ruhe 1976?

$$|x^H Ax - y^H Ay| \leq \operatorname{spr}(A).\sin^2 \theta(x,y)$$

$$\sin^2 \theta(x,y) = 1 - \cos^2 \theta(x,y) = 1 - |x^H y|^2.$$

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$$\boxed{|x^H Ax - y^H Ay| \leq \mathrm{spr}(A).\sin^2 \theta(x,y)}$$

Generalize this to subspaces $\mathcal{X}$ and $\mathcal{Y}$?

Vector notation

$$\boldsymbol{\lambda}(A) \equiv \begin{bmatrix} \lambda_1(A) \\ \cdot \\ \lambda_n(A) \end{bmatrix}, \qquad \boldsymbol{\sigma}(B) \equiv \begin{bmatrix} \sigma_1(B) \\ \sigma_2(B) \\ \cdot \end{bmatrix},$$

$$\boldsymbol{\lambda}(A) = \boldsymbol{\lambda}^{\downarrow}(A), \qquad \boldsymbol{\sigma}(B) = \boldsymbol{\sigma}^{\downarrow}(B),$$

$\boldsymbol{\lambda}^{\downarrow}$ elements ordered "downwards",

$\boldsymbol{\lambda}^{\uparrow}$ elements ordered "upwards".

Angles between subspaces

$$X^H X = Y^H Y = I_k$$

$$\mathcal{X} = \mathrm{Range}(X), \quad \mathcal{Y} = \mathrm{Range}(Y)$$

$\boldsymbol{\theta}(\mathcal{X}, \mathcal{Y})$ vector of angles between $\mathcal{X}$ & $\mathcal{Y}$

$$\boldsymbol{\theta}(\mathcal{X}, \mathcal{Y}) = \boldsymbol{\theta}^{\downarrow}(\mathcal{X}, \mathcal{Y}) = [\theta_1(\mathcal{X}, \mathcal{Y}), \ldots, \theta_k(\mathcal{X}, \mathcal{Y})]^T$$

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These angles are related to singular values:

$$\begin{aligned} \cos \boldsymbol{\theta}(\mathcal{X}, \mathcal{Y}) &\equiv \boldsymbol{\sigma}^{\uparrow}(X^H Y) \\ &= [\sigma_k(X^H Y), \ldots, \sigma_1(X^H Y)]^T \end{aligned}$$

Eigenproblem for $A = A^H \in \mathbb{C}^{n\times n}$

Assume $\quad X, Y \in \mathbb{C}^{n\times k}, \quad X^H X = Y^H Y = I_k$

Subspaces $\quad \mathcal{X} = \mathrm{Range}(X), \quad \mathcal{Y} = \mathrm{Range}(Y)$

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"Ritz values" $\quad \boldsymbol{\lambda}(Y^H AY)$,

Bound $\quad d \equiv |\boldsymbol{\lambda}(X^H AX) - \boldsymbol{\lambda}(Y^H AY)|$?

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$d^{\downarrow} \leq \mathrm{spr}(A)\sin^2\theta(\mathcal{X},\mathcal{Y})$ is <u>false</u>!

Majorization (weak and strong)

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It has recently been used in the analysis of numerical algorithms.

Majorization

$y \in \mathbb{R}^n$ is weakly majorized by $x \in \mathbb{R}^n$, written $y \prec_w x$,

iff

$$\begin{aligned} y_1^\downarrow &\leq x_1^\downarrow \\ y_1^\downarrow + y_2^\downarrow &\leq x_1^\downarrow + x_2^\downarrow \\ \cdots &\leq \cdots \\ y_1^\downarrow + \cdots + y_n^\downarrow &\leq x_1^\downarrow + \cdots + x_n^\downarrow . \end{aligned}$$

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$y \in \mathbb{R}^n$ is strongly majorized by $x \in \mathbb{R}^n$, written $y \prec x$, if also $\sum_{i=1}^n y_i = \sum_{i=1}^n x_i$.

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The set of all $y \in \mathbb{R}^n$ satisfying $y \prec_w x$ is an unbounded convex set. However if x and y are positive vectors, then we also obtain a convex polygon.

Classical majorization results

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Theorem (Ky-Fan): Let $A, B \in \mathbb{R}^{n \times n}$.
Then $\sigma(A + B) \prec_w \sigma(A) + \sigma(B)$.

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Study the convergence of numerical methods which use Ritz approximations for the symmetric (Hermitian) eigenvalue problem

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Study the convergence of numerical methods which use Ritz approximations for the symmetric (Hermitian) eigenvalue problem, e.g.

- Lanczos algorithm 1950
- block Lanczos algorithm (Golub & Underwood, 1977)

a priori bounds

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- it can help our understanding, and
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Three *a priori* approaches for the (sym.) EVP:

- Classical: angles between eigen-vectors and Ritz vectors.
- Saad: angles between eigen-vectors and the trial (Krylov) subspace $\mathcal{Y}$.
- AKPP: angles between the invariant subspace $\mathcal{X}$, and $\mathcal{Y}$.

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THANK YOU!