Harmonic Rayleigh-Ritz for the multiparameter eigenvalue problem

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Outline

- Multiparameter eigenvalue problem (MEP)
- Jacobi–Davidson type methods for MEP
- Harmonic Rayleigh–Ritz for GEP and MEP
- Numerical examples
- Conclusions

Two-parameter eigenvalue problem

- Two-parameter eigenvalue problem:

$$A_1 x = \lambda B_1 x + \mu C_1 x$$
$$A_2 y = \lambda B_2 y + \mu C_2 y, \qquad \text{(MEP)}$$

where A_i, B_i, C_i are $n \times n$ matrices, $\lambda, \mu \in \mathbb{C}$, $x, y \in \mathbb{C}^n$

- Eigenvalue: a pair (λ, μ) that satisfies (MEP) for nonzero x and y.

- Eigenvector: the tensor product $x \otimes y$.

- **Goal: compute eigenvalues (λ, μ) close to a target (σ, τ) and eigenvectors $x \otimes y$.**

Tensor product approach

$$\begin{aligned} A_1 x &= \lambda B_1 x + \mu C_1 x \\ A_2 y &= \lambda B_2 y + \mu C_2 y \end{aligned} \qquad \text{(MEP)}$$

- On $\mathbb{C}^n \otimes \mathbb{C}^n$ of the dimension n^2 we define

$$\begin{aligned} \Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2 \\ \Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2 \\ \Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2. \end{aligned}$$

- MEP is equivalent to a coupled GEP

$$\begin{aligned} \Delta_1 z &= \lambda \Delta_0 z \\ \Delta_2 z &= \mu \Delta_0 z, \end{aligned} \qquad (\Delta)$$

  where $z = x \otimes y$.

- MEP is nonsingular $\Longleftrightarrow$ Δ_0 is nonsingular.
- $\Delta_0^{-1}\Delta_1$ and $\Delta_0^{-1}\Delta_2$ commute.

Right definite problem

(MEP) $\begin{aligned} A_1 x &= \lambda B_1 x + \mu C_1 x \\ A_2 y &= \lambda B_2 y + \mu C_2 y \end{aligned}$ $\begin{aligned} \Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2 \\ \Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2 \\ \Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2 \end{aligned}$ $\begin{aligned} \Delta_1 z &= \lambda \Delta_0 z \\ \Delta_2 z &= \mu \Delta_0 z \end{aligned}$ (Δ)

MEP is right definite when A_i, B_i, C_i are Hermitian and Δ_0 is positive definite.

Atkinson (1972):

Δ_0 positive definite $\Longleftrightarrow (x \otimes y)^* \Delta_0 (x \otimes y) = \begin{vmatrix} x^* B_1 x & x^* C_1 x \\ y^* B_2 y & y^* C_2 y \end{vmatrix} > 0$ for $x, y \neq 0$.

Right definite problem

(MEP) $$\begin{aligned} A_1x &= \lambda B_1x + \mu C_1x \\ A_2y &= \lambda B_2y + \mu C_2y \end{aligned}$$ $$\begin{aligned} \Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2 \\ \Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2 \\ \Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2 \end{aligned}$$ $$\begin{aligned} \Delta_1 z &= \lambda \Delta_0 z \\ \Delta_2 z &= \mu \Delta_0 z \end{aligned}$$ (Δ)

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If MEP is right definite then

- eigenpairs are real
- there exist n^2 linearly independent eigenvectors
- eigenvectors of distinct eigenvalues are Δ_0-orthogonal, i.e., $(x_1 \otimes y_1)^T \Delta_0 (x_2 \otimes y_2) = 0$

Numerical methods

First option: standard algorithms for explicitly computed matrices Δ:

$$(\text{MEP}) \quad \begin{aligned} A_1 x &= \lambda B_1 x + \mu C_1 x \\ A_2 y &= \lambda B_2 y + \mu C_2 y \end{aligned} \qquad \begin{aligned} \Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2 \\ \Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2 \\ \Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2 \end{aligned} \qquad \begin{aligned} \Delta_1 z &= \lambda \Delta_0 z \\ \Delta_2 z &= \mu \Delta_0 z \end{aligned} \quad (\Delta)$$

Algorithms that work with matrices A_i, B_i, C_i:

- Blum, Curtis, Geltner (1978), and Browne, Sleeman (1982): gradient method,
- Bohte (1980): Newton's method for eigenvalues,
- Ji, Jiang, Lee (1992): Generalized Rayleigh Quotient Iteration.
- Continuation method:
  - Shimasaki (1995): for a special class of RD problems,
  - P. (1999): for RD problems, Tensor Rayleigh Quotient Iteration,
  - P. (2000): for weakly elliptic problems.
- Jacobi-Davidson type methods.
  - Hochstenbach, P. (2002): for RD problems,
  - Hochstenbach, Košir, P. (2005): for general nonsingular MEP,
  - Hochstenbach, P. (2007): JD with harmonic extraction.

Jacobi–Davidson method

Subspace methods compute eigenpairs from low dimensional subspaces. They work as follows:

- Extraction: We compute an approximation to an eigenpair from a given search subspace. Usually, we solve the same type of eigenvalue problem as the original one, but of a smaller dimension.
- Expansion: After each step we expand the subspace by a new direction.

As the search subspace grows the eigenpair approximations should converge to an eigenpair.

Jacobi–Davidson method is a subspace method where:

- a new direction to the subspace is orthogonal or oblique to the last chosen Ritz vector,
- approximate solutions of certain correction equations are used for expansion.

JD method can be efficiently generalized for two-parameter eigenvalue problems, while this is not clear for subspace methods based on Krylov subspaces.

JD-like method for the right definite case: extraction

Ritz–Galerkin conditions: search spaces = test spaces: $u \in \mathcal{U}_k$, $v \in \mathcal{V}_k$

$$(A_1 - \sigma B_1 - \tau C_1)u \perp \mathcal{U}_k$$
$$(A_2 - \sigma B_2 - \tau C_2)v \perp \mathcal{V}_k$$

$\Rightarrow$ projected right definite two-parameter eigenvalue problem

$$U_k^T A_1 U_k c = \sigma U_k^T B_1 U_k c + \tau U_k^T C_1 U_k c$$
$$V_k^T A_2 V_k d = \sigma V_k^T B_2 V_k d + \tau V_k^T C_2 V_k d$$

Ritz vectors: $u = U_k c$, $v = V_k d$ for $c, d \in \mathbb{R}^k$

Ritz value: (σ, τ), Ritz pair: $((\sigma, \tau), u \otimes v)$

Will not discuss the correction equation and the deflation.

Works well for exterior eigenvalues.

Two-sided JD-like method for a general problem: extraction

Petrov–Galerkin conditions: search spaces $u_i \in \mathcal{U}_{ik}$, test spaces $v_i \in \mathcal{V}_{ik}$

$$\begin{aligned}(A_1 - \sigma B_1 - \tau C_1)u_1 &\perp \mathcal{V}_{1k}\\ (A_2 - \sigma B_2 - \tau C_2)u_2 &\perp \mathcal{V}_{2k},\end{aligned}$$

$\Rightarrow$ projected two-parameter eigenvalue problem

$$\begin{aligned}V_{1k}^* A_1 U_{1k} c_1 &= \sigma V_{1k}^* B_1 U_{1k} c_1 + \tau V_{1k}^* C_1 U_{1k} c_1\\ V_{2k}^* A_2 U_{2k} c_2 &= \sigma V_{2k}^* B_2 U_{2k} c_2 + \tau V_{2k}^* C_2 U_{2k} c_2,\end{aligned}$$

where $u_i = U_{ik} c_i \neq 0$ for $i = 1, 2$ and $\sigma, \tau \in \mathbb{C}$.

Petrov vectors: $u_i = U_{ik} c_i$, $v_i = V_{ik} d_i$, $c_i, d_i \in \mathbb{C}^k$

Petrov value: (σ, τ), Petrov triple: $((\sigma, \tau), u_1 \otimes u_2, v_1 \otimes v_2)$

Usually performs better than the one-sided method.

Works well for exterior eigenvalues, is less favorable for interior ones.

Rayleigh–Ritz for GEP

For a GEP $Ax = \lambda Bx$ we want an approximate eigenpair (θ, u), where u is in a given search subspace $\mathcal{U}_k$ and θ is close to the given target $\tau \in \mathbb{C}$. Standard Ritz–Galerkin condition

$$Au - \theta Bu \perp \mathcal{U}_k$$

leads to

$$U_k^* A U_k c = \theta\, U_k^* B U_k c,$$

where the columns of U_k form an orthonormal basis for $\mathcal{U}_k$ and $c \in \mathbb{C}^k$.

Rayleigh–Ritz for GEP

For a GEP $Ax = \lambda Bx$ we want an approximate eigenpair (θ, u), where u is in a given search subspace $\mathcal{U}_k$ and θ is close to the given target $\tau \in \mathbb{C}$. Standard Ritz–Galerkin condition

$$r := Au - \theta Bu \perp \mathcal{U}_k$$

leads to

$$U_k^* A U_k c = \theta\, U_k^* B U_k c,$$

where the columns of U_k form an orthonormal basis for $\mathcal{U}_k$ and $c \in \mathbb{C}^k$.

For interior eigenvalues, even for a Ritz value $\theta \approx \tau$, $\|r\|$ can be large and the approximate eigenvector may be poor. As a remedy, the harmonic Rayleigh–Ritz was proposed:

- standard eigenproblem: Morgan (1991), Paige, Parlett, Van der Vorst (1995),
- GEP: Fokkema, Sleijpen, Van der Vorst (1998), Stewart (2001).

Assuming $A - \tau B$ is nonsingular we consider a spectral transformation

$$(A - \tau B)^{-1} Bx = (\lambda - \tau)^{-1} x.$$

The interior eigenvalues $\lambda \approx \tau$ are exterior eigenvalues of $(A - \tau B)^{-1} B$.

Harmonic Rayleigh–Ritz for GEP

To avoid working with $(A-\tau B)^{-1}$ we impose a Petrov–Galerkin condition

$$(A-\tau B)^{-1}Bu-(\theta-\tau)^{-1}u \perp (A-\tau B)^*(A-\tau B)\,\mathcal{U}_k,$$

or, equivalently,

$$Au-\theta Bu=(A-\tau B)u-(\theta-\tau)Bu \perp (A-\tau B)\,\mathcal{U}_k,$$

leading to the projected eigenproblem

$$U_k^*(A-\tau B)^*(A-\tau B)U_kc=(\theta-\tau)\,U_k^*(A-\tau B)^*BU_kc.$$

This approach has two motivations:

- it retrieves exact eigenvectors in the search space;
- a harmonic Ritz pair (θ, u) satisfies (Stewart (2001))

$$\|Au-\tau Bu\| \le |\theta-\tau|\cdot\|Bu\| \le |\theta-\tau|\cdot\|BU_k\|.$$

Harmonic Rayleigh–Ritz for two-parameter eigenvalue problem

GEP: $Ax = \lambda Bx$

subspace is $\mathcal{U}_k$, target is τ

MEP:

$$A_1x = \lambda B_1x + \mu C_1x$$
$$A_2y = \lambda B_2y + \mu C_2y$$

subspace is $\mathcal{U}_k \otimes \mathcal{V}_k$, target is (σ, τ)

Harmonic Rayleigh–Ritz for two-parameter eigenvalue problem

GEP: $Ax = \lambda Bx$

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Rayleigh–Ritz: $Au - \theta Bu \perp \mathcal{U}_k$

MEP:
$$\begin{aligned} A_1 x &= \lambda B_1 x + \mu C_1 x \\ A_2 y &= \lambda B_2 y + \mu C_2 y \end{aligned}$$

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subspace is $\mathcal{U}_k$, target is τ

Rayleigh–Ritz: $Au - \theta Bu \perp \mathcal{U}_k$

MEP:
$$
\begin{aligned}
A_1 x &= \lambda B_1 x + \mu C_1 x \\
A_2 y &= \lambda B_2 y + \mu C_2 y
\end{aligned}
$$

subspace is $\mathcal{U}_k \otimes \mathcal{V}_k$, target is (σ, τ)

Rayleigh–Ritz:
$$
\begin{aligned}
(A_1 - \theta B_1 - \eta C_1)\, u &\perp \mathcal{U}_k \\
(A_2 - \theta B_2 - \eta C_2)\, v &\perp \mathcal{V}_k
\end{aligned}
$$

Harmonic Rayleigh–Ritz for two-parameter eigenvalue problem

GEP: $Ax = \lambda Bx$

subspace is $\mathcal{U}_k$, target is τ

Rayleigh–Ritz: $Au - \theta Bu \perp \mathcal{U}_k$

Spectral transformation: $(A - \tau B)^{-1}Bx = (\lambda - \tau)^{-1}x$

MEP:
$$\begin{aligned} A_1x &= \lambda B_1x + \mu C_1x \\ A_2y &= \lambda B_2y + \mu C_2y \end{aligned}$$

subspace is $\mathcal{U}_k \otimes \mathcal{V}_k$, target is (σ, τ)

Rayleigh–Ritz:
$$\begin{aligned} (A_1 - \theta B_1 - \eta C_1)\, u \quad &\perp \quad \mathcal{U}_k \\ (A_2 - \theta B_2 - \eta C_2)\, v \quad &\perp \quad \mathcal{V}_k \end{aligned}$$

Harmonic Rayleigh–Ritz for two-parameter eigenvalue problem

GEP: $Ax = \lambda Bx$

subspace is $\mathcal{U}_k$, target is τ

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Rayleigh–Ritz:
$$\begin{aligned} (A_1 - \theta B_1 - \eta C_1)\, u \quad &\perp \quad \mathcal{U}_k \\ (A_2 - \theta B_2 - \eta C_2)\, v \quad &\perp \quad \mathcal{V}_k \end{aligned}$$

Spectral transformation: ? ? ?

Harmonic Rayleigh–Ritz for two-parameter eigenvalue problem

GEP: $Ax = \lambda Bx$

subspace is $\mathcal{U}_k$, target is τ

Rayleigh–Ritz: $Au - \theta Bu \perp \mathcal{U}_k$

Spectral transformation: $(A - \tau B)^{-1}Bx = (\lambda - \tau)^{-1}x$

Harmonic Rayleigh–Ritz: $Au - \theta Bu \perp (A - \tau B)\,\mathcal{U}_k$

MEP:

$$\begin{aligned} A_1x &= \lambda B_1x + \mu C_1x \\ A_2y &= \lambda B_2y + \mu C_2y \end{aligned}$$

subspace is $\mathcal{U}_k \otimes \mathcal{V}_k$, target is (σ, τ)

Rayleigh–Ritz:

$$\begin{aligned} (A_1 - \theta B_1 - \eta C_1)\,u &\perp \mathcal{U}_k \\ (A_2 - \theta B_2 - \eta C_2)\,v &\perp \mathcal{V}_k \end{aligned}$$

Spectral transformation: ? ? ?

Harmonic Rayleigh–Ritz for two-parameter eigenvalue problem

GEP: $Ax = \lambda Bx$

subspace is $\mathcal{U}_k$, target is τ

Rayleigh–Ritz: $Au - \theta Bu \perp \mathcal{U}_k$

Spectral transformation: $(A - \tau B)^{-1} Bx = (\lambda - \tau)^{-1} x$

Harmonic Rayleigh–Ritz: $Au - \theta Bu \perp (A - \tau B)\,\mathcal{U}_k$

MEP:
$$\begin{aligned} A_1 x &= \lambda B_1 x + \mu C_1 x \\ A_2 y &= \lambda B_2 y + \mu C_2 y \end{aligned}$$

subspace is $\mathcal{U}_k \otimes \mathcal{V}_k$, target is (σ, τ)

Rayleigh–Ritz:
$$\begin{aligned} (A_1 - \theta B_1 - \eta C_1)\, u &\perp \mathcal{U}_k \\ (A_2 - \theta B_2 - \eta C_2)\, v &\perp \mathcal{V}_k \end{aligned}$$

Spectral transformation: ? ? ?

Harmonic Rayleigh–Ritz:
$$\begin{aligned} (A_1 - \theta B_1 - \eta C_1)\, u &\perp (A_1 - \sigma B_1 - \tau C_1)\, \mathcal{U}_k \\ (A_2 - \theta B_2 - \eta C_2)\, v &\perp (A_2 - \sigma B_2 - \tau C_2)\, \mathcal{V}_k \end{aligned}$$

Details

$$\begin{array}{rcl}(A_1 - \theta B_1 - \eta C_1)\, u & \perp & (A_1 - \sigma B_1 - \tau C_1)\, \mathcal{U}_k \\ (A_2 - \theta B_2 - \eta C_2)\, v & \perp & (A_2 - \sigma B_2 - \tau C_2)\, \mathcal{V}_k\end{array}$$

We call this the harmonic Rayleigh–Ritz extraction for the MEP.

If we compute reduced QR-decompositions

$$(A_1 - \sigma B_1 - \tau C_1)\, U_k = Q_1 R_1, \qquad (A_2 - \sigma B_2 - \tau C_2)\, V_k = Q_2 R_2,$$

then in the extraction we have to solve the projected two-parameter eigenproblem

$$\begin{array}{rcl}R_1\, c & = & (\theta - \sigma)\, Q_1^* B_1 U_k c + (\eta - \tau)\, Q_1^* C_1 U_k c, \\ R_2\, d & = & (\theta - \sigma)\, Q_2^* B_2 V_k d + (\eta - \tau)\, Q_2^* C_2 V_k d.\end{array}$$

We take the eigenpair $((\theta - \sigma, \eta - \tau),\ c \otimes d)$ with minimal $|\theta - \sigma|^2 + |\eta - \tau|^2$.

Motivation

As for the GEP, there are two justifications for the harmonic approach for the MEP:

- upper bounds for the residual norms:

$$\begin{aligned} \|(A_1 - \sigma B_1 - \tau C_1)\, u\| &\leq |\theta - \sigma|\, \|B_1 U_k\| + |\eta - \tau|\, \|C_1 U_k\| \\ \|(A_2 - \sigma B_2 - \tau C_2)\, v\| &\leq |\theta - \sigma|\, \|B_2 V_k\| + |\eta - \tau|\, \|C_2 V_k\| \end{aligned}$$

- if the search space contains an eigenvector, $x = U_k c$, $y = V_k d$, then $((\lambda, \mu), x \otimes y)$ satisfies the harmonic Rayleigh–Ritz equation.

JD-like method with harmonic Rayleigh–Ritz for MEP

1. $s = u_1$ and $t = v_1$ (starting vectors), $U_0 = V_0 = [\ ]$

   for $k = 1, 2, \ldots$

2. $(U_{k-1}, s) \rightarrow U_k, \quad (V_{k-1}, t) \rightarrow V_k$

   Update R_1, R_2, Q_1, and Q_2.

3. Extract appropriate harmonic Ritz pair $((\xi_1, \xi_2), c \otimes d)$ of

$$\begin{aligned} R_1\, c &= \xi_1\, Q_1^* B_1 U_k c + \xi_2\, Q_1^* C_1 U_k c \\ R_2\, d &= \xi_1\, Q_2^* B_2 V_k d + \xi_2\, Q_2^* C_2 V_k d. \end{aligned}$$

4. Take $u = U_k c$, $v = V_k d$ and compute tensor Rayleigh quotient

$$\begin{aligned} \theta &= \frac{(u \otimes v)^* \Delta_1 (u \otimes v)}{(u \otimes v)^* \Delta_0 (u \otimes v)} = \frac{(u^* A_1 u)(v^* C_2 v) - (u^* C_1 u)(v^* A_2 v)}{(u^* B_1 u)(v^* C_2 v) - (u^* C_1 u)(v^* B_2 v)} \\ \eta &= \frac{(u \otimes v)^* \Delta_2 (u \otimes v)}{(u \otimes v)^* \Delta_0 (u \otimes v)} = \frac{(u^* B_1 u)(v^* A_2 v) - (u^* A_1 u)(v^* B_2 v)}{(u^* B_1 u)(v^* C_2 v) - (u^* C_1 u)(v^* B_2 v)} \end{aligned}$$

5. $$\begin{aligned} r_1 &= (A_1 - \theta B_1 - \eta C_1) u \\ r_2 &= (A_2 - \theta B_2 - \eta C_2) v \end{aligned}$$

6. Stop if $(\|r_1\|^2 + \|r_2\|^2)^{1/2} \leq \varepsilon$

7. Solve (approximately) an $s \perp u, t \perp v$ from corr. equation(s)

Saad type theorems

$$A_1 x = \lambda B_1 x + \mu C_1 x \qquad A_2 y = \lambda B_2 y + \mu C_2 y$$

$$\Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2 \qquad \Delta_1 = A_1 \otimes C_2 - C_1 \otimes A_2 \qquad \Delta_2 = B_1 \otimes A_2 - A_1 \otimes B_2$$

$$\Delta_1 z = \lambda \Delta_0 z \qquad \Delta_2 z = \mu \Delta_0 z$$

Let $w := u \otimes v$ be a Ritz vector corresponding to Ritz value (θ, η), and $[w \;\; W_1 \;\; W_2]$ be an orthonormal basis for $\mathbb{C}^{n^2}$ such that $\mathrm{span}([w \;\; W_1]) = \mathcal{U}_k \otimes \mathcal{V}_k$. We define $E_i = [w \;\; W_1]^* \Delta_i [w \;\; W_1]$ for $i = 0, 1, 2$ and assume that E_0 is invertible.

The components θ_j and η_j of the Ritz values (θ_j, η_j) are eigenvalues of the commuting matrices $E_0^{-1} E_1$ and $E_0^{-1} E_2$, respectively. From

$$(E_1 - \theta E_0)\, e_1 = 0 \quad \text{and} \quad (E_2 - \eta E_0)\, e_1 = 0$$

it follows that

$$E_0^{-1} E_1 = \begin{bmatrix} \theta & f_1^* \\ 0 & G_1 \end{bmatrix} \quad \text{and} \quad E_0^{-1} E_2 = \begin{bmatrix} \eta & f_2^* \\ 0 & G_2 \end{bmatrix},$$

where the eigenvalues of commuting matrices G_1 and G_2 form the remaining $k^2 - 1$ Ritz values $(\theta_j, \eta_j) \neq (\theta, \eta)$.

Theorem for the standard extraction

Let $((\theta, \eta),\ u \otimes v)$ be a Ritz pair and $((\lambda, \mu),\ x \otimes y)$ an eigenpair.
Let $E_i = [w\ \ W_1]^* \Delta_i [w\ \ W_1]$ for $i = 0, 1, 2$ and assume E_0^{-1} is invertible. Then

$$\varphi(\sin(u,x),\, \sin(v,y)) \le \left(1 + \frac{\gamma^2}{\delta^2}\right) \cdot \varphi(\sin(\mathcal{U}_k, x),\, \sin(\mathcal{V}_k, y)),$$

where

$$\begin{aligned}
\varphi(a,b) &= a^2 + b^2 - a^2 b^2, \\
\gamma &= \|E_0^{-1}\| \Big(\|P(\Delta_1 - \lambda\Delta_0)(I-P)\|^2 + \|P(\Delta_2 - \mu\Delta_0)(I-P)\|^2 \Big)^{1/2}, \\
\delta &= \sigma_{\min}\left(\begin{bmatrix} G_1 - \lambda I \\ G_2 - \mu I \end{bmatrix} \right),
\end{aligned}$$

and P is the orthogonal projection onto $\mathcal{U}_k \otimes \mathcal{V}_k$.

Saad type theorem for the harmonic extraction

Let

$$\begin{aligned} \widetilde{A}_i &= (A_i - \sigma B_i - \tau C_i)^* A_i, & \widetilde{\Delta}_0 = \widetilde{B}_1 \otimes \widetilde{C}_2 - \widetilde{C}_1 \otimes \widetilde{B}_2, \\ \widetilde{B}_i &= (A_i - \sigma B_i - \tau C_i)^* B_i, & \widetilde{\Delta}_1 = \widetilde{A}_1 \otimes \widetilde{C}_2 - \widetilde{C}_1 \otimes \widetilde{A}_2, \\ \widetilde{C}_i &= (A_i - \sigma B_i - \tau C_i)^* C_i, & \widetilde{\Delta}_2 = \widetilde{B}_1 \otimes \widetilde{A}_2 - \widetilde{A}_1 \otimes \widetilde{B}_2. \end{aligned}$$

Let $\widetilde{w} := \widetilde{u} \otimes \widetilde{v}$ be a harmonic Ritz vector corresponding to the harmonic Ritz value $(\widetilde{\theta}, \widetilde{\eta})$, and let $[\widetilde{w}\ \ \widetilde{W}_1\ \ \widetilde{W}_2]$ be an orthonormal basis for $\mathbb{C}^{n^2}$ such that $\mathrm{span}([\widetilde{w}\ \ \widetilde{W}_1]) = \mathcal{U}_k \otimes \mathcal{V}_k$.

If $\widetilde{E}_i = [\widetilde{w}\ \ \widetilde{W}_1]^* \widetilde{\Delta}_i [\widetilde{w}\ \ \widetilde{W}_1]$ then $\widetilde{E}_0^{-1}\widetilde{E}_1 = \begin{bmatrix} \widetilde{\theta} & \widetilde{f}_1^* \\ 0 & \widetilde{G}_1 \end{bmatrix}$ and $\widetilde{E}_0^{-1}\widetilde{E}_2 = \begin{bmatrix} \widetilde{\eta} & \widetilde{f}_2^* \\ 0 & \widetilde{G}_2 \end{bmatrix}$.

Analogous to the previous theorem:

$$\varphi(\sin(\widetilde{u}, x),\, \sin(\widetilde{v}, y)) \le \left(1 + \frac{\widetilde{\gamma}^2}{\widetilde{\delta}^2}\right) \cdot \varphi(\sin(\mathcal{U}_k, x),\, \sin(\mathcal{V}_k, y)),$$

where $\widetilde{\gamma}$ and $\widetilde{\delta}$ are defined analogous to the previous theorem.

Numerical example: a right definite case

$n = 1000.$

We compute 100 eigenvalues closest to the origin.

Maximum dimension of the search space before restart is 14.

	Ritz extraction				Harmonic Ritz extraction			
GMRES	iter	time	in 50	in 100	iter	time	in 50	in 100
8	876	144	49	96	346	98	50	97
16	347	80	50	95	127	50	49	85
32	277	89	50	93	149	86	50	91
64	276	105	50	91	137	77	47	75

The values in the above table are:

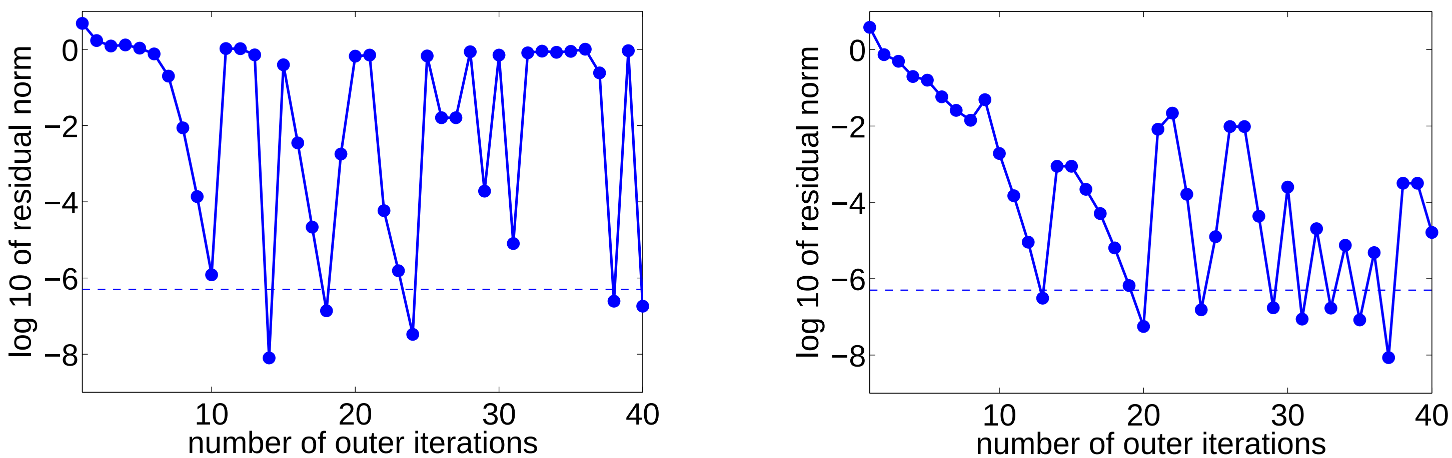
- *iter*: the number of outer iterations,
- *time*: time in seconds,
- *in 50, in 100*: the number of the computed eigenvalues that are among the 50 and 100 closest eigenvalues to the target, respectively.

Numerical example: a general case

Non right definite problem, $n = 1000$. We want to compute 50 eigenvalues closest to the origin using at most 2500 outer iterations. Maximum dimension of the search space is 14.

	One-sided Ritz			Two-sided Ritz			Harmonic Ritz				
GMRES	eigs	in 10	in 30	eigs	in 10	in 30	iter	time	in 10	in 30	in 50
8	17	10	17	12	9	12	226	119	10	30	46
16	19	10	19	19	10	19	106	73	10	30	44
32	20	10	20	22	10	22	89	87	10	29	40
64	22	10	22	30	10	29	93	118	10	28	40

The convergence graphs for the two-sided Ritz extraction (left) and the harmonic extraction (right) for the first 40 outer iterations using 8 GMRES steps in the inner iteration.



Conclusions

Although there seems to be no straightforward generalization of a spectral transformation for the MEP, the harmonic approach can be generalized to the MEP together with Saad's theorem.

Our recommendations for the numerical computation:

- interior eigenvalues: one-sided JD combined with harmonic Ritz extraction,
- exterior eigenvalues: standard Ritz extraction and one-sided JD (for right definite problems) or two-sided JD (for general problems).

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