A Hybrid Kaczmarz - CG Algorithm for Inconsistent Systems arising in Image Reconstruction

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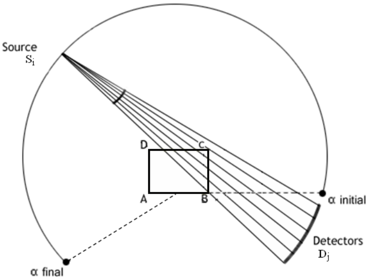
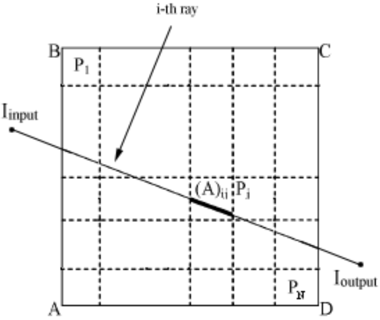
CT fan beam scanning procedure





Fan beam scanning **Construction of A and b**

A: an $M \times N$ matrix; b: measurements of I_{input} and I_{output}

$M =$ number of sources $\times$ number of detectors (left figure)

$N =$ number of pixels (right figure)

Mathematical formulation: $Ax = b, \ x = (x_1, \ldots, x_N)^T$;

$x_j =$ "value of the attenuation function on P_j"

Properties of A and b

- $A : M \times N$ is rectangular, big, sparse, rank-deficient, $\exists!\ x_{LS}$
- in practice $b \notin R(A) \Rightarrow$, i.e. $Ax = b$ is inconsistent $\Longrightarrow$

$$\| Ax - b \| = \min\{\| Az - b \|, z \in \boldsymbol{R}^N\} \qquad (*)$$

- Tikhonov-like regularization

$$\| Ax - b \|^2 + \delta^2 \| Lx \|^2 = \min! \Leftrightarrow \left\| \begin{bmatrix} A \\ \delta L \end{bmatrix} x - \begin{bmatrix} b \\ 0 \end{bmatrix} \right\|^2 = \min!$$

still inconsistent (!), but $x_{LS}(L;\delta) \approx x_{LS}$ for an appropriate choice of L and δ

Motivation - 1

- 2005-2006: common research on image reconstruction with medical applications with colleagues from Technical Faculty of University Erlangen - Nürnberg (prof. Ulrich Rüde, prof. Joachim Hornegger, dr. Harald Köstler and dr. Marcus Prümmer + some master and Ph.D. students)
- 2D and 3D images provided by Siemens Medical Solutions Dept. Erlangen
- during 2006: a master thesis in Constanta (Tiberius Duluman; implementation of algorithms)
- Data for a common 3D rec. probl. : $N = 256^3 = 16.777.216$, $M = 150 \mathrm{sources} \times 1024^2 \mathrm{detectors} = 157.286.400$; the matrix A cannot be any more stored in the computer memory (not even in a compressed form !); an alternative - storing A on the disk and reading it (and/or A^T) in each iteration $\Rightarrow$ too much time consuming !

Motivation - 2

Three requests have been formulated:

- R_{regen}: A has to be regenerated row-by-row in each iteration; the computations in the iterations of the algorithm have to be "compatible" with such a procedure
- R_{eff}: the algorithm has to produce an efficient reconstruction
- R_{inc}: the algorithm has to be convergent in the inconsistent case

Candidates: Kaczmarz Extended (KE) and Generalized CG (CG)

The Generalized CG algorithm

Genaralized Conjugate Gradient (CG(A; b; ·))

Let $x^0 \in \boldsymbol{R}^n$, $r^0 = b - Ax^0$; for $k = 0, 1, \ldots$ do

$$\begin{aligned}
r^0 &= b - Ax^0, \\
p^0 &= A^T r^0, \\
\alpha_k &= \parallel A^T r^k \parallel^2 / \parallel Ap^k \parallel^2, \\
x^{k+1} &= x^k + \alpha_k p^k, \\
r^{k+1} &= b - Ax^{k+1} = r^k - \alpha_k Ap^k, \\
\beta_k &= \parallel A^T r_{k+1} \parallel^2 / \parallel A^T r^k \parallel^2, \\
p^{k+1} &= A^T r^{k+1} + \beta_k p^k.
\end{aligned} \tag{1}$$

Convergence: Kammerer and Nashed, SIAM N.A., 9(1972), 165-181

Theorem

For any $x^0 \in \boldsymbol{R}^N$ the above sequence $(x^k)_{k \geq 0}$ converges to an element of LSS(A; b). If $x^0 = 0$, then $\lim_{k \to \infty} x^k = x_{LS}$.

Kaczmarz Extended

Let $x^0 \in \boldsymbol{R}^N, y^0 = b$; for $k = 0, 1, \ldots$ do

$$\begin{aligned} y^{k+1} &= (\varphi_1 \circ \cdots \circ \varphi_N)(y^k) \\ b^{k+1} &= b - y^{k+1} \\ x^{k+1} &= (f_1 \circ \cdots \circ f_M)(b^{k+1}; x^k), \end{aligned}$$

$$\varphi_j(y) = y - \frac{\langle y, A^j \rangle}{\|A^j\|^2} A^j, \quad f_i(\beta; x) = x - \frac{\langle x, A_i \rangle - \beta_i}{\|A_i\|^2} A_i$$

Convergence: C. Popa, IJCM 55(1995), 77-89

Theorem

Suppose that $A_i \neq 0$, $A^j \neq 0, \forall i, j$. Then, for any $x^0 \in \boldsymbol{R}^N$ the above sequence $(x^k)_{k \geq 0}$ converges to an element of LSS(A; b). If $x^0 = 0$, then $\lim_{k \to \infty} x^k = x_{LS}$.

- KE - (very) good results in image reconstruction (belongs to the class of ART methods)
- CG - poor results in image reconstruction; **comments by prof. Gabor T. Herman (personal communication)**: *The CG works as advertised: it gets very near to the stated minimizer in an often surprisingly few iterations. However, the results tend to be worse (from the image reconstruction from projections point of view) than those produced by ART. The problem is that the (even regularized) normal equations do not properly capture what seems to be important for image reconstruction; as opposed to this, early iterative steps of ART seem to do that. This is due to some magic that I have never succeeded to understand.*

CG versus KE: computational effort

- for (very) big dimensions M and N, the matrix A has to be regenerated (row-by-row) in each CG or KE iteration
- CG and KE use in each iteration beside A, A^T but differently
  1. for CG: compute $v = A^T y$ by

     $v = 0$
     for $i = 1 : M$
       generate the $i -$ th row A_i
       $u_i = \langle x, A_i \rangle$
       $v = v + y_i \ A_i$
     endfor
  2. for KE: for $j = 1 : N$
       $aux = \frac{\langle y, A^j \rangle}{\|A^j\|^2}$
       $y = y - \ aux \cdot A^j$
     endfor

  Unfortunately, formulas (2) request that we regenerate **all the** M rows of the matrix A for **each** $j = 1, \ldots, N$ in **each** KE iteration, which is a huge computational effort !!!

Some experiments

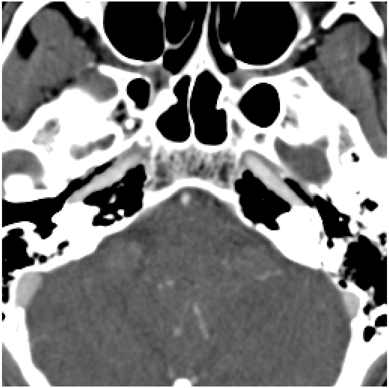


Figure: Image x^{exact} (1024×1024)

Image provided by J. Hornegger and M. Prümmer, University of Erlangen-Nürnberg, Germany; tests on Intel P IV, 3.00GHz, 1GB RAM

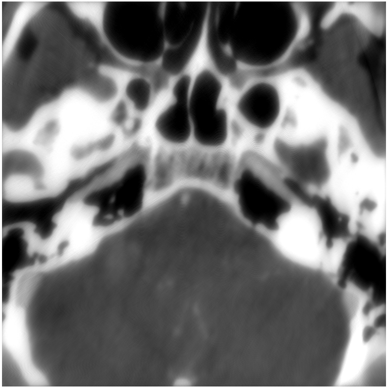
Implementation details

- For CG: the rows of A have been generated 2 times in each iteration
- For KE: A and A^T ahve been stored in a compressed form on the disk and then read in each iteration
- The total execution time has been measured
- simulation (consistent case): $b = Ax^{exact}$, then $Ax = b$ was solved for x_{LS}

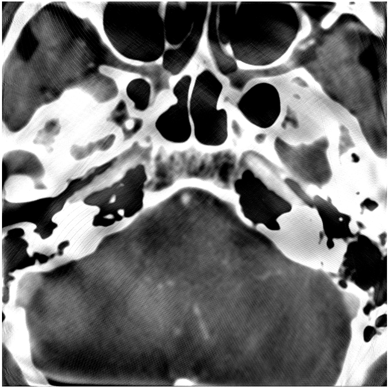
Consistent case - underdetermined

490 sources $\times$ 2048 detectors, 10 iterations

M = 490 x 2048 = 1.003.520, N = 1024 x 1024 = 1.048.576



CG - Time: 55 min.

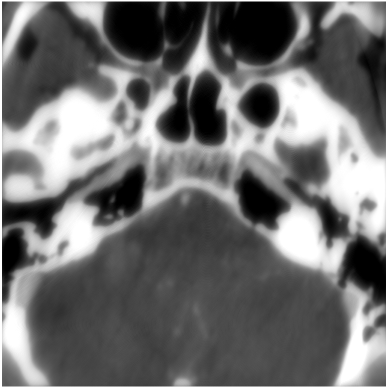


KE - Time: 625 min.

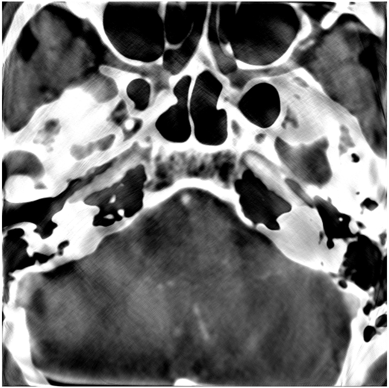
Consistent case - overdetermined

1140 sources $\times$ 2048 detectors, 10 iterations

M = 1140 x 2048 = 2.334.720, N = 1024 x 1024 = 1.048.576



CG - Time: 130 min.



KE - Time: 1155 min.

Replay

KE and CG versus requests			
Algorithm	R_{regen}	R_{eff}	R_{inc}
KE	N	Y	Y
CG	Y	N	Y
KE+CG=KECG	Y	Y	Y

The KECG algorithm

Algorithm KECG Let $x^0 \in \boldsymbol{R}^N, y^0 = b$; for $k = 0, 1, \ldots$ do

$$
\begin{aligned}
y^{k+1} &= CG(A^T; 0; y^k) \quad (**) \\
b^{k+1} &= b - y^{k+1} \\
x^{k+1} &= (f_1 \circ \cdots \circ f_M)(b^{k+1}; x^k)
\end{aligned}
$$

Note. (**) means CG applied to $A^T y = 0$, with $y^0 = b$ which gives $\lim_{k \to \infty} y^k = P_{N(A)^T}(b))$.

Theorem

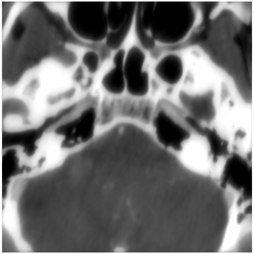
(C. Popa, 2007) *Suppose that $A_i \neq 0, i = 1, \ldots, M$. Then, for $x^0 = 0$ the sequence $(x^k)_{k \geq 0}$ generated with the KECG algorithm converges to x_{LS}.*

Consistent case; underdetermined)

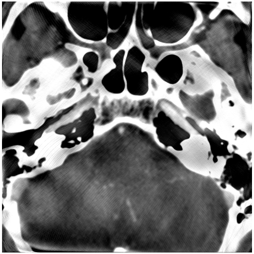
490 sources $\times$ 2048 detectors, 10 iterations

for KECG the rows of A have been regenerated 3 times per iteration

M = 490 x 2048 = 1.003.520, N = 1024 x 1024 = 1.048.576



CG - 55 min.



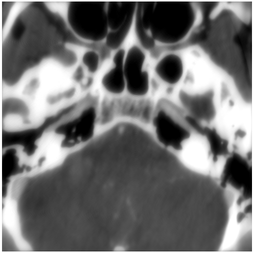
KE - 625 min.

KECG - 97 min.

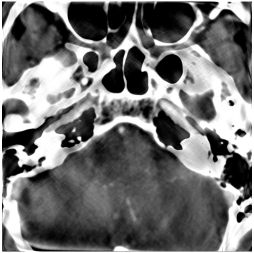
Consistent case overdetermined)

1140 sources $\times$ 2048 detectors, 10 iterations

M = 1140 x 2048 = 2.334.720, N = 1024 x 1024 = 1.048.576



CG - 130 min.



KE - 1155 min.

KECG - 225 min.

Inconsistent case - Perturbation

The consistent system $b^0 = Ax^{exact}$ was perturbed as

$$b = b^0 + pert, \quad pert = 10\% \cdot \parallel b^0 \parallel \cdot rand,$$

with *rand* a randomly generated vector with unitary norm. Then, the least squares problem $\parallel Ax - b \parallel = \min!$ was solved by regularization

$$\parallel Ax - b \parallel^2 + \delta^2 \parallel Lx \parallel^2 = \min! \Leftrightarrow \left\| \begin{bmatrix} A \\ \delta L \end{bmatrix} x - \begin{bmatrix} b \\ 0 \end{bmatrix} \right\|^2 = \min!$$

Inconsistent case - regularization

For each $i \in 1 \dots n$, let H_i be the set of horizontally neighbour pixels of i, V_i the set of vertically neighbour pixels and D_i the set of diagonally neighbour pixels of i. For each $j \in 1 \dots n$

$$(L)_{ij} = \begin{cases} (L)_{ij} = w_h, & \text{if } j \in H_i \\ (L)_{ij} = w_v, & \text{if } j \in V_i \\ (L)_{ij} = w_d, & \text{if } j \in D_i \\ \sum_{k=1}^{n} |(L)_{ik}|, & \text{if } j = i \text{ and } k \neq i \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Note. In our experiments we used $w_h = -1, w_v = -1, w_d = -1/\sqrt{2}$ and $\delta^2 = 20$; the matrix L from (2) is positive semidefinite.

Inconsistent + regularization

10 KECG iterations

Underdetermined: 103 min.

Overdetermined: 237 min.

Acknoledgements, conclusions and future work

Acknoledgements. The computer programmes for the numerical experiments have been made by my Master student in Constanta, Tiberius Duluman.

Conclusions. We designed a hybrid algorithm - KECG - which combines in an efficient way both good parts of its components:

- the good quality of the reconstructed image, specific to KE
- the possibility to regenerate (only three times) in each KECG iteration the rows of the matrix A, specific to CG

Future work. Improving the implementation of the KECG reconstruction algorithm (essentially the scanning procedure and computation of the coefficients in the rows of A)