Robust Algorithms for Chebyshev Polynomials and Related Approximations

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Chebyshev Polynomials and their Relatives

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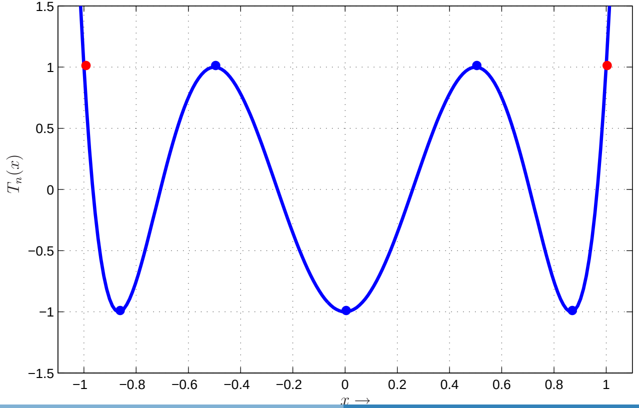
Chebyshev polynomials - approximation view

$$(1 - x^2)\left(\frac{dy}{dx}\right)^2 = n^2\,(1 - y^2)$$



Parametric solutions of differential equation

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = n^2(1-y^2)$$

$$\Downarrow$$

$$\frac{dy}{\sqrt{1-y^2}} = n\,\frac{dx}{\sqrt{1-x^2}}$$

$x = \cos\Phi$ De Moivre's formula

$y = \cos n\,\Phi$ $\Rightarrow$ $y(x) = T_n(\cos\Phi)$

The second order differential equation

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = n^2\,(1-y^2)$$

$$\Downarrow$$

$$(1-x^2)\,\frac{d^2y}{dx^2} - x\,\frac{dy}{dx} + n^2\,y = 0$$

$$y(x) \equiv T_n(x) = \sum_{k=0}^{n} t(k)x^k$$

.. is a polynomial of variable x

Recursive evaluation of the coefficients $t(k)$

given n

initialisation $t(n) = 2^{n-1}$

$t(n-1) = 0$

recursive body

(for $k = n-2$ *to* *0*

$$t(k) = -\frac{(k+2)(k+1)}{n^2-k^2}\, t(k+2)$$

end)

- Algorithm produces the coefficients $t(k)$ for Chebyshev polynomials $T_n(x)$ as expected.

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
T_0	1	0	0	0	0	0	0	0	0	0
T_1	0	1	0	0	0	0	0	0	0	0
T_2	-1	0	2	0	0	0	0	0	0	0
T_3	0	-3	0	4	0	0	0	0	0	0
T_4	1	0	-8	0	8	0	0	0	0	0
T_5	0	5	0	-20	0	16	0	0	0	0
T_6	-1	0	18	0	-48	0	32	0	0	0
T_7	0	- 7	0	56	0	-112	0	64	0	0
T_8	1	0	-32	0	160	0	-256	0	128	0
T_9	0	9	0	-120	0	432	0	-576	0	256

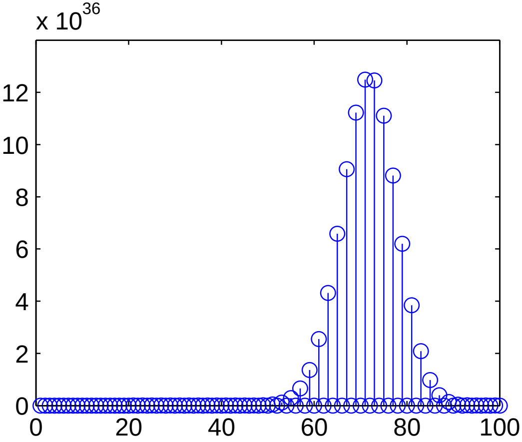
Comments

- The second order differential equation converts parametric representation to the explicit form
- In contrast to the explicit formula

$$T_n(x) = \frac{n}{2} \sum_{m=0}^{n} \frac{(-1)^m (n-m-1)!}{m!\,(n-2m)!} (2\,x)^{n-2m}$$

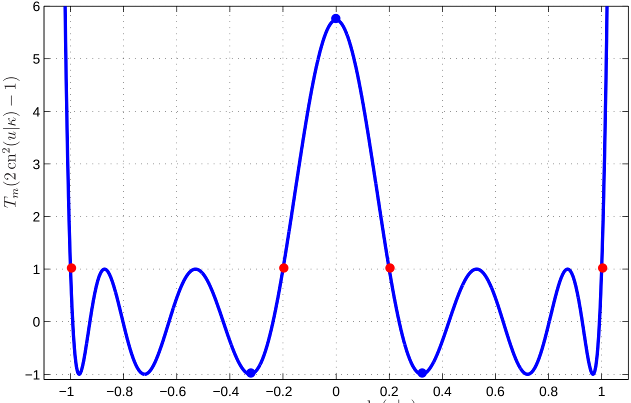
  the algorithm computes $t(k)$ for quite high order polynomials ($n \approx 100's$)
- The maximum of the coefficients appears at $\approx \frac{\sqrt(2)}{2} \times n$

Absolute values of the coefficients $n = 100$



Symmetrical Zolotarev polynomials-approximation

$$(1 - x^2)(x^2 - \kappa'^2)\left(\frac{dy}{dx}\right)^2 = 4\,m^2 x^2\,(1 - y^2)$$



Parametric solutions of differential equation

$$(1-x^2)(x^2-\kappa'^2)\left(\frac{dy}{dx}\right)^2 = 4\,m^2x^2\,(1-y^2)$$

$$\Downarrow$$

$$\frac{dy}{\sqrt{1-y^2}} = 2m\,\frac{xdx}{\sqrt{(1-x^2)(x^2-\kappa'^2)}}$$

$x = \mathrm{dn}(u|\kappa)$ Jacobi elliptic functions

$y = T_m(2\,\mathrm{cn}^2(u|\kappa)-1) \quad \Rightarrow y(x) = T_m\left(\frac{2x^2-1-\kappa'^2}{1-\kappa'^2}\right)$

The second order differential equation

$$(1-x^2)(x^2-\kappa'^2)\left(\frac{dy}{dx}\right)^2 = 4\,m^2x^2\,(1-y^2)$$

$$\Downarrow$$

$$x(x^2-\kappa'^2)\left[(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}\right]+\kappa'^2(1-x^2)\frac{dy}{dx}+4\,m^2x^3\,y=0$$

$$y(x) \equiv T_m\left(\frac{2x^2-1-\kappa'^2}{1-\kappa'^2}\right) = \sum_{k=0}^{m} b(2k)x^{2k}$$

.. is a polynomial of variable x

Recursive evalution of the coefficients $b(2k)$

given m

initialisation $b(2m) = \dfrac{2^{2m-1}}{(1-\kappa'^2)^m}, \; b(2m+2) = 0$

recursive body

(for $k = m-1$ *to 0*

$$b(2k) = -(1+\kappa'^2)\frac{(2k+2)(2k+1)}{4m^2-4k^2}\,b(2k+2) + \kappa'^2\frac{(2k+4)(2k+2)}{4m^2-4k^2}\,b(2k+4)$$

end)

An alternative representation

$$y(x) \equiv T_m\left(\frac{2x^2-1-\kappa'^2}{1-\kappa'^2}\right) = \sum_{k=0}^{m} a(2k)T_{2k}(x)$$

.. is developed in terms of Chebyshev polynomials

Inserting $y(x) = \sum_{k=0}^{m} a(2k)T_{2k}(x)$ in differential equation

$$x(x^2-\kappa'^2)\left[(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx}\right] + \kappa'^2(1-x^2)\frac{dy}{dx} + 4\,m^2x^3y = 0$$

we obtain...

Recursive algorithm for the coefficients $a(2k)$

$$
\begin{array}{rl}
\textit{given} & m \\
\textit{initialisation} & a(2m) = \dfrac{1}{(1-\kappa'^2)^m} \\
 & a(2m+2) = a(2m+4) = 0 \\
\textit{recursive body} & \\
\textit{(for} & k = m-1 \;\; \textit{to} \;\; \textit{0} \\
 & -\left[m^2-k^2\right] a(2k) = \\
 & +\left[3(m^2-(k+1)^2)+(2k+2)(2k+1)\kappa'^2\right] a(2k+2) \\
 & +\left[3(m^2-(k+2)^2)+(2k+4)(2k+5)\kappa'^2\right] a(2k+4) \\
 & +\left[m^2-(k+3)^2\right] a(2k+6) \\
\textit{end}\,) &
\end{array}
$$

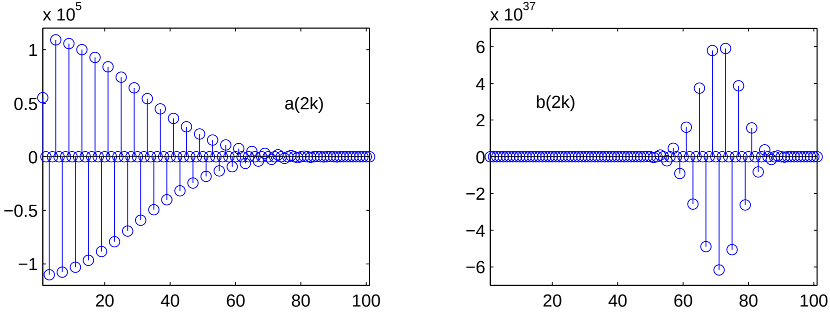
Comments

- Properties of Jacobi elliptic functions as

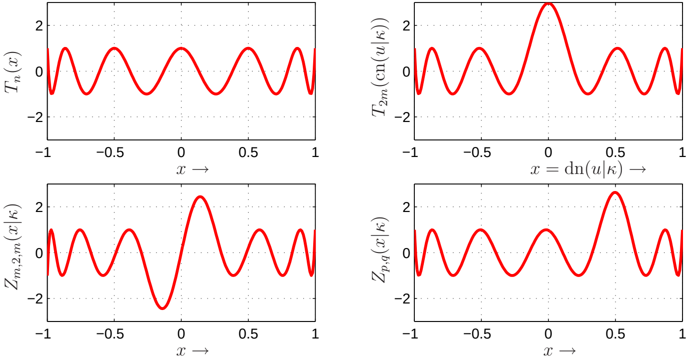
$$\kappa'^2 + \kappa^2 \mathrm{cn}^2(u|\kappa) = \mathrm{dn}^2(u|\kappa)$$

  convert parametric representation to the explicit form
- The second order differential equation produces recursive algorithms for coefficients $b(2k)$ and $a(2k)$
- The dynamic range of coefficients $a(2k)$ is far better than $b(2k)$ and it enables to evaluate safely the symmetrical Zolotarev polynomial of order ($n \approx 1000's$)

Dynamic range of both representations $m = 50$



Equiripple polynomials



Differential equations - approximation view

$$f(x)\left(\frac{dy}{dx}\right)^2 = n^2 g(x)(1-y_i^2) \qquad (1)$$

y_a :	$f(x) = 1 - x^2$	$g(x) = 1$
y_b :	$f(x) = (1-x^2)(x^2-\kappa'^2)$	$g(x) = x^2$
y_c :	$f(x) = (1-x^2)(x^2-x_p^2)(x^2-x_s^2)$	$g(x) = (x^2-x_m^2)^2$
y_d :	$f(x) = (1-x^2)(x-x_p)(x-x_s)$	$g(w) = (x-x_m)^2$

Linear differential equation

By derivation of eq. (1) we obtain

$$\frac{d^2y}{dx^2} + \frac{1}{2}\left[\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}\right]\frac{dy}{dx} + n^2\frac{g(x)}{f(x)}y = 0\,,$$

or in *canonical form*

$$\frac{d^2y}{dx^2} - \frac{d}{dx}\ln\sqrt{\frac{g(x)}{f(x)}}\frac{dy}{dy} + n^2\frac{g(x)}{f(x)}y = 0\,.$$

Wronskian and general solutions

Two linearly independent solutions of eq. (2) $y_1(x)$ a $y_2(x)$, form Wronskian determinant

$$\mathcal{W}(y_1(x), y_2(x)) \equiv y_1(x)y_2'(x) - y_1'(x)y_2(x) =$$

$$= exp^{\int \frac{d}{dx} \ln \sqrt{\frac{g(x)}{f(x)}} dx} = \sqrt{\frac{g(x)}{f(x)}} .$$

Solutions

In order to avoid relatively complicated parametric representation...

$$Z_{p,q}(\mathcal{A}_{\frac{p}{n}}(u|\kappa)) = (-1)^p \frac{1}{2}\left[\left(\frac{\vartheta_1\left(v - \frac{\pi}{2}\frac{p}{p+q}\right)}{\vartheta_1\left(v + \frac{\pi}{2}\frac{p}{p+q}\right)}\right)^n + \left(\frac{\vartheta_1\left(v + \frac{\pi}{2}\frac{p}{p+q}\right)}{\vartheta_1\left(v - \frac{\pi}{2}\frac{p}{p+q}\right)}\right)^n\right]$$

$$\mathcal{A}_{\frac{p}{n}}(u|\kappa) = \frac{1}{2}\left[\frac{\vartheta_1\left(v - \frac{\pi}{2}\frac{p}{p+q}\right)}{\vartheta_1\left(v + \frac{\pi}{2}\frac{p}{p+q}\right)} + \frac{\vartheta_1\left(v + \frac{\pi}{2}\frac{p}{p+q}\right)}{\vartheta_1\left(v - \frac{\pi}{2}\frac{p}{p+q}\right)}\right]$$

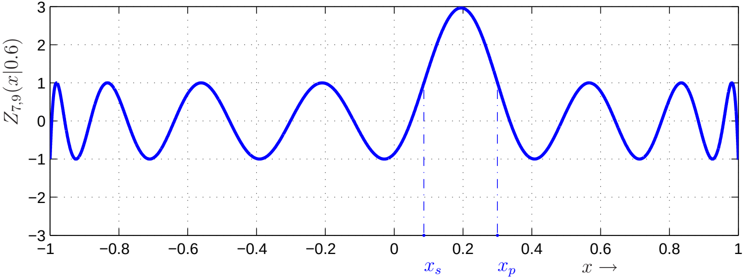
... the recursive algorithms

for the coefficients $b(m)$ and $a(m)$ have been developed.

$$Z_{p,q}(x|\kappa) = \sum_{m=0}^{n} b(m) x^m$$

$$Z_{p,q}(x|\kappa) = \sum_{m=0}^{n} a(m) T_m(x)$$

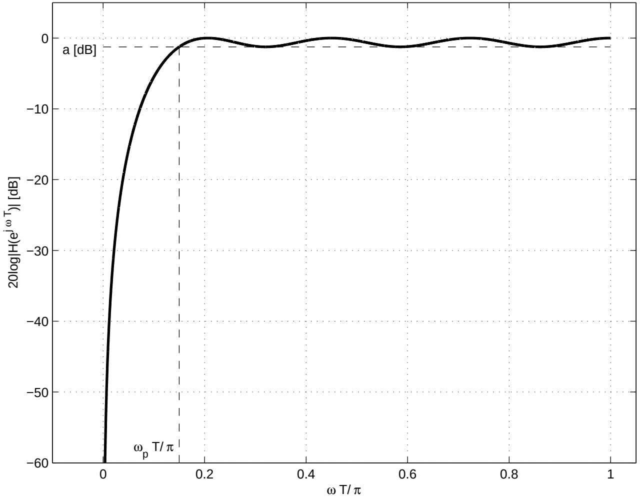
Zolotarev polynomial



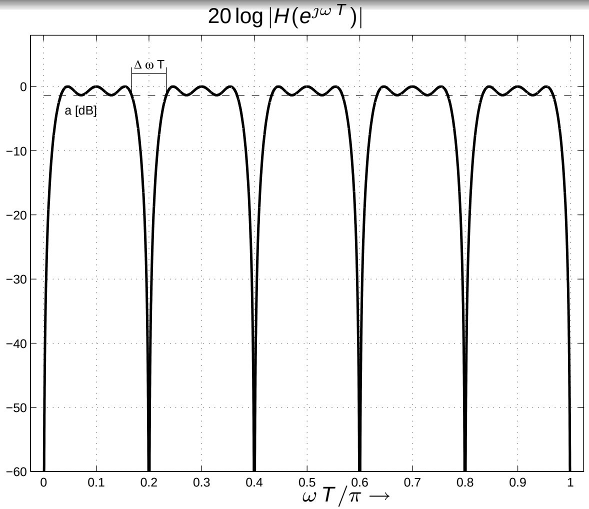
Application in Filter Design

- Design of digital FIR filters: DC-Notch FIR filters and Comb FIR filters
- Based on differential equation developed for Chebyshev polynomials as $T_n(\lambda w + \lambda - 1)$ and $T_n\left[\lambda T_r(w)\right]$

Example of DC-Notch: $\omega_p T = 0.15\pi$, $a = -1.2446$ dB



Example of Comb: $\Delta\omega_p T = 0.066\pi$, $a = -0.65$ dB



Differential Equation for $T_n(\lambda w + \lambda - 1)$

$$\left[1 - w^2 + 2\,\frac{1-\lambda}{\lambda}(1-w)\right]\frac{d^2F(w)}{dw^2} - \left(w - \frac{1-\lambda}{\lambda}\right)\frac{dF(w)}{dw} + n^2F(w) = 0$$

... by substitution of $F(w) = \sum_{m=0}^{n} \alpha(m)\,T_m$ we obtain

Recursive algorithm for coefficients

given n (integer value), λ (real value)

initialization $\alpha(n) = \lambda^n\,, \quad \alpha(n+1) = \alpha(n+2) = \alpha(n+3) = 0$

body

(*for* $k = 2 \ldots n+1$)

$$\alpha(n+1-k) =$$

$$\{\ 2\,[(k-1)(2n+1-k) - ((1-\lambda)/\lambda)(n+1-k)(2n+1-2k)]\,\alpha(n+2-k)$$
$$+\,4\,((1-\lambda)/\lambda)(n+2-k)\ \alpha(n+3-k)$$
$$-\,2\,[(k-3)(2n+3-k) - ((1-\lambda)/\lambda)(n+3-k)(2n+7-2k)]\ \alpha(n+4-k)$$
$$+\,(k-4)(2n+4-k)\ \alpha(n+5-k)\ \ \}\ /\ k(2n-k)$$

(*end loop on* k)

$$\alpha(0) = \frac{\alpha(0)}{2}$$

Differential Equation for $T_n\,[\lambda\,T_r(w)]$

$$U_{r-1}(w)\left(\kappa^2 - T_r^2(w)\right)\left[(1-w^2)\frac{d^2F_{nr}(w)}{dw^2} - w\frac{dF_{nr}(w)}{dw}\right]$$
$$-r(1-\kappa^2)T_r(w)\frac{dF_{nr}(w)}{dw}$$
$$+n^2r^2U_{r-1}(w)\left(1-T_r^2(w)\right)F_{nr}(w) = 0$$

... by substitution of $F_{nr}(w) = \sum_{m=0}^{n\times r} \tilde{a}(m)\,T_m(w)$ we obtain

Basic recursive algorithm for $\tilde{a}(nr - 2\mu r) = \alpha(n - 2\mu)$

given — n (even integer), r (integer), $\lambda > 1$ (real)

initialization

$$\kappa = \frac{1}{\lambda}$$
$$\alpha(n) = \lambda^n$$
$$\alpha(n+2) = \alpha(n+4) = \alpha(n+6) = 0$$

body

(*for* $\mu = 1 \ldots \frac{n}{2}$)

$$\begin{aligned}
&\alpha(n-2\mu) = \\
&\{ \quad \alpha(n-2(\mu-1)) \\
&\quad \times \left[(1-\kappa^2)(n-(2\mu-1))(n-(2\mu-2)) + 3(\mu-1)(n-(\mu-1))\right] \\
&\; -\alpha(n-2(\mu-2)) \\
&\quad \times \left[(1-\kappa^2)(n-(2\mu-4))(n-(2\mu-5)) + 3(\mu-2)(n-(\mu-2))\right] \\
&\; +\alpha(n-2(\mu-3))(\mu-3)(n-(\mu-3)) \quad \} \; / \; \mu(n-\mu)
\end{aligned}$$

(*end loop on* μ)

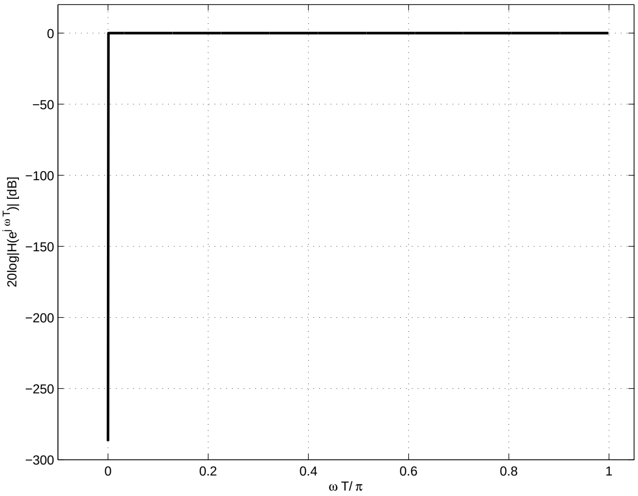
$$\alpha(0) = \frac{\alpha(0)}{2}$$

Robustness of Algorithms

Strange filter specification $\omega_p T = 0.00001\pi$, $a = -0.01$ dB requires to evaluate expression

$$1 - \frac{T_{259524}\,(1.00000000024674w + 0.00000000024674) + 1}{T_{259524}(1.00000000049348) + 1}$$

Robustness of Algorithms



1. Chebyshev polynomials and their relatives were presented in unified approach
2. recursive algorithms were developed
3. the role of the linear differential equations was emphasized
4. numerical robustness was shown

References

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Thank you for your attention !