What does mathematical fuzzy logic offer to description logic?

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Abstract

Continuous t-norm based fuzzy predicate logic is surveyed as a generalization of classical predicate logic; then a kind of fuzzy description logic based on our fuzzy predicate logic is briefly described as a powerful but still decidable formal system of description logic dealing with vague (imprecise) concepts.

Description logic is a florishing domain of research (see [2]) and has been for long developed in such a way that it is naturally embeddable into the classical (two)valued, Boolean) predicate logic. Early papers on a possible fuzzy description logic, notably [14, 11, 12] work with a rather minimalistic system of fuzzy logic. In my paper [7] I develop a system (ALC-like) of fuzzy description logic based on the formal system of fuzzy logic from my monograph [6]. The presented paper is a *companion* of [6], not containing any proofs but concentrating to a presentation of fuzzy predicate logic as a natural and rich generalization of classical predicate logic (Section 1), a presentation of fuzzy description logic as a natural and powerful generalization of "classical" description logic (Section 2) and some examples and some discussion (Section 3).

1 From classical logic to fuzzy logic

We start with quickly surveying the basic notions of classical predicate logic (which is undoubtedly the queen of all logics). The reader is assumed to know them and hoped to accept a slightly non-traditional presentation prepared for generalization to fuzzy logic.

A *language* is given by *predicates* $P, Q, \ldots$, each with its arity (number of arguments – unary, binary, ..., n-ary) and *object constants* $a, b, c, \ldots$ *Logical symbols* are *object variables* $x, y, \ldots$, *connectives* (conjunction $\wedge$, disjunction $\vee$, implication $\rightarrow$, equivalence $\equiv$, negation $\neg$), *quantifiers* (universal $\forall$, existential $\exists$) and (possibly) truth constants $\top$ (truth), $\bot$ (falsity). Formulas are built from these in the obvious way (atomic having the form $P(t_1, \ldots, t_n)$ where

t_i's are variables or constants; $\top$ and $\bot$ are formulas; other formulas are built using connectives and quantifiers). For example, $(\forall x)(P(x,c) \vee (\exists y)Q(x,y))$ is a formula; it is closed, contains no free variables (all variables are quantified).

There are two truth values: 1 – true, 0 – false. An *interpretation* of such a language is a structure $\mathbf{M} = (M, (r_P)_{P\,predicate}, (v_c)_{c\,constant})$ where M is a non-empty set (the *domain*), for each predicate P of arity n, r_P is the characteristic function of an n-ary relation on M named by P, i.e., $r_P : M^n \to \{0,1\}$. For each constant c, v_c is an element of M.

Given $\mathbf{M}$, for each formula $\varphi(x_1,\ldots,x_n)$ with free (non-quantified) variables $x_1,\ldots,x_n$ and for each n-tuple $u_1,\ldots,u_n \in M$, $\|\varphi(u_1,\ldots,u_n)\|_{\mathbf{M}}$ is the truth value of $\varphi(x_1,\ldots,x_n)$ in $\mathbf{M}$ for the elements $u_1,\ldots,u_n$; it is 1 ($u_1,\ldots,u_n$ satisfy φ in $\mathbf{M}$) or 0 (they do not satisfy. This is defined inductively, e.g. for an atomic formula $P(x,y)$ and $u,v \in M$, (u,v) satisfy $P(x,y)$ if $r_P(u,v) = 1$, thus $\|P(u,v)\|_{\mathbf{M}} = r_P(u,v)$. For connectives one uses the well-known truth tables, e.g. (φ stands for $\varphi(u_1,\ldots)$ etc.)

$\|\varphi \wedge \psi\|_M = 1$ iff $\|\varphi\|_M = \|\psi\|_M = 1$, thus $\|\varphi \wedge \psi\|_{\mathbf{M}} = \min(\|\varphi\|_{\mathbf{M}}, \|\psi\|_{\mathbf{M}})$;
$\|\varphi \to \psi\|_{\mathbf{M}} = 1$ iff $\|\psi\|_{\mathbf{M}} = 1$ or $\|\varphi\|_{\mathbf{M}} = 0$, i.e., iff $\|\varphi\|_{\mathbf{M}} \leq \|\psi\|_{\mathbf{M}}$;
$\|\neg\varphi\|_{\mathbf{M}} = 1$ iff $\|\varphi\|_{\mathbf{M}} = 0$; $\|\top\|_{\mathbf{M}} = 1$, $\|\bot\|_{\mathbf{M}} = 0$ for all $\mathbf{M}$.
$\|(\forall x)\varphi(x,\ldots)\|_{\mathbf{M}} = 1$ iff for all $v \in M$, $\|\varphi(v,\ldots)\|_{\mathbf{M}} = 1$,
$\|(\exists x)\varphi(x,\ldots)\|_{\mathbf{M}} = 1$ iff some $v \in M$ $\|\varphi(v,\ldots)\|_{\mathbf{M}} = 1$, thus
$\|(\forall x)\varphi(x,\ldots)\|_{\mathbf{M}} = \min\{\|\varphi(v,\ldots)\|_{\mathbf{M}} \mid v \in M\}$,
$\|(\exists x)\varphi(x,\ldots)\|_{\mathbf{M}} = \max\{\|\varphi(v,\ldots)\|_{\mathbf{M}} \mid v \in M)$.

Clearly, if φ is closed (has no free variables) then $\|\varphi\|_{\mathbf{M}}$ is just the truth value of φ in $\mathbf{M}$; if $\|\varphi\|_{\mathbf{M}} = 1$ we say that φ is *true* in $\mathbf{M}$ and denote by $\mathbf{M} \models \varphi$.

A formula φ is a *tautology* if it is true in all interpretations. Just mention the existence of *axioms* and *deduction rules* giving the notion of a formula *provable* in the predicate calculus. The *completeness theorem* says that a formula is provable in the predicate calculus iff it is a tautology.

To close this telegraphic summary of the classical predicate calculus let us mention that one can choose some connectives to be basic (or starting) and define the others from them. For example, one can take $\wedge$ and $\neg$ for starting and define $\varphi \vee \psi$ to be $\neg(\neg\varphi \wedge \neg\psi)$, define $\varphi \to \psi$ to be $\neg\varphi \vee \psi$ and define $\varphi \equiv \psi$ to be $(\varphi \to \psi) \wedge (\psi \to \varphi)$. Or take $\to$ and $\neg$ to be starting and define $\varphi \wedge \psi$ to be $\neg(\varphi \to \neg\psi)$, define $\varphi \vee \psi$ to be $\neg\varphi \to \psi$ etc. This is OK since the semantics of the defining and defined formula is the same.

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Fuzzy logic is the logic of vague (imprecise) notions; formulas of fuzzy logic may be not just true of false but may be partially true, true in a degree. Mathematical fuzzy logic (or logic in a narrow sense) is a formal system like classical logic but having more than two truth values that are ordered (possibly partially ordered), a logic with a comparative notion of truth. (Think of the truth degree of a sentence like "John is young" etc.) As such, fuzzy logic is a kind of multiple-valued logic (many-valued logic) with some specific properties and aims. The *standard domain of truth degrees* is the real unit interval $[0,1]$. The definition of a *language* is the same as in classical logic; a (standard) *interpretation* of a language is a structure $\mathbf{M} = (M, (r_P)_{P\,predicate}, (v_c)_{c\,constant})$ where M and v_c are as above but for each P, r_P is a *fuzzy relation* on M, i.e. a mapping $r_P : M^n \to [0,1]$ assigning to each n-tuple $(v_1,\ldots,v_n)$ of elements of M the *degree* $r_P(v_1,\ldots,v_n)$ in which the tuple is in the relation. A trivial example: $M = \{1,2,3\}$, one binary predicate L (read $L(x,y)$ "x likes y") and r_P is given by the following table:

L	1	2	3
1	1	0.2	0
2	0.7	0.8	0.8
3	0	0.1	0.5

Starting connectives are *conjunction, implication* (binary) and the truth constants $\top$ (truth), $\bot$ (falsity). Since we shall have two different conjunctions we start with the conjunction denoted by &; implication will be $\to$. Since early days of fuzzy logic, for the truth function of & one takes a *continuous t-norm*, which is a binary operation on $[0,1]$ continuous as a real-valued function and satisfying the following for each $x,y,z \in [0,1]$:

associativity	$x*(y*z) = (x*y)*z$,
commutativity	$x*y = y*z$
monotony	$x \leq y$ implies $x*z \leq y*z$
zero and unit	$0*x = 0, 1*x = 1$.

Three most important continuous t-norms are

Łukasiewicz	$x*y = \max(0, x+y-1)$
Gödel	$x*y = \min(x,y)$
product	$x*y = x \cdot y$ (real product)

Each continuous t-norm is "composed" from a linearly ordered at most countable system of copies of these three t-norms (Łukasiewicz, Gödel, product) – in a precise well defined sense (Mostert-Shields theorem). For a detailed formulation see e.g. [6]; here we only mention that the system may and may not have a least (first) element.

Thanks to continuity each continuous t-norm has its *residuum* $x \Rightarrow y = \max\{z | x*z \leq y\}$ (left continuity suffices). The operation $\Rightarrow$ is the *truth function of implication* given by the t-norm. This implication has very good properties, in particular, for each continuous t-norm $*$, $x \Rightarrow y = 1$ iff $x \leq y$. For $x > y$ different t-norms give different residua, notably: for $x > y$,

Łukasiewicz	$x \Rightarrow y = 1 - x + y$
Gödel	$x \Rightarrow y = y$
product	$x \Rightarrow y = y/x$.

Given this, we may define the truth degree of each formula $\varphi(x_1, \ldots, x_n)$ given by elements $u_, \ldots, u_n \in M$ denoted $\|\varphi(u_1, \ldots, u_n)\|^*_{\mathbf{M}}$ (since it depends on the choice of our t-norm) in analogy to the classical logic as follows:

For an atomic formula $P(x_1, \ldots, x_n)$, $\|P(u_1, \ldots, u_n)\|^*_{\mathbf{M}} = r_P(u_1, \ldots, u_n)$, and similarly if the atomic formula contains some constants, e.g. for $P(x, c, y)$ and $u_1, u_2 \in M$, $\|P(u_1, c, u_2)\|^*_{\mathbf{M}} = r_P(u_1, v_c, u_2)$ (where v_c is the interpretation of c in $\mathbf{M}$).

$$\|\top\|^*_{\mathbf{M}} = 1,\ \|\bot\|^*_{\mathbf{M}} = 0,\ \|\varphi \,\&\, \psi\|^*_{\mathbf{M}} = \|\varphi\|^*_{\mathbf{M}} * \|\psi\|^*_{\mathbf{M}},\ \|\varphi \to \psi\|^*_{\mathbf{M}} = \|\varphi\|^*_{\mathbf{M}} \Rightarrow \|\psi\|^*_{\mathbf{M}}.$$

(In our example above, $\|L(2,2)\|^*_{\mathbf{M}} = 0.8, \|L(2,1)\|^*_{\mathbf{M}} = 0.7$, thus for $*$ being Ł (Łukasiewicz) $\|L(2,2) \to L(2,1)\|^{Ł}_{\mathbf{M}} = 1 - 0.8 + 0.7 = 0.9$; for Gödel you get $\|L(2,2) \to L(2,1)\|^*_{\mathbf{M}} = 0.7$ etc.)

Some defined connectives:

$$\begin{gathered}\neg\varphi \text{ is } \varphi \to \bot,\\ \varphi \wedge \psi \text{ is } \varphi \,\&\, (\varphi \to \psi),\\ \varphi \vee \psi \text{ is } ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi),\\ \varphi \equiv \psi \text{ is } (\varphi \to \psi) \,\&\, (\psi \to \varphi).\end{gathered}$$

Here $\neg$ is *negation*, $\wedge$ is min-conjunction, $\vee$ is max-disjunction, $\equiv$ is equivalence. For any choice of $*$, the truth function of $\wedge$ is minimum (i.e. $\|\varphi \wedge \psi\|^*_{\mathbf{M}} = \min(\|\varphi\|^*_{\mathbf{M}}, \|\psi\|^*_{\mathbf{M}})$, the truth function of $\vee$ is maximum; negation depends on $*$, in particular:

$$\|\neg\varphi\|^{Ł}_{\mathbf{M}} = 1 - \|\varphi\|^{Ł}_{\mathbf{M}}$$

but for $*$ being G of Π (product) we get Gödel negation: if $\|\varphi\|^G_{\mathbf{M}} = 0$ then $\|\neg\varphi\|^G_{\mathbf{M}} = 1$, but $\|\varphi\|^G_{\mathbf{M}} > 0$ implies $\|\neg\varphi\|^G_{\mathbf{M}} = 0$. (negation of 0 is 1, negation of a positive value is 0). Clearly, $\|\varphi \equiv \psi\|^*_{\mathbf{M}} = 1$ iff $\|\varphi\|^*_{\mathbf{M}} = \|\psi\|^*_{\mathbf{M}}$.

For quantifiers the definition is as follows:

$$\|(\forall x)\varphi(x, \ldots)\|^*_{\mathbf{M}} = \inf\{\|\varphi(v, \ldots)\|^*_{\mathbf{M}} \mid v \in M\},$$

$$\|(\exists x)\varphi(x, \ldots)\|^*_{\mathbf{M}} = \sup\{\|\varphi(v, \ldots)\|^*_{\mathbf{M}} \mid v \in M\},$$

thus the truth degree of a universally quantified formula is the infimum of truth degrees of its instances and similarly for existential quantification and supremum. If M is finite, we may replace "infimum" by "minimum" and "supremum" by "maximum"; for infinite M this may not be the case. For example let M be the set of positive natural numbers $1, 2, 3, \ldots$ and let $r_{Sm}(n) = \frac{1}{n}$ (read Sm "small"); then (for any $*$) $\|(\forall x)Sm(x)\|^*_{\mathbf{M}} = \inf_n \frac{1}{n} = 0$, but $\|Sm(n)\|^*_{\mathbf{M}}$ is positive for each n. (This is related to the so-called Sorites paradox, solved in fuzzy logic.) A formula φ is a $*$-tautology if $\|\varphi\|^*_{\mathbf{M}} = 1$ for each $\mathbf{M}$; φ is a *standard tautology* if it is a $*$-tautology for each continuous t-norm $*$. Analogously to classical logic, some standard tautologies are taken for axioms of the basic fuzzy predicate logic BL$\forall$; deduction rules are as in classical logic (modus ponens and generalization.) This gives the notion of *provability* in BL$\forall$. It is complete with respect to a more general semantics over so-called BL-algebras which are some algebras of truth functions, algebras given by continuous t-norms being particular BL-algebras. (BL-algebras are particular *residuated lattices*, we shall not go into any details here, see [6].) Note that the set of all standard predicate tautologies is not (effectively) axiomatizable.

The following are some few examples of standard tautologies:

$$
\begin{gathered}
(\varphi \to (\psi \to \chi)) \equiv (\psi \to (\varphi \to \psi)) \\
(\varphi \to (\psi \to \chi)) \equiv ((\varphi \,\&\, \psi) \to \chi) \\
(\varphi \to \psi) \to (\neg\psi \to \neg\varphi) \\
(\varphi \to \psi) \to ((\varphi \,\&\, \chi) \to (\psi \,\&\, \chi)) \\
(\forall x)(\varphi \to \psi) \to ((\forall x)\varphi \to (\forall x)\psi) \\
(\forall x)(\varphi \to \psi) \to ((\exists x)\varphi \to (\exists x)\psi) \\
(\exists x)\varphi \to \neg(\forall x)\neg\varphi
\end{gathered}
$$

Now let us present examples of formulas that are *not* standard tautologies, i.e. for some continuous t-norm $*$ they are not $*$-tautologies. (All of them are tautologies of classical logic.) In brackets, Ł$, G, \Pi$ means that the formula *is* a $*$-tautology for $*$ being Łukasiewicz, Gödel or product t-norm.

$$
\begin{array}{cc}
\neg\neg\varphi \equiv \varphi & (\text{Ł}) \\
(\varphi \,\&\, \varphi) \equiv \varphi & (G) \\
\neg(\varphi \wedge \neg\varphi) & (G, \Pi) \\
\varphi \vee \neg\varphi & (\text{none}) \\
(\exists x)\varphi \equiv \neg(\forall x)\neg\varphi & (\text{Ł}) \\
(\forall x)\varphi \equiv \neg(\exists x)\neg\varphi & (\text{Ł})
\end{array}
$$

(The reader may verify easily that these formulas are tautologies of the logics indicated remembering the definition of Łukasiewicz negation and of Gödel negation as well as the fact that in Gödel logic the conjunctions & and $\wedge$ have the same semantics.)

Similarly as in classical logic, using fuzzy logic one gets some feeling (or practice) in recognizing well-known tautologies and well-known non-tautologies (but, I repeat, there is no algorithm to decide on any given formula if it is a standard tautology; similarly for (non-)tautologies of a fixed continuous t-norm).

We shall also use the following terminology: an interpretation $\mathbf{M}$ is a $*$-*model* of a closed formula φ if $\|\varphi\|^*_{\mathbf{M}} = 1$. $\mathbf{M}$ is a $*$-model of a set T of closed formulas if $\|\varphi\|^*_{\mathbf{M}} = 1$ for each $\psi \in T$. Finally, T $*$-*entails* φ if each $*$-model of T is an T-model of φ.

To close this section let us mention a "minimalistic" fuzzy logic KD used early papers in fuzzy logic (and much later in early papers on fuzzy description logic). It uses only connectives $\wedge, \vee$ (min-conjunction, max-disjunction), Łukasiewicz negation $\|\neg\varphi\|^{\mathrm{KD}}_{\mathbf{M}} = 1 - \|\varphi\|^{\mathrm{KD}}_{\mathbf{M}}$ and so-called Kleene-Dienes implication $\|\varphi \to \psi\|^{\mathrm{KD}}_{\mathbf{M}} = \max(1 - \|\varphi\|^{\mathrm{KD}}_{\mathbf{M}}, \|\psi\|^{\mathrm{KD}}_{\mathbf{M}})$, inspired by mechanical use of the classical tautology $(\varphi \to \psi) \equiv (\neg\varphi \vee \psi)$. Observe that connectives of KD are definable from connectives given by the Łukasiewicz t-norm; but the KD-implication does not have good properties. For example, observe that in our example above (with the predicate "likes"), for any continuous t-norm $*$, the formula $(\forall x, y)(L(x, y) \to L(x, x))$ has the truth value 1 (since for any u, v, $r_L(u, v) \le r_L(u, u)$, i.e. every x likes himself at least as much as he likes anybody else), i.e. $\|(\forall x, y)(L(x, y) \to L(x, x)\|^{*}_{\mathbf{M}} = 1$, but $\|(\forall x, y)(L(x, y) \to L(x, x))\|^{\mathrm{KD}}_{\mathbf{M}} = \frac{1}{2}$ since in particular $\|L(3, 3) \to L(3, 3)\|^{\mathrm{KD}}_{\mathbf{M}} = (\frac{1}{2} \Rightarrow_{\mathrm{KD}} \frac{1}{2}) = \frac{1}{2}$. This seems to be rather counter-intuitive.

2 From description logic to fuzzy description logic

We shall restrict ourselves to the description logic $\mathcal{ALC}$ and its fuzzy counterpart. Recall that in $\mathcal{ALC}$ concepts are built from finitely many unary predicates (atomic concepts), and finitely many binary predicate (roles) using connectives $\wedge, \vee, \neg$ and quantifier constructs: if C, D are concepts then $C \wedge D$, $C \vee D$, $\neg C$ are concepts, if C is a concept and R is role then $(\forall R.C)$, $(\exists R.C)$ are concepts. From the point of view of classical predicate logic, concepts correspond to particular formulas with one free variable: if A is an atomic concept, take $A(x)$; if $C(x)$ and $D(x)$ are defined then $C(x) \wedge D(x)$, $C(x) \vee D(x)$, $\neg C(x)$ have clear meaning. The quantifier constructs are understood as follows.

$$\begin{array}{lll} (\forall R.C)(x) & \text{means} & (\forall y)(R(x, y) \to C(y)), \\ (\exists R.C)(x) & \text{means} & (\exists y)(R(x, y) \wedge C(y)). \end{array}$$

Also for each concept C and a constant a, $C(a)$ has clear meaning. Saying that a concept is valid we mean that $(\forall x)C(x)$ is a predicate tautology; saying that it is satisfiable we mean that $C(a)$ has a model (an interpretation $\mathbf{M}$ such that $\mathbf{M} \models C(a)$). Saying that C is subsumed by D (in symbols, $C \sqsubseteq D$) we mean that $(\forall x)(C(x) \to D(x))$ is a tautology. Since in classical logic implication is definable from $\wedge$ and $\neg$, for any concepts C, D we have the concept $C \to D$ equivalent to the concept $\neg C \vee D$ and we see that $C \sqsubseteq D$ (C is subsumed by D) iff the concept $C \to D$ is valid (i.e. the formula $(\forall x)(C(x) \to D(x))$ is a tautology). It is known that the question if a given concept is valid is decidable and the same for the question if a given concept is satisfiable (see e.g. [10]) consequently, also subsumption $C \sqsubseteq D$ is decidable.

Note also the finite model property: C is valid iff $(\forall x)C(x)$ is true in all *finite* interpretation (having finite domain); and C is satisfiable iff $C(a)$ is true in a finite interpretation.

For simplicity we restrict ourselves just to those problems, not discussing termonological axioms and axioms-facts. Our aim is to show on this simplest fragment how it can be combined with *fuzzy logic* and what are the problems with this.

Thus take the same language of finitely many atomic concepts and finitely many roles. Concepts are built from atomic concepts using $\bot, \&, \rightarrow$ (i.e. $\bot$ is a concept; each atomic concept is a concept; if C, D are concepts then $C \,\&\, D$, $C \rightarrow D$ are concepts) and using the quantifier constructs $(\forall R.C)$, $(\exists R.C)$. Translation to predicate formulas is clear, only take $(\exists R.C)(x)$ to be $(\exists y)(R(x,y) \,\&\, C(y))$ (use the &-conjunction). Also here observe that defined connectives can be used to construct concepts, thus if C, D are concepts then so are $C \wedge D$, $C \vee D$, $\neg C$, $C \equiv D$.

Now choose a continuous t-norm $*$; you may ask whether a concept C is $*$-valid (i.e. $\|(\forall x)C(x)\|^*_{\mathbf{M}} = 1$ for each $\mathbf{M}$), whether it is $*$-satisfiable (there is some $\mathbf{M}$ with $\|C(a)\|^*_{\mathbf{M}} = 1$ i.e. $C(a)$ has a $*$-model), whether C is $*$-subsumed by D and similar. We present here (without proofs) the main results of the paper [7] and illustrate them by some examples.

First we shall introduce some important notions. Recall that the truth degree of an universally quantified formula is defined as the infimum of the set of truth values of its instances and that this set need not have a minimal element. Similarly for existentially quantified formulas supremum and maximum. Let us *define*: let $\varphi(x, y_1, \ldots, y_n)$ be a formula $\mathbf{M}$ an interpretation and $u_1, \ldots, u_n \in M$. An object $v \in M$ is a $*$-*witness* in $\mathbf{M}$ for $(\forall x)\varphi(x, y_1, \ldots, y_n)$ and $u_1, \ldots, u_n$ if $\|(\forall x)\varphi(x, u_1, \ldots, u_n)\|^*_{\mathbf{M}} = \|\varphi(v, u_1, \ldots, u_n)\|^*_{\mathbf{M}}$; similarly for $(\exists x)\varphi(x, y_1, \ldots, y_n)$, i.e. $\|(\exists x)\varphi(x, u_1, \ldots, u_n)\|^*_{\mathbf{M}} = \|\varphi(v, u_1, \ldots, u_n)\|^*_{\mathbf{M}}$. An interpretation $\mathbf{M}$ is $*$-*witnessed* if each formula beginning by a quantifier has a $*$-witness in $\mathbf{M}$ for any evaluation of its free variables by elements of M.

Surely, there are non-witnessed interpretations (we saw some above); each finite interpretation (with a finite domain) is witnessed.

For the Łukasiewicz t-norm Ł the following holds true: A formula φ is Ł-true in some interpretation iff it is Ł-true in some witnessed interpretation; and φ is Ł-true in all interpretations iff it is Ł-true in all witnessed interpretations (see [7]); but for other t-norms it is not the case. Observe the formula $\neg(\forall x)P(x) \,\&\, \neg(\exists x)\neg P(x)$, which has a G-model (and a Π-model) but no witnessed G-model (Π-model): To get a G-model, take the interpretation of Sm above: $r_{Sm}(n) > 0$ for all $n \in M$ but $\inf_n r_{Sm}(n) = 0$. Thus $\|(\forall x)Sm(x)\|^G_{\mathbf{M}} = 0$, $\|(\exists x)\neg Sm(x)\|^G_{\mathbf{M}} = 0$, hence $\|\neg(\forall x)Sm(x) \,\&\, \neg(\exists x)\neg Sm(x)\|^G_{\mathbf{M}} = 1$. In Sect. 3 we show that the formula has no witnessed model.

This leads us to the investigation of witnessed models of concepts. (For the aims of description logic non-witnessed models appear to be pathological.) We get the following:

Theorem 1 Let C be a concept, $*$ a continuous t-norm.

(1) $C(a)$ has a witnessed $*$-model iff it has a finite $*$-model

(2) $(\forall x)C(x)$ is true in all witnessed $*$-interpretations iff it is $*$-true in all finite $*$-interpretations.

Observe that e.g. for C being $A \equiv \neg A$ the formula $C(a)$ has a finite Ł-model (in which $\|A(a)\|^{\L}_{\mathbf{M}} = \frac{1}{2}$) but $C(a)$ has no G-model (Π-model); if C is $A \equiv (A \,\&\, A)$ then $(\forall x)C(x)$ is true in all G-interpretations but not in all Ł-interpretations, neither in all Π-interpretations.

For a full proof of the theorem see [7]; here we only sketch the main idea. Given a formula $C(a)$ (where C is a concept and a is a constant) we construct a finite number of new unary predicates and new object constants, from which we construct effectively (using an algorithm) a finite theory T whose axioms are closed quantifier-free formulas (propositional combinations of atoms) and a closed quantifier-free formula $prop(C(a))$ such that for each continuous t-norm $*$ and each $*$-model $\mathbf{M}$ of T, $\|prop(C(a))\|^*_{\mathbf{M}} = \|C(a)\|^*_{\mathbf{M}}$. In particular, due to the very simple (propositional) form of $prop(C(a))$, any $*$-model $\mathbf{M}$ of T determines a *finite* $*$-model $\mathbf{M}_0$ of T consisting just of (interpretations of) constants occuring in T and $\|prop(C(a))\|^*_{\mathbf{M}_0} = \|propC(a)\|^*_{\mathbf{M}}$. Moreover, each witnessed $*$-interpretation of the language of $C(a)$ expands to a model of T. Now $C(a)$ has a $*$-model iff $T \cup \{prop(C(a))\}$ has a finite $*$-model from constants as above; this reduces the problem to $*$-satisfiability of a finite set of propositional formulas. $(\forall x)C(x)$ is a $*$-tautology iff T (propositionally) $*$-entails $prop(C(a))$ (i.e. $prop(C(a))$ is true in each $*$-model of T. This gives the result.

We shall not present the algorithm for constructing T here, see [7]. But below we illustrate it by giving some simple illustrative examples.

Our construction reduces the problems of witnessed $*$-satisfiability of C (i.e. $C(a)$ having a witnessed $*$-model) and witnessed $*$-validity of C ($(\forall x)C(x)$ being a witnessed $*$-tautology) to problems of propositional $*$-satisfiability and propositional $*$-entailment from a finite set of assumptions. These problems are decidable (see [7] for references), which gives us the following:

Theorem 2 (1) The problem of witnessed $*$-satisfiability (equivalently, of finite $*$-satisfiability) of a concept is decidable.

(2) The same for the problem of witnessed (finite) $*$-validity of a concept.

For $*$-satisfiability we can say more. By the structural characterization of continuous t-norms (Mostert-Shields theorem, see [6]), continuous t-norms can be divided into two disjoint groups:

- beginning by Łukasiewicz, i.e. for some $0 < a \leq 1$, the restriction of $*$ to $[0, a]^2$ is isomorphic to Łukasiewicz t-norm, and
- having Gödel negation ($\|\neg\varphi\|^*_{\mathbf{M}} = 1$ for $\|\varphi\|^*_{\mathbf{M}} = 0$ and $\|\neg\varphi\|^*_{\mathbf{M}} = 0$ if $\|\varphi\|^*_{\mathbf{M}} > 0$).

Theorem 3 (1) For any $*$ beginning by Łukasiewicz and each concept C, C is $*$-satisfiable iff C is Ł-satisfiable (Ł denoting Łukasiewicz t-norm).

(2) For any other $*$ (i.e. $*$ having Gödel negation) and each concept C, C is $*$-satisfiable iff C is satisfiable in Boolean logic. (Again see [7].)

3 Examples and comments

We first illustrate the transformation of $C(a)$ to a formula $prop(C(a))$ and the corresponding theory T by giving two examples. They are rather particular since they do not contain nested quantifier constructs. What we show here is just one step sufficient here but needing iteration in the general case.

Example 1 Take the concept C defined as $\forall R.(A \equiv \neg A) \to \neg\exists R.A$. Is it witnessedly $*$-valid? Equivalently: is $\forall R(A \equiv \neg A)$ $*$-subsumed by $\neg\exists R.A$? We consider the formula $C(a)$. In each witnessed $*$-interpretation, the two quantifier concepts have witnesses, i.e. for some c, d, the following formulas are $*$-true.

$$\forall R.(A \equiv \neg A)(a) \equiv [R(ac) \to (A(c) \equiv \neg A(c))],$$
$$\exists R.A(a) \equiv (R(a,d) \,\&\, A(d))$$

Since c witnesses the formula $(\forall x)(R(a,x) \to (A(x) \equiv \neg A(x))$, d must satisfy (write Rac for $R(a,c)$ etc.)

$$(1) \qquad [Rac \to (Ac \equiv \neg Ac)] \to [Rad \to (Ad \equiv \neg Ad)]$$

and similarly, since d witnesses $(\exists x)(R(a,x) \,\&\, A(x))$, c must satisfy

$$(2) \qquad (Rac \,\&\, Ac) \to (Rad \,\&\, Ad).$$

The formula $C(a)$ becomes equivalent to

$$(3) \qquad [Rac \to (Ac \equiv \neg Ac)] \to \neg(Rad \,\&\, Ad).$$

$C(a)$ is witnessedly $*$-valid iff (1) and (2) $*$-entail (3). (Indeed, if they do then take any $\mathbf{M}$ and introduce witnesses c and d, getting a model of (1) and (2); (3) follows and replacing witnesses by the corresponding quantified formulas you get $*$-truth of $C(a)$. Conversely, if you can evaluate all atoms involved by truth values in such a way that (1) and (2) are $*$-true but (3) not, you get a $*$-interpretation (having just three elements named a, c, d) in which $C(a)$ has value less then 1.

Thus first take Łukasiewicz. Put $\|Rac\| = \|Rad\| = 1$, $\|Ac\| = \|Ad\| = \frac{1}{2}$. Then (1), (2) are Ł-true, but (3) gets value $[1 \to (\frac{1}{2} \equiv \frac{1}{2})] \to \neg(1 \,\&\, \frac{1}{2}) = 1 \to \frac{1}{2} = \frac{1}{2} < 1$.

But for Gödel any formula $\varphi \equiv \neg\varphi$ is always false (has the value 0), thus (1) gives $(Rac \to \bot) \to (Rad \to \bot)$, hence $\neg Rac \to \neg Rad$ and (3) becomes equivalent to $\neg Rac \to \neg(Rad \,\&\, Ad)$. Since $\neg Rad \to \neg(Rad \,\&\, Ad)$ is a tautology (for each $*$), (1) implies $\neg Rac \to \neg(Rad \,\&\, Ad)$ which is (3). $C(a)$ is a (witnessed) G-tautology. (Note that $C(a)$ is even a tautology of Gödel logic with general models.)

Example 2 Let C be $\neg\forall R.A \,\&\, \neg\exists R.\neg A$. Let us show that $C(a)$ is not witnessedly $*$-satisfiable, whatever $*$ you take. (But recall that $C(a)$ is G-satisfiable and Π-satisfiable in a non-witnessed model, see above.) Assume witnesses c, d for $\forall R.C$ and $\exists R.\neg C$ respectively. If $\mathbf{M}$ is a witnessed model for $C(a)$, it satisfies

$$\neg(Rac \to Ac), \tag{1}$$

$$\neg(Rad \,\&\, \neg Ad), \tag{2}$$

and by witnessing,

$$(Rac \to Ac) \to (Rad \to Ad), \tag{3}$$

$$(Rac \,\&\, Ac) \to (Rad \,\&\, \neg Ad). \tag{4}$$

From (1) we get $\neg Ac$ (since $\neg(Rac \to Ac) \to \neg Ac$ is a tautology for each $*$); from (4) we get by equivalent transformations

$$\neg Ac \to [Rac \to (Rad \,\&\, \neg Ad)],$$

hence we get $Rac \to (Rad \,\&\, \neg Ad)$ and by (2) we get $\neg Rac$, hence $Rac \to Ac$, which is a contradiction with (1). $C(a)$ has no *witnessed* model.

$*$

Now let us return to the title of the paper: *What does mathematical fuzzy logic offer to description logic?* I hope the reader will agree that it offers:

- rich languages, allowing many choices (approaches), but still decidable (whereas of course for each continuous t-norm, the full predicate logic given by $*$ is undecidable)
- precise syntax and semantics (plus deductive systems, not described here, see [6])
- interesting research problems (see below).

In this context let us mention Straccia's paper [13] where the author develops a description logic based on some continuous t-norms and makes several restricting assumptions among them he assumes the inter-definability of quantifiers: $(\forall x)\varphi(x) \equiv \neg(\exists x)\neg\varphi(x)$ and dually. His assumptions are satisfied by Łukasiewicz t-norm and one can show that each continuous t-norm satisfying them is (isomorphic to) Łukasiewicz. (If $*$ is not isomorphic to Ł then either for some $a < 1$ its restriction to $[0, a]^2$ is isomorphic to Ł or it has Gödel negation, then take any $0 < u < 1$. Take an interpretation with $M = \{a\}$ and $r_P(a) = u$. Then $\|(\forall x)P(x)\|^*_{\mathbf{M}} = \frac{1}{2}$, $\|\neg P(a)\|^*_{\mathbf{M}} = 0$, thus $\|(\exists x)\neg P(x)\|^*_{\mathbf{M}} = 0$,

$\|\neg(\exists x)\neg P(x)\|^*_{\mathbf{M}} = 1 = \frac{1}{2}$.) Łukasiewicz fuzzy logic is very important but one should know whether one restricts himself to it or one admits more possibilities. This is only a remark illustrating that knowledge of mathematical fuzzy logic may be helpful for developing fuzzy description logic.
What is (still) missing? (Problems.)

- Allowing truth constants into the language. This works well expecially for Łukasiewicz logic but can be done in the general. This needs further study.
- Allowing generalized quantifiers as "many", e.g. (Many$R.C$)(a) could say "many objects R-related to a have C". This *differs* from the fuzzy $(\forall x)(R(a,x) \to C(x))$. Cf. [6].
- Problems of computational complexity for our fuzzy description logic, in analogy to "classical" description logic
- Problems of implementation.

This shows that, as said above, our approach to fuzzy description logic is a reasonable and interesting research topic. Any comments are welcome.

Acknowledgement

This paper is a part of the project number 1ET100300419 of the Program of the Information Society – Thematic Program II of the national Research Program of the Czech Republic. The work was partly supported by the Institutional Research Plan AV0Z10300504.

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