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Abstracts

Preferences, Games and Decisions

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Abstracts

János Fodor, Erich Peter Klement, Marc Roubens
Editors

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ABSTRACTS

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Since their inception in 1979 the Linz Seminars on Fuzzy Sets have emphasized the development of mathematical aspects of fuzzy sets by bringing together researchers in fuzzy sets and established mathematicians whose work outside the fuzzy setting can provide direction for further research. The seminar is deliberately kept small and intimate so that informal critical discussion remains central. There are no parallel sessions and during the week there are several round tables to discuss open problems and promising directions for further work.

LINZ 2006 will be already the 27th seminar carrying on this tradition, will be devoted to the mathematical aspects of “Preferences, Games and Decisions”. As usual, the aim of the Seminar is an intermediate and interactive exchange of surveys and recent results.

János Fodor
Erich Peter Klement
Marc Roubens

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Contents

Fuzzy Truth, Partial Truth, and Games

Joseph M. Barone

321 East 43rd Street, #209
New York, NY 10017 USA
secretary@nafips.org

We begin with the category **Sets²** as defined in [1] (see esp. p. 25 and pp. 35-36). This is the category whose objects are functions from one set to another, denoted by $\sigma: X \to X'$. The arrows then are commutative squares between the objects (the σ's). For **Sets²**, the subobject classifier Ω (the set of truth values) becomes the arrow $\sigma: \{0, 1, 2\} \to \{0, 1\}$ which sends 0 to 0 (true, σ0), 1 to 0 (true, σ1) and 2 to 1 (false, σ2).To understand this, note that a subobject of an object $\sigma: X \to X'$ would be an arrow $\sigma: S \to S'$ where $S \subseteq X$, $S' \subseteq X'$ and $\sigma(S) \subseteq S'$. The characteristic function then maps $X(x) \to X'(x)$ to σ0 if $x \in S$, to σ1 if $x \notin S$ but $\sigma(x) \in X'$, and to σ2 if $x \notin S$ and $\sigma(x) \notin X'$. Thus, as is noted in [1], the characteristic function tells us whether "x is in S" is true always, just at 1, or never. The question is exactly how do we come to this particular subobject classifier? Why, for instance, could we not simply use (say) $\rho: \{0, 1\} \xrightarrow{id} \{0, 1\}$ by analogy to the Ω for sets ({0, 1})?

The obvious answer is that using ρ as the characteristic function would conflate the status of the elements in X that come to be in S when σ is applied to X (the elements true "just at 1") with the status of the elements that do not come to be so (the elements that are true from the beginning). But why should this matter? Because then there can be no suitable characteristic function from $X \to X'$ to $\{0, 1\} \xrightarrow{id} \{0, 1\}$for a subobject whose underlying morphism takes some x not in S to S'. We need the characteristic function to take all members of S to 0 (true) and all members of S' to 0 (also true) so that the "truth function" square will commute. But this will not be the case: one side of the square will take $\sigma: S \to S'$ to 1 to $0 \to 0$ (true) only for $x \in S$, while the other side will take the subset $\sigma: S \to S'$ to $\sigma: X \to X'$ which will need to be mapped to $\{0, 1\} \xrightarrow{id} \{0, 1\}$. All we can do here is map S to 0 and S' to 0, but this will "leave out" $x \to x'$ ($x \notin S$, $x' \in S'$) since it would require a morphism in Ω which takes 1 to 0, and we have not provided one. Thus the structure of **S²** itself dictates the Ω chosen in [1]. Remember also in this regard that the characteristic morphism (function) ϕ_S must be a component of a pullback, that is, that the monomorphism is the pullback of *true* $(1 \to \Omega)$ along ϕ_S. In **Sets**, for instance, it is easy to see that this is correct. Choose a monomorphism (subset) S of a set X and call the characteristic function of X for subset S X_S. Then clearly all other commutative squares to Ω through X_S will factor through S since their origins will be subobjects (subsets) of S. It is not possible in **Sets** for $S' \to 1$ (true)$\to \Omega$ to commute with $S' \to X$ $(X_S)\to \Omega$ if S' is not a subset of S.

It is useful to examine another category, a sort of recursive version of **S²**, which requires an infinite number of truth values, i.e., an infinite Ω which is nonetheless unique. Consider the category
ε
of endomaps of sets, from [2], whose objects are single sets equipped with an endomap and denoted (X; α). By reasoning very similar to that described above for **S²**, it is apparent that for any subobject $(S, \alpha|S)$ of (X; α) (i.e., a subset of X closed under α), an element in X may be taken to S by one application of α (x is in S), by two (i.e., $\alpha \circ \alpha$), by three ($\alpha \circ \alpha \circ \alpha$), by any number of applications ($\alpha \circ \alpha \circ \ldots \circ \alpha$) or never. And each of these iterations creates another subobject of (X; α) with the same subset S, so all of these must be classified Thus, an element of X must be classified, relative to a particular subobject, by the number of iterations of the endomorphisn required to get it into the

subset of the subobject, and the unique Ω therefore is (N; γ), where N is the set of natural numbers and γ takes the natural number n to n-1. Thus $x \in X$ is mapped to 0 in Ω if it is an "original" member of S in X, to 1 if it becomes a member (i.e., a generalized element - see, e.g., [2], pp. 8-9) of S after one iteration of α, and so on. An x in X mapped to ∞ never becomes a member of S (no matter how many times α is iterated), so ∞ represents completely false. There are, in other words, an infinite number of truth values for the category $\boldsymbol{\varepsilon}$, yet Ω is nevertheless unique and motivated entirely by the structure of the category. Following [2], we refer to the degrees of truth assigned to elements of $\mathbf{S}^2$ or $\boldsymbol{\varepsilon}$ as "partial truths."

Before turning specifically to fuzzy sets and their subobject classifiers, we provide some background to make sure all preliminaries are clear. First of all, by a presheaf on a category **C** we mean a *contravariant* functor from **C** to **Sets** (i.e., $\mathbf{Sets}^{\mathbf{C}\mathrm{op}}$, as in [1]) and not a covariant functor from **C** to **Sets** as in [2]. Thus, generally speaking, any reference to [2] will require an implied *mutatis mutandis*. Since we will want to think of fuzzy sets as functors (presheaves, members of $\mathbf{Sets}^{\mathbf{C}\mathrm{op}}$), it is important to keep in mind that in $\mathbf{Sets}^{\mathbf{C}\mathrm{op}}$, all subfunctors are subobjects and conversely (see [1], p. 36). By straightforward application of the Yoneda lemma (as in [1], p. 37), the set of "truth values" for an object C of **C** in a presheaf must be (isomorphic to) the set of subfunctors of $\mathrm{Hom}_{\mathbf{C}}(-, C)$. Of course, this is $\boldsymbol{\Omega}$(C) and will be sufficient to classify only $\mathrm{Hom}_{\mathbf{C}}(-, C)$; to classify subobjects of objects in $\mathbf{Sets}^{\mathbf{C}\mathrm{op}}$ in the general case we need $\boldsymbol{\Omega}$(.) for each object in **C**.

A helpful way to view the elements of $\boldsymbol{\Omega}$(C) is as sieves (ibid.). A sieve S on C in **C** is a set of arrows in **C** with codomain C s.t. if f is in S and $f \bullet h$ is defined then $f \bullet h$ is also in S. For a poset, a sieve is simply a set of elements $B \leq C$ s.t. if $A \leq B \in S$ then $A \in S$. In any locally small category, the sieves on C are the same as the subfunctors of $\mathrm{Hom}_{\mathbf{C}}(-, C)$ ([1], p. 38). Now take any presheaf $\mathbf{Sets}^{\mathbf{C}\mathrm{op}}$, any member functor P and subfunctor Q (not necessarily hom-functors), and any morphism f: $A \to C$ in **C**. Then f determines a function P(f): $P(C) \to P(A)$ in **Sets**. For any given x in P(C), P(f) may take x into Q(A) or it may not, and the set $\{f \mid x \bullet f \in Q(\mathrm{dom}(f))\}$ is a sieve on C where f ranges over all morphisms with codomain C. This sieve is the "set of all those paths f to C which translate the element x of P(C) into the subfunctor Q." ([1], p. 39) It is also, therefore, the "truth value" of x. If $x \in Q(C)$, then this operation will yield the maximal sieve on C; if $x \notin Q(C)$, then this operation will yield some other sieve on C which is not maximal and which may be the empty sieve.

We now turn specifically to subobject classifiers for fuzzy sets. Let us begin with the subobject classifier for the categories **Set H**and **Mod H** as described by Wyler in [3] (p. 255). Recall that an H-set is a pair (|A|, δ_A), where H is a complete Heyting algebra, |A| is a set and δ_A is a symmetric and transitive mapping from |A| x |A| to H. Degree of membership in an H-set is given by $\varepsilon_A = \delta_A(x, x)$. Thus H-fuzzy sets are (can be) totally fuzzy sets (both membership and equality are fuzzy). An H-valued fuzzy relation f: $A \to B$ is an extensional mapping f: $|A| \times |B| \to H$. The category **Set H** is defined, then, to be the category with H-sets as objects and H-valued relations as morphisms, and **Mod H** to be the category with H-sets as objects and H-valued relations as morphisms, but only those induced mappings of the underlying sets. We shall focus on **Set H** here.

The easiest way to describe subobject classification in **Set H** is in terms of H-subset structures. For a given H-set A in **Set H**, an H-subset structure of A is a mapping $\alpha : A \to H$ where $\alpha(x) \leq \varepsilon_A(x)$ and $\alpha(x) \wedge \delta_A(x, x') \leq \alpha(x')$ $\forall x, x'$ in |A| ([3], p. 249). An H-subset A_α of A, then, with the given H-subset structure α, is the H-set (|A|, δ_α), where $\delta_\alpha(x, x') = \alpha(x) \wedge \delta_A(x, x')$. An injective morphism can easily be constructed which takes A_α to A. Now take the set of truth values Ω to be the elements of the complete Heyting algebra H. Then the characteristic morphism ch j_α for A_α is the morphism induced by α, that is, ch j_α (x, a) = $\varepsilon_{A\alpha}(x) \wedge \delta_\Omega(\alpha(x), a)$. This means that for each x in |A|, its truth value is a fuzzy set on H whose membership value at each a in H is either $\varepsilon_{A\alpha}(x)$ or a (cf. [3], p. 255; this simplification follows in particular from the fact that all relations in **Set H** must be total, i.e., $\vee_{y\in|B|}f(x, y) = \varepsilon_A(x)$ ([3], p. 244)). The min operator and the definition of an H-subset structure guarantee that the fuzzy truth value of any H-subset A_α of A will be ≤ the fuzzy truth value of the H-set A taken as a subset of itself. These truth values are, of course, equivalent to sieves in the functor category $\mathbf{Sets}^{\mathbf{Hop}}$. Note that this is "sheaf-theoretic" truth; truth here is not "partial" in the sense of truth in $\mathbf{S}^2$ or $\boldsymbol{\varepsilon}$ as described above. We shall refer to this kind of truth here as "fuzzy truth" to distinguish it from the notion of partial truth just mentioned. A crucial difference here is that fuzzy truth does not seem to reflect the idea of "stages of truth" or temporal truth ([2], esp. Section 3.3).

With this background, we now ask what might be the relevance of partial truth and fuzzy truth to the areas of decision theory and game theory. This issue has been raised in a general way by Voinov in [4]. He observes that in many decision making contexts, strict "numerifications" of similarity data (e.g., projections onto metric spaces) are not appropriate or illuminating. "Nearness" relations (basically, suitable subsets of the cartesian product) , on the other hand, may capture much more closely the actual similarities of the system under study, but they are likely to have more rudimentary mathematical structure (say the structure of a pre-uniformity). Another aspect of decision making , Voinov points out, is the notion of cognitive spaces or "regions of evidence" relative to which similarities and other kinds of evidence may be interpreted. These spaces are usually explained "modally," i.e., as manifestations of possible worlds in modal logics, including fuzzy modal logics. The difficulty here, in Voinov's opinion, is that once again the construction of possible worlds and making choices among them requires arbitrary application of numerical techniques.

Voinov suggests that the correct level of mathematical generality for both "connectivity" (possible world choices) and "similarity grade" relationships is that of a topos. More than that, he notes that this incorporates both notions into a single mathematical framework, since the object structure of the topos (the topology) provides a framework for possible worlds and their interrelationships, while the subobjects and subobject classification provide a basis for similarity grades. In other words, in a topos, there is no need to provide separate formalisms for the set of possible worlds and the set of truth values.

Now let us consider briefly fuzzy games, in particular the variety known as fuzzy moves [5]. The (crisp) theory of moves (TOM) was originally developed by S. J. Brams [6]; a nice description of the basic framework may be found in [7]. In "basic" TOM, each game is 2x2, each player is assigned two actions (strategies), and at her turn, a player may move (go to the next strategy) or not move (retain the current strategy). An equilibrium, known as a non-myopic equilibrium (NME) is achieved when a player decides not to move. Players are given a starting state, and are expected to look ahead as far as necessary to determine their next move. This means, as Ghosh and Sen [7] point out, that it is not actually necessary to play the game to determine the NME. It is important to note that only relative payoffs need be used, and also to note that while access to complete information is assumed in the original formulation, it is possible (see [7]) to allow the players to learn their opponent's preferences dynamically. In [6], Brams exhaustively enumerates all 78 possible 2x2 games, and so 2x2 TOM games are usually referred to by their position in Brams' list, e.g. "game 23". It is apparent that a 2x2 TOM can be thought of as a member of the category $\boldsymbol{\varepsilon}$ of endomaps of sets described above.

In [5], Kandel and Zhang suggest that TOM games can be made more complete by adding a fuzzy component. This component consists of an assignment of a value in [0, 1] for each player to each payoff in the 2x2 game; the assigned value represents the overall desirability of the state for that player. This permits a more comprehensive and (according to [5]) more realistic situation in which players attempt to maximize both the order (local) payoff and the fuzzy (global) payoff for a game. The payoff for a given state (a^{-}_{ij}), then, is a "transformation function" [5] of the ordinal (now fuzzified) payoffs of the players and the global (fuzzy) goal α for player A or β for player B, i.e., for player A, $a^{-}_{ij} = F(a_{ij}, b_{ij}, \alpha)$ where i, j ε {1, 2}. The strategies at each row and column intersection in the crisp 2x2 game (the pair of ordinals (a_{ij}, b_{ij}) for row i and column j are transformed in the fuzzy game, then, to (a^{-}_{ij}, b^{-}_{ij}), and the decision to move or not to move is made on the basis of these new (fuzzy) values.

It is apparent that totally fuzzy sets, and hence the category **Set H**, are appropriate models for and generalizations of the theory of fuzzy moves (TFM) of [5] just described. For player A, the membership value of a state represents its global goal, i.e., $\mu(a_{ij}) = \varepsilon(a_{ij}) = \delta(a_{ij}, a_{ij}) = \alpha$ and $\mu(b_{ij}) = \delta(b_{ij}, b_{ij}) = \beta$, while $\delta(a_{ij}, a_{kl})$ represents the local payoff or preference of a_{ij} relative to a_{kl} expressed in terms of equality, i.e., if $\delta(a_{ij}, a_{kl})$ is small, then state a_{ij} and state a_{kl} are far apart from each other in (ordinal) value, and vice versa. The original (crisp) ordinal ranks may then be recovered from the membership values and the equalities taken together. Given two H-sets A and B, the global payoffs a^{-}_{ij} are now given by morphisms in **Set H**, from A to B for the A component of the payoff, and from B to A for the B component. The required properties of a morphism in **Set H** ensures that these global payoffs will be functionally related to the global goals and the local payoffs. Thus, for instance, the requirement that morphisms be total guarantees that the value in H of any f: A $\rightarrow$ B will be $\leq \varepsilon(a)$ for every a in |A| (see [3], pp. 244-245).

It should be emphasized that totally fuzzy sets in **Set H** constitute both a model and a generalization of TFM, or at least of the normal TFM game with normal global goal (see [5], p. 165). Thus, for instance, TFM allows just a single global goal, whereas (obviously) each state (member of |A|) in A can have a different global goal. On the other hand, using sets and morphisms in **Set H** for TFM imposes certain constraints on various aspects of TFM which are not necessarily inherent in the formulation of [5]. One such constraint, viz., that morphisms be total, was mentioned in the previous paragraph. Similarly, the requirement of extensionality ([3], p. 243) guarantees that the value in H of any f: A $\rightarrow$ B will always be $\leq \varepsilon(b)$. These constraints seem reasonable and natural, but may yield a somewhat different set of global preferences than the original TFM. Finally, it should be noted that given Proposition 74.6 and Corollary 74.6.1 of [3] (p. 251), there is an important connection between H-subsets in **Set H** and morphisms in **Set H**. This suggests an important truth-functional connection between "allowed" global payoffs and the underlying local payoffs and global goals. This in turn may lead to a statement of conditions for stronger kinds of equilibria than NME in TFM games, as well as a clear statement of the way fuzzy truth (in the sense the term was used above) may subsume the partial truth of [2].

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Fuzzy Transitive Relations

Ismat Beg and Samina Mazhar
Centre for Advanced Studies in Mathematics and
Department of Mathematics,
Lahore University of Management Sciences,
54792-Lahore, Pakistan
ibeg@lums.edu.pk

Max-min transitivity faces a lot of criticism (see [1, 2, 3]). This criticism seems to be fair particularly because the existing definition decides whether a relation is transitive or not? In this note we are more interested in knowing how much transitive it is? This is what one should call fuzzy transitivity. This transitivity can be extended from crisp transitivity by fuzzifying the operators involved.

If one looks at the definition of crisp transitivity: A binary relation *R* on a set *X* is transitive if and only if $(\forall(x, y, z)\in X^3)((xRy \wedge yRz)\Rightarrow xRz)$. The two operators used are conjunction and implication, both of which have been extended to their fuzzy counterparts by this time (implicators were not converted to fuzzy ones at the time when Zadeh [4] defined his max-min transitivity). To get fuzzy transitivity one should fuzzify the operators used.

DEFINITION 1: (*pointwise*) Let *R* be a fuzzy relation on *X*. A *transitivity function* $Tr: X \times X \to [0,1]$ is defined by

$$\mathrm{Tr}(x, z) = \inf_{y\in Y} I(T(R(x, y),R(y, z)), R(x, z)).$$

Where *I* is the implicator corresponding to the t-norm *T*.

Transitivity function Tr, so defined is a pointwise defined function, which assigns a degree of transitivity to the relation at each point of $X \times X$. According to Definition 1, the given transitive fuzzy relation have different degrees of transitivity at different pairs of points. This fact when incorporated into similarity or indistinguishbility may be interpreted as “different pairs of points may be less or more similar under the same relation”. Now, the problem is to decide how much transitive a fuzzy relation is? The most natural answer seems to take the inf over all the points $x, y, z \in X$.

DEFINITION 2: Let *R* be a fuzzy relation on *X*. *Fuzzy transitivity of R* is a function defined as

$$Tr(R) = \inf_{(x,y,z)\in X^3} I(T(R(x, y),R(y, z)), R(x, z)) = \inf_{(x,z)\in X^2} \mathrm{Tr}(x,z).$$

Where *I* is the implicator corresponding to the t-norm *T*.

REMARK 3: A fuzzy relation *R* will be called *non-transitive* if $Tr(R) = 0$, and it will be called *strongly transitive* if $Tr(R) = 1$. The first case is basically dealing with crisp non-transitive relations as a case of the fuzzy one, and in the later case *R* is going to have a transitivity value 1 at each of it's triplet of points which means crisp transitivity

REMARK 4: Where is Zadeh's definition of transitivity placed in this situation?

Before going onwards let us name for the sake of convenience in calculation:

$$a = R(x, y),\ b = R(y, z),\ \text{and } c = R(y, z).$$

For any $x, y, z \in X$.

$$Tr(x, y, z) = I(T(a, b), c),$$

(where *a, b, c* are points dependent). Zadeh [4] assumed the first variable of the implicator to be smaller than the second one i.e. $T(a,b) \leq c$ which leads to a value 1 for almost all the implicators we prefer to use. So while working with these implicators Zadeh's definition is the second case stated in remark 3 i.e. the crisp transitivity. Some of the implicators may give values other than 1 in case of taking max-min transivity valid. The study of such situations is in progress.

REMARK 5: What happens to the so defined equivalence relation?

Next step after having defined a fuzzy transitivity, is fuzzy equivalence relation. That is equivalence up to a certain degree. That should be the exact picture of indistinguishbility i.e.; How indistinguishable two points are with respect to the pseudo-metrics associated with equivalence relation? If they are totally indistinguishable than an interesting fact is that similarity so defined conforms following three properties:

1. If *R* is a similarity relation then $\forall\ (x,y,z) \in X^3$, at least two of the degrees *R(x,y), R(x,z)* and *R(y,z)* are equal.
2. *R* is a similarity relation if and only if for all $\alpha \in]0,1]$ the α-cut R_α is an equivalence relation.
3. *R* is a similarity relation if and only if complement of *R* is a *[0,1]*–valued pseudo-ultra metric on *X*.

There is another beautiful aspect that the whole theory defined earlier remains valid and becomes a part of the new one for example transitive closures will now be defined as the fuzzy binary relation with a value of transitivity 1 which contains the given relation. Moreover for a fuzzy binary relation *R* with transitivity value equal to 1, $\overline{R} = R$ holds.

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An interdisciplinary approach to coalition formation

Rudolf Berghammer[1], Patrik Eklund[2], Marc Roubens[3,4], Agnieszka Rusinowska[5,6], and Harrie De Swart[7]

[1] Institute of Computer Science, University of Kiel
24098 Kiel, Germany
rub@informatik.uni-kiel.de
[2] Umeå University, Department of Computing Science
90187 Umeå, Sweden
peklund@cs.umu.se
[3] University of Liège, Institute of Mathematics, B37, 4000 Liège, Belgium
m.roubens@ulg.ac.be
[4] Faculté Polytechnique de Mons
7000 Mons, Belgium
[5] Radboud University Nijmegen, Nijmegen School of Management
6500 HK Nijmegen, The Netherlands
a.rusinowska@fm.ru.nl
[6] Warsaw School of Economics, Department of Mathematical Economics
02-554 Warsaw, Poland
arusin@sgh.waw.pl
[7] Tilburg University, Faculty of Philosophy
5000 LE Tilburg, The Netherlands
H.C.M.deSwart@uvt.nl

This paper concerns an interdisciplinary approach to coalition formation. We apply the MacBeth software (see also [7]), relational algebra, the RelView tool (see [3]), graph theory, bargaining theory, social choice theory (see [4]), and consensus reaching (see also [6]) to the model of coalition formation introduced in [9].

In [9], the notion of a feasible stable government is central. Roughly speaking, a feasible government is a pair consisting of a (majority) coalition of parties and a policy supported by this coalition. Different governments may have different utilities (values) for different parties. Stability of a feasible government means that it is not dominated by another feasible one.

MacBeth, which stands for 'Measuring Attractiveness by a Categorical Based Evaluation Technique', is an interactive approach to quantify the attractiveness of different alternatives, in such a way that the measurement scale constructed is an interval scale; see [1], [2] or www.m-macbeth.com. The MacBeth technique may be applied to many real life situations, and it appears to be a very useful tool also for coalition formation. In [7], we present an application of the MacBeth approach to the model of coalition formation presented in [9]. The MacBeth software increases the applicability of the coalition formation model considerably. We use the MacBeth technique to quantify the attractiveness and repulsiveness of feasible governments to parties. Using MacBeth, every party judges the difference of attractiveness between each two policies on a given issue (including the majority coalitions). The MacBeth software signals when the matrix of judgements of a party becomes inconsistent, and it gives suggestions to make it consistent. We use the MacBeth software to calculate the utilities of governments to parties. Based on these utilities, stable governments are determined. In the original model presented in [9], a party is assumed to express its preferences very precisely. However, the MacBeth tool enables us also to deal with fuzzy preferences.

Since some decades relational algebra is used successfully for formal problem specification, prototyping, and algorithm development. Also, relational algebra seems to be promising for computeraided investigations of coalition formation. In [3], we present an application of relational algebra to coalition formation. We formulate the notions of feasibility, dominance, and stability for governments in relation-algebraic terms. Feasibility of a government can be described by two relations, which state whether a party accepts a coalition and whether a party supports a policy. Stability can be defined in terms of the 'is-part-of' relations between parties and governments, the dominance relation on governments, and a list of relations comparing governments with respect to the utility of parties. This enables us to use RelView, a tool for the visualization and manipulation of relations and for prototyping and relational programming, to compute the dominance relation and the set of all feasible stable governments. To illustrate the power of the approach, we solve an example based on the structure of the Polish government after the 2001 elections.

A stable government is by definition not dominated by any other government. However, it may happen that all governments are dominated. In [4], we deal with the problem what to do when there is no un-dominated government. We combine concepts from graph theory, bargaining theory and social choice theory to solve this problem. Using graph-theoretic terms, the non-existence of a stable government means that the dominance graph does not possess a source. Using concepts of graph theory (initial strongly connected components, minimum feedback vertex sets), we present a procedure for choosing one government if the set of all stable governments is empty. As in [3], also in [4] the decisive parts of our procedure are formulated as relational expressions and programs, respectively, so that RelView can be used for executing them and for visualizing the results. Given a dominance graph without a source, first we compute all initial strongly connected components. Next, for each initial strongly connected component, we compute the set of all minimum feedback vertex sets, where a feedback vertex set is a minimal set of vertices which breaks all cycles. Next, we choose a specific minimum feedback vertex set according to the following rule. First, we choose the set(s) for which the number of ingoing arcs is maximal. Since an ingoing arc denotes that a government is dominated, such a choice means selecting governments dominated most frequently. Next, if there are at least two such sets, we choose the one(s) for which the number of outgoing arcs is minimal, meaning the choice of the governments which dominate other governments least frequently. Next, we break all cycles by removing the chosen set of governments. One may say that we remove governments which are least attractive for two reasons: because they are most frequently dominated and they dominate other governments least frequently. According to our procedure, if there is more than one initial strongly connected component, we select the final stable government (from the results of the procedure described above) by applying bargaining or some well-known social choice rules. Concerning the application of bargaining, we use several bargaining games (defined in [8]) and choose the government which is a subgame perfect equilibrium result. Concerning the application of social choice theory, we apply the plurality rule, the majority rule, the Borda rule, or approval voting. Of course, some of these applications may also lead to a non-unique solution. In this case, we propose to combine several techniques and to apply a several-steps method consisting of, for instance, a social choice rule in the first step, and a bargaining game in the second step.

In the model presented in [9], a party evaluates all governments the party belongs to with respect to some criteria. We allow the criteria to be of unequal importance to a party. These criteria concern majority coalitions and policy issues. The parties' preferences are supposed to be constant, and no possibility of adjusting the preferences of a party is considered. In [6], we introduce a dynamic model of coalition formation in which parties may compromise in order to reach consensus. We apply a consensus model analyzed in [5], where the authors study the problem of formalizing consensus, within a set of decision makers trying to agree on a mutual decision. By combining some notions of both the consensus model [5] and the model of a stable government [9], a new consensus model of political decision-making is constructed. Parties may be advised to adjust their preferences, i.e., to change their evaluation concerning some government(s) or/and the importance of the criteria, in order to obtain a better political consensus. If parties are willing to compromise, it is always possible to reach consensus, and to create a feasible government. In the procedure there is an ‘outsider’, called the chairman, who advises parties how to adjust their preferences.

First, each feasible coalition tries to reach consensus within this coalition about the government to be formed. Parties consider only feasible governments, i.e., governments acceptable for all parties belonging to the coalition involved, and if there is only one feasible government they can form, they agree. If the parties from a given coalition manage to reach consensus, the coalition proposes to form the government agreed upon. This consensus government is stable in the given coalition with respect to the set of all feasible governments formed by that coalition.

If there are at least two coalitions that succeed in reaching consensus, that is, if at least two governments are proposed, we select the governments which are ‘internally stable’. Next, if there are at least two such internally stable governments, some extra procedures are applied in order to choose one of these governments. The protocol given in [6] can be mechanized, resulting in a decision support system for coalition-government formation. The informational requirements of the proposed protocol are demanding, but the MacBeth software can deliver all information needed in a very rational way.

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The Möbius Value: A Generalized Solution for Cooperative Games⋆

Antoine Billot[1,2] and Jacques-François Thisse[2]

[1] Université Paris 2 and Institut Universitaire de France
billot@u-paris2.fr
[2] CORE Université catholique de Louvain, Belgium, and CERAS-ENPC (Paris)

Abstract. All quasivalues rest on a set of three basic axioms (efficiency, null player, and additivity), which are augmented with positivity for random order values, and with positivity and partnership for weighted values. We introduce the concept of Möbius value associated with a sharing system and show that this value is characterized by the above three axioms. We then establish that (i) a Möbius value is a random order value if and only if the sharing system is stochastically rationalizable and (ii) a Möbius value is a weighted value if and only if the sharing system satisfies the Luce choice axiom.

1 Introduction

The general question raised by any cooperative game can be described as follows: how should the utility sets available to all coalitions be used to determine an outcome from the set of feasible solutions? So far, no single solution-concept has emerged that satisfies everyone's sense of equity (Moulin, 1988). Yet, there seems to be a large agreement to consider the Shapley value as one of the most appealing solutions (Shapley, 1953). However, when players do not stand behind the veil of ignorance, this solution is no longer valid. Various concepts have then been proposed to deal with social and economic contexts in which players have idiosyncratic rights in sharing the final outcome (see Monderer and Samet, 2002, for a recent survey). All these solutions rest on a common set of three basic axioms (efficiency, null player, and additivity), which are augmented with positivity by Weber (1988) in the case of random order values, and with positivity and partnership by Kalai and Samet (1987) for weighted values. In this paper, we restrict ourselves to these three basic axioms only and characterize the set of corresponding values that we call *Möbius values*.

The extensions of the Shapley value allow for a redistribution of the total worth according to two dimensions: the marginal contribution of each player within all possible coalitions and a *sharing system* which is given a priori. The idea behind the sharing system is that the reward of a player may be related to her marginal contribution to each coalition in various ways. This aims at capturing the fact that a society may be governed according to a large variety of distributive rules, which are themselves based on various principles of justice (Bentham, Rawls, etc.). For example, in the theory of cooperative values, Kalai and Samet (1987) attribute a given weight to each player that expresses her power within each coalition whereas, in Weber (1988), the weight depends on the relative place of the player in society endowed with different orderings. As will be shown in this paper, the additional axioms that have been introduced in the literature (positivity and partnership) do actually restrict in a fairly strong manner the admissible sharing systems. More precisely, we will see that *the existing values*, called quasivalues, *are such that the sharing rule within a particular coalition is constrained by the way the sharing rule is defined within all broader coalitions (and vice versa)*. Put differently, saying how to share within the grand coalition tells us how to share within all subcoalitions. In practice, the

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existence of such a master sharing rule may be problematic because it requires the implicit agreement of all players about it. By contrast, *our approach allows for sharing systems that are independent of the existence of such a master sharing rule and is, therefore, more general*. Yet, we need some minimal requirement linking sharing in a coalition and sharing in its subcoalitions. In this respect, we suggest that the way the worth of a coalition is shared is such that, in all its subcoalitions, players' shares cannot be lower than what they are in the referential coalition. Given this caveat, sharing within a particular coalition need not be related to the way the worth of any subcoalition or supercoalition is distributed among its members. In other words, by being the members of a coalition, the corresponding players find themselves in a particular sharing context that defines what each of them will receive. It is in that sense that the Möbius value allows for sharing when context matters.

In our paper, the Möbius value of a player is given by a linear combination of the pure contribution of her cooperation within all coalitions including her; the coefficient associated with each coalition is the share that this player can claim in this coalition. By "pure contribution of cooperation" (**PCC**), we mean the net reward of cooperation within a coalition after having discounted for what cooperation brings about in all possible proper subcoalitions. Formally, the **PCC** of a coalition is the Möbius inverse of the characteristic function of the game. Focussing on the **PCC** of a coalition, instead of the marginal contribution of its members, concurs with our idea that a coalition defines a specific sharing context, which is a priori independent of all possible subcoalitions. Moreover, the coefficients of the linear combination define a probability over the corresponding coalition, but they need not be "consistent" across coalitions. By contrast, we identify two forms of consistency of the sharing system used in quasivalues. First, *a random order value is such that the sharing system is stochastically rationalizable*, that is, there exists a probability distribution defined over all orderings on the set of players which yields the sharing system (Block and Marschack, 1960). Second, *a weighted value*, as introduced by Kalai and Samet (1987), *is such that the sharing system satisfies the more demanding condition given by the Luce choice axiom used in discrete choice theory* (Luce, 1959). This axiom says that each sharing rule may be viewed as the Bayesian restriction of a master distribution defined on the set of players. Hence, our approach to cooperative values allows us to characterize each quasivalue by means of restrictions imposed on the corresponding sharing system.[3]

The remainder of this paper is organized as follows. Definitions and notation are given in Section 2. The concept of a Möbius value is defined and axiomatically characterized in Section 3 (Theorem 1). The relationships with quasivalues are explored in Section 4 where the following results are proven: (i) a Möbius value is a random order value if and only if the sharing system is stochastically rationalizable (Theorem 3) and (ii) a Möbius value is a weighted value if and only if the sharing system satisfies the Luce choice axiom (Theorem 4). In Section 4, we prove that a Möbius value is positive if and only if the game is monotone (Theorem 5) and that the set of Möbius values is the core if and only if the game is convex (Theorem 7). Section 6 concludes.

2 The Pure Contribution of Cooperation in a TU-Game

A *cooperative game with transferable utility* (TU-game) is a pair (Z, ν) where Z, the *grand coalition* with $\sharp Z = n$, is defined by a finite set of *players* and ν, the *characteristic function*, is defined by a mapping from 2^Z to $\mathbb{R}$ such that $\nu(\emptyset) = 0$. Any subset Y of Z is called a *coalition* and for any nonempty coalition Y, we denote $Z \backslash Y$ by $\overline{Y}$, $Y \backslash \{i\}$ by Y_{-i}, $Y \cup \{i\}$ by Y_{+i} and $2^Y \backslash \emptyset$ by $2^Y_{-\emptyset}$.

[3] These results also uncover some new connections between cooperative values and probabilistic discrete choices, a topic which has already been under investigation (Monderer, 1992; Gilboa and Monderer, 1992).

The set of TU-games whose set of players is Z is given by the vector space $\mathbb{R}^{2^Z_{-\emptyset}}$. A characteristic function ν is *monotone* if $\nu(X) \leq \nu(Y)$ for every $X \subset Y$ and *convex* if $\nu(X \cup Y) + \nu(X \cap Y) \geq \nu(X) + \nu(Y)$ for every pair $X, Y \in Z$. For convenience, all properties that are satisfied by ν on Z are said to be satisfied by the TU-game itself.

A *solution* of the game (Z, ν) is a mapping $\varphi : \mathbb{R}^{2^Z_{-\emptyset}} \to \mathbb{R}^n$. A solution $\varphi(\nu)$ is said to be *positive* when $\varphi_i(\nu) \geq 0$ for all $i \in Z$.

The above concepts are standard and we now introduce one of the new tools of this paper.

Consider any TU-game (Z, ν). Then, for any nonempty coalition Y, following Shapley (1953), there exists a unique set of coefficients $\left(\Gamma_\nu(X) : X \in 2^Y_{-\emptyset}\right)$ such that:

$$\nu(Y) = \sum_{X \in 2^Y_{-\emptyset}} \Gamma_\nu(X) \tag{1}$$

that are given by

$$\Gamma_\nu(Y) = \sum_{X \in 2^Y_{-\emptyset}} (-1)^{y-x} \nu(X) \tag{2}$$

where y and x stand for the cardinalities of Y and X, respectively. These coefficients may be interpreted as follows. Set $\nu(i) \equiv \nu(\{i\})$. If $Y = \{i, j\} \subset Z$, the worth $\nu(Y)$ may be different from $[\nu(i) + \nu(j)]$. In such a context, two cases may arise. In the first, the cooperation is "negative" because the two players are worse off when they cooperate. In the second, the cooperation is "positive" because the two players are better off when they cooperate. In both cases, it is natural to express the *pure contribution of cooperation* **(PCC)** $\Gamma_\nu(Y)$ *of* Y, also called the *dividend* of Y in Harsanyi (1963), by the difference

$$\Gamma_\nu(Y) = \nu(Y) - [\nu(i) + \nu(j)]\,. \tag{3}$$

In other words, $\Gamma_\nu(Y)$ measures the exact contribution of the cooperation inside of Y because we have accounted for the individual worthies. When $Y = \{i, j, k\}$, one might think that $\Gamma_\nu(Y)$ would be given by $\Gamma_\nu(Y) = \nu(Y) - [\nu(i) + \nu(j) + \nu(k)]$. However, this expression already includes the **PCC** of each pair $\{i, j\}$, $\{i, k\}$ and $\{j, k\}$ to the **PCC** of Y. Given (3), the **PCC** of $Y = \{i, j, k\}$ should instead be defined as follows:

$$\begin{aligned} \Gamma_\nu(Y) = &\{\nu(Y) - [\nu(i) + \nu(j) + \nu(k)]\} \\ &- \{\Gamma_\nu(i,j) + \Gamma_\nu(i,k) + \Gamma_\nu(j,k)\}\,. \end{aligned} \tag{4}$$

More generally, in view of these expressions, we define the **PCC** of a TU-game as the mapping $\Gamma_\nu : 2^Z_{-\emptyset} \to \mathbb{R}$ such that, for each coalition $Y \subset Z$, (1) and (2) hold. In words, $\Gamma_\nu(Y)$ can be interpreted as *the contribution of cooperation within the coalition Y independently of what cooperation brings about in all possible subcoalitions that could have been formed before the coalition Y is determined*. Stated differently, $\Gamma_\nu(Y)$ measures the total benefit generated by the coalition Y once we have accounted for all the possible subcoalitions formed by any proper subset of players.[4]

The **PCC** Γ_ν is equivalent to the Möbius inverse of the characteristic function ν (Rota, 1964; Chateauneuf and Jaffray, 1992). Note that, for any $Y \in 2^Z_{-\emptyset}$, we have:

$$\nu(Y) = \sum_{i \in Y} \nu(i) + \sum_{\substack{X \subset Y \\ x \geq 2}} \Gamma_\nu(X) \tag{5}$$

[4] The **PCC** of a coalition is the game-theoretic counterpart of the "contextual utility" as defined by Billot and Thisse (1999) in discrete choice theory and of the "evidence of an event" in Dempster-Shafer's theory of belief functions.

which means that the worth of a coalition is equal to the sum of the individual worthies plus the sum of the PCCs of all possible subcoalitions. In particular, for the grand coalition, we have:

$$\nu(Z) = \sum_{Y \subset Z} \Gamma_\nu(Y)$$

that is, the worth of the grand coalition is equal to the sum of the pure contributions of all possible coalitions.

In what follows, we show that the **PCC** of a coalition may be negative even when the TU-game is monotone. The same example is used throughout the paper.

Example 1: Consider the TU-game (Z, ν) such that $Z = \{1,2,3\}$ whereas its characteristic function v is defined by

$$\begin{cases} \nu(Z) = 8, \\ \nu(Z_{-i}) = 7 - i, \qquad \forall i \in Z \\ \nu(i) = i, \qquad \forall i \in Z. \end{cases}$$

This characteristic function is monotone and convex. The associated **PCCs** can be computed as follows:

$$\begin{cases} \Gamma_\nu(Z) = \Gamma_\nu(123) = 8 - (6+5+4) + (1+2+3) = -1, \\ \Gamma_\nu(Z_{-1}) = \Gamma_\nu(23) = 6 - (2+3) = 1, \\ \Gamma_\nu(Z_{-2}) = \Gamma_\nu(13) = 5 - (1+3) = 1, \\ \Gamma_\nu(Z_{-3}) = \Gamma_\nu(12) = 4 - (1+2) = 1, \\ \Gamma_\nu(1) = 1, \\ \Gamma_\nu(2) = 2, \\ \Gamma_\nu(3) = 3. \end{cases}$$

This implies that (i) the **PCC** of a pair Z_{-i} is greater than that of the grand coalition Z, (ii) the **PCC** of a pair is constant whoever is in the pair, and (iii) the **PCC** of the grand coalition is negative. Note also that

$$\begin{aligned} \nu(Z) &= \sum_{Y \subset Z} \Gamma_\nu(Y) \\ &= -1 + (1+1+1) + (1+2+3) \\ &= 8 \end{aligned}$$

while

$$\begin{cases} \nu(Z_{-1}) = \nu(23) = 1+2+3 = 6, \\ \nu(Z_{-2}) = \nu(13) = 1+1+3 = 5, \\ \nu(Z_{-3}) = \nu(12) = 1+1+2 = 4. \end{cases}$$

3 Möbius Values

3.1 Definition

The *sharing rule* of a coalition $Y \in 2^Z_{-\emptyset}$ is a probability distribution $p_Y : 2^Y \to [0,1]$ where $p_Y(i)$ corresponds to the share player $i \in Y \in 2^Z_{-\emptyset}$ may claim in coalition Y, which satisfies the following two conditions:

Individual sharing consistency : For all coalitions $X \subset Y \in 2^Z_{-\emptyset}$ and all player $i \in X$, $p_Y(i) \leq p_X(i)$.

Negligible player condition : If $p_{\{i,j\}}(i) = 0$ for some $i, j \in Y$, then for all coalitions $X \subset Y \in 2^Z_{-\emptyset}$, $p_Y(X) = p_{Y_{-i}}(X_{-i})$.

The first condition implies that, when a coalition shrinks, an individual's shares never decreases. This precludes sharing context in which the departure of one individual from a group would reduce the share the others can claim. The second condition means that a player gets a zero share in any coalition when she gets such a share in a 2-person coalition. The latter condition implies that $p_Y(i) = 0$ for each negligible player, whereas $p_{\{i\}}(i) = 1$ for each player $i \in Z$.

A *sharing system*, denoted $(Z, \mathcal{P})$, is then defined by the set Z of players and by a mapping which associates each coalition $Y \in 2^Z_{-\emptyset}$ with a sharing rule p_Y.

Apart from the two conditions above, the sharing rule p_Y depends only upon the particular redistribution context defined by the coalition. Hence, for any coalition X different from Y, p_X need not be functionally related to p_Y in the system $(Z, \mathcal{P})$. As will be seen below, this makes our approach to cooperative values more general than standard quasivalues (Monderer and Samet, 2002).

Consider a TU-game (Z, ν). We define the *Möbius value* of the player $i \in Z$ associated with the sharing system $(Z, \mathcal{P})$, denoted $\varphi_i(\nu, \mathcal{P})$ by

$$\varphi_i(\nu, \mathcal{P}) = \sum_{\substack{Y \in 2^Z_{-\emptyset} \\ Y \ni i}} p_Y(i) \Gamma_\nu(Y). \tag{6}$$

In words, the Möbius value of player i is given by a linear combination of the **PCCs** of all nonempty coalitions Y including i, where the coefficient $p_Y(i)$ associated with the coalition Y is the share that player i can claim in this coalition.[5] When the expression above holds for all sharing systems, we discard $\mathcal{P}$ in $\varphi_i(\nu, \mathcal{P})$; similarly, we denote p_Z by p.

Remark 1: For any negligible player i, we have $\varphi_i(\nu) = \nu(i)$.

Example 2: Consider a sharing rule $\mathcal{P}^*$ given by $p^*(1) = p^*(2) = p^*(3) = 1/3$, $p^*_{12}(1) = 1/3$ while $p^*_{12}(2) = 2/3$, $p^*_{13}(1) = 2/3$ while $p^*_{13}(3) = 1/3$ and $p^*_{23}(2) = 2/3$ while $p^*_{23}(3) = 1/3$. Then, the associated Möbius value $\varphi_i(\nu, \mathcal{P}^*)$ defined by (17) leads to

$$\begin{aligned}\varphi_1(\nu, \mathcal{P}^*) &= \Gamma_\nu(1)\, p^*_1(1) + \Gamma_\nu(12)\, p^*_{12}(1) + \Gamma_\nu(13)\, p^*_{13}(1) \\ &\quad + \Gamma_\nu(Z)\, p^*(1) \\ &= 1 + \frac{1}{3} + \frac{2}{3} - \frac{1}{3} = \frac{5}{3} \simeq 1.65,\end{aligned}$$

$$\begin{aligned}\varphi_2(\nu, \mathcal{P}^*) &= \Gamma_\nu(2)\, p^*_2(2) + \Gamma_\nu(12)\, p^*_{12}(2) + \Gamma_\nu(23)\, p^*_{23}(2) \\ &\quad + \Gamma_\nu(Z)\, p^*(2) \\ &= 2 + \frac{2}{3} + \frac{2}{3} - \frac{1}{3} = 3,\end{aligned}$$

and

$$\begin{aligned}\varphi_3(\nu, \mathcal{P}^*) &= \Gamma_\nu(3)\, p^*_3(3) + \Gamma_\nu(13)\, p^*_{13}(3) + \Gamma_\nu(23)\, p^*_{23}(3) \\ &\quad + \Gamma_\nu(Z)\, p^*(3) \\ &= 3 + \frac{1}{3} + \frac{1}{3} - \frac{1}{3} = \frac{10}{3} \simeq 3.35.\end{aligned}$$

[5] A related, but different, approach in terms of interaction among players forming a coalition is developed by Grabisch and Roubens (1999).

The Möbius solution of our example is therefore given by the triplet

$$(1.65, 3, 3.35).$$

Note that, in this example, p^*_{12}, p^*_{13} and p^*_{23} are *not* Bayesian restrictions of p^* onto the subsets $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$. In other words, the sharing rule p^*_{12}, p^*_{13} and p^*_{23} are independent of the master rule p^*.

3.2 Axioms

We introduce the following three axioms to characterize the Möbius value as defined by (6).

Axiom 1 (φ-Efficiency) : *Let (Z, ν) be any TU-game. Then,*

$$\varphi_Z(\nu) = \sum_{i \in Z} \varphi_i(\nu) = \nu(Z).$$

This means that the solution of the grand coalition is equal to its worth.

Axiom 2 (φ-Null Player) : *For each player $i \in Z$, if for each coalition $Y \subset Z_{-i}$ we have $\Gamma_\nu(Y_{+i}) = 0$, then*

$$\varphi_i(\nu) = 0.$$

This axiom says that the solution of an individual is zero when her **PCC** of any coalition she belongs to is always zero. Note that $\nu(i) = 0$ and $\nu(Y_{+i}) = \nu(Y)$ when i is a null player.[6]

Axiom 3 (φ-Linearity) : *Let (Z, ν) and (Z, μ) any two TU-games and $\alpha \in \mathbb{R}$. Then,*

$$\varphi(\alpha\nu + \mu) = \alpha\varphi(\nu) + \varphi(\mu).$$

For any $X \subset Z$, consider a *X-unanimity TU-game* (Z, ν^X) for which the characteristic function ν^X is defined as follows:

$$\nu^X(Y) = \begin{cases} 1 & \text{if } X \subset Y, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Lemma 1. : *For any coalition $X \subset Z$, the **PCC** Γ_{ν^X} associated with the unanimity TU-game (Z, ν^X) is such that:*

$$\Gamma_{\nu^X}(Y) = \begin{cases} 1 & \text{if } X = Y, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2. : *For any TU-game (Z, ν), we have:*

$$\nu = \sum_{X \in 2^Z_{-\emptyset}} \Gamma_\nu(X)\, \nu^X.$$

Proofs are straightforward and omitted. Note that Shapley (1953, p. 311) already proved that every characteristic function can be decomposed in a unique way as a linear combination of unanimity games (our Lemma 2).

[6] This axiom is weaker than the dummy axiom used by Shapley (1953) and Weber (1988). See Nowak and Radzik (1994) and Monderer and Samet (2002) for a discussion of the null player axiom *vs* the dummy axiom.

Lemma 3. : *Let i be any player of Z. Under* A1-A3, *for any nonempty Y, we have:*

$$\varphi_i\left(\alpha \nu^Y\right) = \begin{cases} \alpha p_Y(i) & \text{if } i \in Y, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: By A3, we may assume without loss of generality that $\alpha = 1$. First, consider a given coalition Y. If $i \notin Y$, then, for each subcoalition X such that $i \notin X$, (7) implies that $\nu^Y(X_{+i}) = \nu^Y(X) = 0$. Thus, any player $i \in Z \backslash Y$ is a null player for ν^Y. Hence, by A2, we have $\varphi_j\left(\nu^Y\right) = 0$ for all $i \notin Y$.

Second, if $i \in Y$, it follows from (6) that

$$\varphi_i\left(\nu^Y\right) = \sum_{\substack{X \subset Y \\ X \ni i}} p_X(i) \Gamma_{\nu^Y}(X).$$

Now, Lemma 1 implies that $\Gamma_{\nu^Y}(X) = 0$ for all $X \neq Y$ and $\Gamma_{\nu^Y}(X) = 1$ for $X = Y$. Consequently, we obtain for all $i \in Y$:

$$\varphi_i\left(\nu^Y\right) = p_Y(i) \Gamma_{\nu^Y}(Y) = p_Y(i).$$

□

This result also shows the existence of a one-to-one correspondence between the Möbius values and the sharing systems.

We may now state one of our main results.

Theorem 1. : *Any solution $\varphi(\nu)$ of the TU-game (Z, ν) is a Möbius value if and only if $\varphi(\nu)$ satisfies the axioms* A1-A3.

Proof: (Sufficiency) Using Lemma 2 and A3, we have:

$$\varphi_i(\nu) = \sum_{Y \in 2^Z_{-\emptyset}} \varphi_i\left[\Gamma_\nu(Y) \nu^Y\right].$$

Hence, from Lemma 3 and A3, it follows that

$$\begin{aligned} \varphi_i(\nu) &= \sum_{Y \subset Z_{-i}} \sum_{X \subset Y} \varphi_i\left[\Gamma_\nu(X_{+i}) \nu^{X_{+i}}\right] \\ &= \sum_{X \ni i} \Gamma_\nu(X) p_X(i) \end{aligned}$$

which is identical to (6).

(Necessity) The proof is straightforward.

□

3.3 The Shapley Value as a Uniform Möbius Value

The Shapley value of a TU-game (Z, ν), denoted $S(\nu)$, allocates the worth $\nu(Z)$ among all players $i \in Z$ as follows:

$$S_i(\nu) = \frac{1}{n!} \sum_{\substack{X \subset Z \\ i \in X}} (x-1)!(n-x)![\nu(X) - \nu(X_{-i})]. \tag{8}$$

The standard interpretation of the Shapley value is as follows. Assume that the players in Z are randomly ordered as $(i_1, i_2, \cdots, i_n)$ such that each ordering is equally probable. The Shapley value $S_i(\nu)$ is then the average of player i's marginal contributions $\nu(X) - \nu(X_{-i})$ taken over all coalitions $X \subset Z$.

The probability of any coalition X is defined by the probability that the predecessors of i in the random ordering $(i_1, i_2, \cdots, i_n)$ are the elements of X.

Our next result suggests another interpretation: when player i cooperates within a coalition X whose **PCC** equals $\Gamma_\nu(X)$, player i gets the same "share" from this coalition than any other member of X. In other words, the sharing of $\Gamma_\nu(X)$ is *uniform* within X. Hence, the Shapley value of player i is the unweighted and normalized sum of all coalition worthies. The associated sharing system is denoted $(Z, \mathcal{U})$ where $\mathcal{U} = \left(u_Y : Y \in 2^Z_{-\emptyset}\right)$ and u_Y the uniform probability distribution over Y.

As shown by Denneberg and Grabisch (1999, Theorem 4.1), this result can be proven by using symmetry, whereas Grabisch (1997) gives a direct proof in a setting involving interactions among individuals. Yet, in order to illustrate the nature of individual contributions to a coalition, we give in Appendix B a different proof that does not rely on symmetry.

Theorem 2. : *Let $(Z, \mathcal{U})$ be the uniform sharing system. Then, the corresponding Möbius value of the TU-game (Z, ν) is the Shapley value:*

$$\varphi_i(\nu, \mathcal{U}) = \sum_{\substack{Y \in 2^Z_{-\emptyset} \\ Y \ni i}} \frac{\Gamma_\nu(Y)}{y} = S_i(\nu) \quad \textit{for all players } i \in Z.$$

In other words, the Shapley value corresponds to a sharing of the **PCC**s which is uniform across players. This interpretation is perfectly consistent with the axiom of anonymity (or symmetry) which defines the Shapley value (Shapley, 1953): players are a priori given the same share in *all* possible coalitions. This should not come as a surprise since, on the one hand, we know from Kalai and Samet (1987) that the Shapley value is a weighted value with identical weights and, on the other hand, that the uniform distribution satisfies the Luce choice axiom that characterizes weighted values (see our Theorem 4 below).

Example 3: Consider the uniform sharing rule given by $u_X(i) = 1/x$ for all $i \in X$, all $X \subset Z$. Then, the Shapley value $S_i(\nu)$ defined by (17) leads to

$$\begin{aligned} S_1(\nu) &= \Gamma_\nu(1)\, u_1(1) + \Gamma_\nu(12)\, u_{12}(1) + \Gamma_\nu(13)\, u_{13}(1) \\ &\quad + \Gamma_\nu(Z)\, u(1) \\ &= 1 \times 1 + 1 \times \frac{1}{2} + 1 \times \frac{1}{2} - 1 \times \frac{1}{3} \\ &= 1 + \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{5}{3} \simeq 1.66 > 1, \end{aligned}$$

$$\begin{aligned} S_2(\nu) &= \Gamma_\nu(2)\, u_2(2) + \Gamma_\nu(12)\, u_{12}(2) + \Gamma_\nu(23)\, u_{23}(2) \\ &\quad + \Gamma_\nu(Z)\, u(2) \\ &= 2 + \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{8}{3} \simeq 2.66 > 2 \end{aligned}$$

and

$$\begin{aligned} S_3(\nu) &= \Gamma_\nu(3)\, u_3(3) + \Gamma_\nu(13)\, u_{13}(3) + \Gamma_\nu(23)\, u_{23}(3) \\ &\quad + \Gamma_\nu(Z)\, u(3) \\ &= 3 + \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{11}{3} \simeq 3.66 > 3. \end{aligned}$$

The Shapley solution of our example is therefore given by the triplet

$$(1.66, 2.66, 3.66).$$

4 Relationships between Möbius Values and Quasivalues

4.1 Random Order Values

Weber (1988) has introduced a generalization of the Shapley value, called *random order (or probabilistic) values*, by weighting the marginal contributions $\nu(Y_{+i}) - \nu(Y)$ of player i by the probability π_Y^i of joining any coalition Y in Z_{-i}:

$$\phi_i(\nu) = \sum_{Y \subset Z_{-i}} \pi_Y^i [\nu(Y_{+i}) - \nu(Y)]. \tag{9}$$

Then, Weber (1988) has proved that a solution is a quasivalue if and only if it is a random order value.

When comparing (6) and (9), we first note that the coefficients π_Y^i in (9) are interpreted by Weber as the probability for i to become a member of Y (or to join Y) while, in the present paper, $p_Y(i)$ in (6) is defined as the share attributable to player i when i is a member of the coalition Y. The two interpretations are therefore different. Second, the marginal contribution $\nu(Y_{+i}) - \nu(Y)$ differs from the pure contribution $\Gamma_\nu(Y_{+i})$ of coalition Y_{+i}. So, the connection between the two values is not clear (at least to us). Hence, our research strategy is naturally to uncover the relationships between (6) and (9) through their respective coefficients. More precisely, we are interested in determining *the connections between the share a player may obtain within a particular coalition and the probability she has to join this coalition*.

Definition (6) may be rewritten in terms of marginal contribution as follows:

$$\begin{aligned}
\varphi_i(\nu) &= \sum_{Y \in 2^Z_{-\emptyset}} \Gamma_\nu(Y)\, p_Y(i) \\
&= \sum_{\substack{Y \subset Z \\ i \in Y}} p_Y(i) \sum_{X \subset Y} (-1)^{y-x} \nu(X) \\
&= \sum_{X \subset Z} \nu(X) \sum_{Y \supset X_{+i}} (-1)^{y-x}\, p_Y(i)
\end{aligned}$$

that is

$$\varphi_i(\nu) = \sum_{Y \subset Z_{-i}} \left\{ \sum_{X \supset Y_{+i}} (-1)^{x-(y+1)} p_X(i) \right\} [\nu(Y_{+i}) - \nu(Y)]. \tag{10}$$

This shows that the Möbius value involves coefficients γ_Y^i of the marginal contributions of i to Y of the type

$$\gamma_Y^i \equiv \sum_{X \supset Y_{+i}} (-1)^{x-y-1} p_X(i), \text{ for all } Y \subset Z_{-i} \tag{11}$$

which are not here primitives of the game, as they are in the various extensions of the Shapley value (Monderer and Samet, 2002). Furthermore, γ_Y^i need not be a probability and may even be negative. As will been seen, all quasivalues are special cases of the Möbius value in which the coefficients γ_Y^i take a particular form. Stated differently, *all quasivalues are special Möbius values associated with specific sharing systems*. In particular, the Möbius value is a random order value if and only if the coefficients γ_Y^i are probabilities. In this case, whenever the game is monotone, the positivity axiom for quasivalues - which one can find from Kalai and Samet (1987) to Monderer and Samet (2002) through Weber (1988) - always holds. Hence, it remains to identify the restrictions to be imposed on the sharing system for a Möbius value to have probabilistic coefficients.

If $Y = Z_{-i}$, then the coefficient π^i_Y for player i to join the coalition Y is identical to her share $p(i)$. Consider now $Y = Z_{-ij}$. Once i has joined Y, either i belongs to the coalition Z_{-j} or to the coalition Z because Y_{+i} is a subset of both. Since $Y = Z_{-ij}$, the weight for i to join Y is therefore given by:

$$\pi^i_Y = p_{Z_{-j}}(i) - p(i). \tag{12}$$

In other words, π^i_Y is the coefficient of joining the coalition Y without being in the coalition Z. If $Y = Z_{-ijk}$, one might think that π^i_Y is such that

$$\pi^i_Y = p_{Z_{-jk}}(i) - p_{Z_{-j}}(i) - p_{Z_{-k}}(i) - p(i).$$

However, this expression does not account for the fact that, when i belongs to Z_{-j} (resp. Z_{-k}), this may be because she has joined Z_{-ij} (resp. Z_{-ik}). Deleting these occurrences, we obtain:

$$\pi^i_Y = p_{Z_{-jk}}(i) - \left[p_{Z_{-j}}(i) - \pi^i_{Z_{-ij}}\right] - \left[p_{Z_{-k}}(i) - \pi^i_{Z_{-ik}}\right] - p(i).$$

Given (12), this may be rewritten as follows:

$$\pi^i_Y = p_{Z_{-jk}}(i) - p_{Z_{-j}}(i) - p_{Z_{-k}}(i) + p(i).$$

More generally, for all $i \in Z$ and all $Y, X \subset Z_{-i}$, the coefficient for i to join Y is given by:

$$\pi^i_Y = \sum_{X \supset Y} (-1)^{x-y} p_{X_{+i}}(i).$$

It is readily verified that this expression can be also written as follows:

$$\pi^i_Y = p_{Y_{+i}}(i) - \sum_{X \supsetneq Y} \left[p_{X_{+i}}(i) - \pi^i_X\right]$$

where $\pi^i_Z \equiv 0$.

The difference $p_{X_{+i}}(i) - \pi^i_X$ may be viewed as the *net share* of player i for being in X_{+i}, once π^i_X is interpreted as the (normalized) "cost" she bears to join the coalition X. Then, the coefficient for i to join the coalition Y is equal to her share in the coalition Y_{+i} minus the sum of the net shares that i belongs to all the supercoalitions $X_{+i} \supset Y$. Put differently, π^i_Y *is the coefficient to join Y directly and not through any of its supercoalitions*. Using again (1) and (2), we then have: for all $i \in Z$, all $Y, X \subset Z_{-i}$,

$$\pi^i_Y = \sum_{X \supset Y} (-1)^{x-y} p_{X_{+i}}(i) \tag{13}$$

if and only if

$$p_{Y_{+i}}(i) = \sum_{X \supset Y} \pi^i_X. \tag{14}$$

Remark 2: Expressions (13) and (14) can be viewed as, respectively, the Möbius and the *co-Möbius inverse* of a set function v^i (which is unrelated to the characteristic function ν), such that the following two conditions hold (see Appendix A for more details):

$$\begin{cases} p_{Y_{+i}}(i) = \sum_{X \supset \overline{Y}} (-1)^{z-x} v^i(X), \\ v^i(Y) = \sum_{X \supset \overline{Y}} (-1)^x p_{X_{+i}}(i). \end{cases} \tag{15}$$

Hence, the properties of the Möbius and co-Möbius inverse of v^i can be used for studying the relationships between π^i_Y and $p_{Y_{+i}}(i)$ where the former corresponds to the Möbius inverse and the latter to the co-Möbius (see, e.g. the proof of Theorem 5).

Remark 3: Expression (14) may be given the following interpretation: the share of player i in Y_{+i} is equal to the sum of the coefficients that this player has to join all supercoalitions of Y, that is, *her share must cover exactly the sum of the costs that she would incur by joining all the supercoalitions of Y.*

Remark 4: In the special case where there exist some players i such that $p(i) = 0$, then (13) implies $\pi^i_Y = 0$ for all coalitions $Y \neq \emptyset$. In other words, all such players always stay alone because $\pi^i_\emptyset = 1$.

Equations (13) and (11) imply

$$\gamma^i_Y = \pi^i_Y.$$

However, for π^i_Y to be a probability, the sharing system $(Z, \mathcal{P})$ must satisfy some additional conditions that we now investigate. Following Block and Marschak (1960) and Falmagne (1978), we say that the sharing system $(Z, \mathcal{P})$ is *stochastically rationalizable* if and only if the Block-Marschak polynomials of $(Z, \mathcal{P})$ are all nonnegative. Recall that the Block-Marschak polynomials of $(Z, \mathcal{P})$ are defined for all subsets $Y \subset Z_{-i}$ by the expression:

$$K(i,Y) = \sum_{k=0}^{y} (-1)^k \sum_{X \in \mathcal{F}(Y, y-k)} p_{\overline{X}}(i)$$

where $\mathcal{F}(Y, y-k)$ is the family of subsets of Y whose cardinal is equal to $y-k$ and $\overline{X}$ the complement of X in Z. We thus have:

Theorem 3. : *For any TU-game (Z, ν), the Möbius value is a random order value, i.e.*

$$\varphi_i(\nu) = \phi_i(\nu),$$

if and only if the sharing system $(Z, \mathcal{P})$ is stochastically rationalizable.

Proof: Expressions (13) and (14) define a one-to-one correspondence between the two sets of coefficients γ^i_Y and π^i_Y. To prove that the coefficients π^i_Y correspond to Weber's probabilities, it remains to show, on one hand, that they are all nonnegative and, on the other hand, that $\sum_{Y \subset Z_{-i}} \pi^i_Y = 1$.

Let X and Y be any two subsets of Z such that $i \notin X$ and $i \in Y$. We have

$$\begin{aligned} K(i,Y) &= \sum_{k=0}^{y} (-1)^k \sum_{X \in \mathcal{F}(Y, y-k)} p_{\overline{X}}(i) \\ &= \sum_{X \subset Y} (-1)^{y-x} p_{\overline{X}}(i) \\ &= \sum_{\overline{X} \supset \overline{Y}} (-1)^{\bar{x}-\bar{y}} p_{\overline{X}}(i) \\ &= \pi^i_{\overline{Y}}. \qquad \text{by (13)} \end{aligned}$$

Since Y is arbitrary, π^i_Y is nonnegative if and only if the sharing system (Z, p) is stochastically rationalizable. Moreover, it is readily verified that $\sum_{Y \subset Z_{-i}} \pi^i_Y = p_i(i) = 1$, which ends the proof.

□

Corollary 1. : *Any Block-Marschak polynomial $K(i,Y)$ of a choice probability system $(Z,\mathcal{P})$ corresponds to the coefficient $\pi^i_{\overline{Y}}$ as defined by (13).*

Theorem 3 is consistent with the following result derived by Monderer (1992): for any random order value, there exists a rationalizable system of choice probabilities defined on Z consistent with the probabilities π^i_Y in (9). Note also that the stochastic rationality of the sharing system $(Z,\mathcal{P})$ is equivalent to the positivity axiom. Then, *a solution satisfying A1-A3 whose sharing system is stochastically rationalizable is a quasivalue*.

Observe that (13) allows for the computation of the coefficients used by Weber from the individual shares. This, in turn, permits the study of the likelihood of various coalitions and, therefore, to analyze the occurrence of coalition formation and to perform some "comparative statics" on the sharing rule. Everything else equal, *the smaller* (resp. the larger) *a player's share, the higher* (resp. the lower) *her probability to stand alone*, a situation which involves no coalitional cost. Likewise, the smaller (resp. the larger) a player's share, the higher (resp. the lower) her probability to be joined by players with larger shares. Unfortunately, it seems hard to say something about players with intermediate shares without specifying the connections between the sharing system $\mathcal{P}$ and the characteristic function ν.

4.2 Weighted Values

Kalai and Samet (1987) have considered a subset of quasivalues defined as follows. Set a *weight system* $w = (w_X)_{X\in 2^Z_{-\emptyset}}$ such as

$$w_X(i) = \frac{w_Y(i)}{w_Y(X)}$$

for all $Y \supset X$, all $i \in X$ and $w_Y(X) > 0$. It is worth noting that a weight system w is *strictly* positive. The associated *weighted value* ϕ^w is then defined for any unanimity game ν^X by

$$\phi^w_i(\nu^X) = \begin{cases} w_X(i) & \text{if } i \in X, \\ 0 & \text{otherwise.} \end{cases}$$

In words, a player belonging to coalition X receives her weight within this coalition. Moreover, a coalition Y is said to be a coalition of partners or a *p-type coalition* in (Z,ν) if, for every subcoalition $X \subset Y$ and each $W \subset \overline{Y}$, $\nu(W \cup X) = \nu(W)$. In other words, players are called partners when they refuse to cooperate outside the coalition of partners. A value ϕ satisfies the *partnership axiom* if, whenever Y is a p-type coalition:

$$\phi_i(\nu) = \phi_i(\phi_Y(\nu)\nu^Y) \qquad \text{for all } i \in Y \tag{16}$$

where ϕ_Y is the share attributed to the coalition Y. This axiom, introduced by Kalai and Samet (1987), requires that if subcoalitions of Y are irrelevant, then it makes no difference either players of Y receive their individual shares in ν, or they altogether receive their group share in ν and determine their individual shares later. Kalai and Samet (1987) then proves that a weighted value is a quasivalue that satisfies the partnership axiom. Hence, we need to identify the properties of the sharing rules which characterize a Möbius value as a weighted value, i.e. to interpret the partnership axiom in terms of shares.

Lemma 4. *: For any TU-game (Z,ν), a Möbius value satisfies the partnership axiom if and only if the sharing system $(Z,\mathcal{P})$ satisfies the Luce choice axiom: for all $Y \in 2^Z_{-\emptyset}$*

$$p(i) = p(Y) \times p_Y(i) \qquad \textit{for all } i \in Z \textit{ such that } 0 < p(i) < 1.$$

Proof: As noticed by Chun (1991, p.186), it is always possible to define the weight system w by $w_i = \varphi_i\left(\nu^Z\right)$ where ν^Z is the characteristic function of the unanimity game $\left(Z, \nu^Z\right)$. Accordingly, since $w_Y(X) > 0$ for all nonempty coalitions $X \subset Y \subset Z$, we have $\varphi_i\left(\nu^Z\right) > 0$ for all $i \in Z$. Furthermore, A1 implies that $\sum_{i \in Z} \varphi_i\left(\nu^Z\right) = 1$. As a result, we can identify the weight system w with a strictly positive sharing rule p such that $\varphi_i\left(\nu^Z\right) = p(i) > 0$ for all players $i \in Z$. Let $\left(Z, \nu^Y\right)$ be a unanimity game such that $Y \subset Z$ and $Y \neq Z$. The coalition Y being a p-type coalition for ν^Z, we have for any player $i \in Y$: $\varphi_i\left(\nu^Z\right) = \varphi_i(\varphi_Y\left(\nu^Z\right)\nu^Y)$. Using A3, this expression becomes $\varphi_i\left(\nu^Z\right) = \varphi_Y\left(\nu^Z\right) \times \varphi_i(\nu^Y)$, i.e.

$$\varphi_i(\nu^Y) = \frac{\varphi_i\left(\nu^Z\right)}{\varphi_Y\left(\nu^Z\right)} = \frac{p(i)}{p(Y)}.$$

Now, by Lemma 3, we have $\varphi_i(\nu^Y) = p_Y(i)$ and, then, the Luce choice axiom holds.

□

We are now able to establish the following result:

Theorem 4. : *For any TU-game (Z, ν), the Möbius value is a weighted value, i.e.*

$$\varphi_i(\nu) = \phi_i(\nu)$$

if and only if the sharing system $(Z, \mathcal{P})$ satisfies the Luce choice axiom.

Monderer and Samet (2002, Th. 5) have proved that a weighted value is a random order value that satisfies the partnership axiom, a result consistent with our Theorem 4. Hence, since a random order value is a Möbius value with a stochastically rationalizable sharing system (our Theorem 3), we know, using Luce and Suppes (1965), that the necessary and sufficient condition for the sharing system $(Z, \mathcal{P})$ to satisfy the Luce choice axiom is (1) to be stochastically rationalizable and (2) to satisfy the following condition:

$$\pi_Y^i = p_{Y_{+i}}(i) \times p_{Y_{+ij}}(j) \times p_{Y_{+ijk}}(k) \ldots$$

which always holds for weighted values.

Remark 5: Example 2 in Section 3.1 is associated with a sharing system that does not satisfy stochastic rationality (because $\gamma_\emptyset^2 = -1/6$) nor the Luce choice axiom (because $p^*(2) \neq p^*(23) \times p_{23}^*(2)$). Hence, it is neither a random order value nor a weighted value, but a Möbius value.

5 Properties of the Möbius Value

5.1 Monotone TU-Games

Most variations on the Shapley value assume that the positivity axiom holds: whenever the game is monotone, each individual value is positive (Monderer and Samet, 2002). Hence, the literature seems to focus on values for which the monotonicity of the game would be a sufficient condition for positivity. We show below that monotonicity is both a necessary and sufficient condition for any Möbius value to be positive. This implies that the positivity axiom may be replaced by the assumption of game monotonicity in the study of Möbius values.

Theorem 5. : *Any Möbius value $\varphi(\nu)$ is positive if and only if the TU-game (Z, ν) is monotone.*

Proof: It is sufficient to show that $\varphi_i(\nu) \geq 0$ for any player $i \in Z$. Using (10), we get:

$$\varphi_i(\nu) = \sum_{X \subset Z_{-i}} \underbrace{[\nu(X_{+i}) - \nu(X)]}_{(A)} \underbrace{\sum_{Y \supset X_{+i}} (-1)^{y-(x+1)} p_Y(i)}_{(B)}.$$

As (A) is positive if only if the game (Z, ν) is monotone, we just have to show that (B) is positive for each characteristic function ν, each player i and each coalition Y. Using Remark 2, we see that (B) corresponds to the definition of the Möbius inverse of a particular set function v^i, so that $p_{Y+i}(i)$ is the co-Möbius of that same function v^i. Now, we know that (i) a Möbius inverse is always nonnegative if and only if v^i is ∞-monotone, that is, v^i is a belief function and (ii) a characteristic function v^i is a belief function if and only if its co-Möbius is decreasing (see Shafer, 1976; Grabisch *et al.*, 2000). The individual sharing consistency condition shows that $p_Y(i)$ satisfies this last condition.

□

Since a quasivalue is defined by a solution characterized by the axioms A1-A3 as well as by positivity (Weber, 1988), it then follows from Theorem 1 that *a quasivalue is a Möbius value that satisfies the positivity axiom*. This proves our claim that quasivalues are special cases of Möbius values.

5.2 Convex TU-Games

We know from Shapley (1971) that the core of a convex game is nonempty. The following result shows that all the Möbius values belong to the core for a convex game.

Theorem 6. : *Any Möbius value $\varphi^p(\nu)$ is in the core of the TU-game (Z, ν) if and only if this game is convex.*

The proof is given in Appendix C.

We may now show that the set of Möbius values is identical to the core of a convex game. Indeed, when the game is convex, all the Möbius values belong to the core as shown by Theorem 6. Hence, for a nonconvex game, the set of random order values is a proper subset of Möbius values and, when the game is convex, we have the following result:

Theorem 7. : *For any TU-game (Z, ν), the set of all Möbius values is equal to its core if and only if the game is convex.*

Theorems 3 and 6 together with Weber's Theorem 14 imply that the core of a convex game being equal to the set of random order values, then the set of stochastically rationalizable Möbius values is equal to that of Möbius values, i.e. is equal to the core itself.

6 Concluding Remarks

Our approach to cooperative values allows us to shed new light on cooperative game theory. Indeed, we have shown that the weighted values correspond to the most constrained class of solutions. They are axiomatically characterized by Kalai and Samet (1987) through efficiency (A1), null-player (A2), additivity, positivity and partnership. Since positivity and additivity imply homogeneity (as shown by Kalai and Samet, 1987, p.213), the first two axioms may be replaced by linearity (A3) while positivity may be replaced by the stochastic rationality of the sharing system and partnership by the Luce choice axiom. Hence, our main results may be summarized as follows.

– For *any sharing system*, a solution that satisfies A1-A3 is *a Möbius value*.
– For *any stochastically rationalizable sharing system*, a solution that satisfies A1-A3 is *a random order value* (i.e. a solution that satisfies A1, A2, additivity and positivity).
– For *any sharing system satisfying the Luce choice axiom*, a solution that satisfies A1-A3 is *a weighted value* (i.e. a solution that satisfies A1, A2, additivity, positivity and partnership).

Some questions remain open. First, is there always an element in the nonempty core of a nonconvex game that can be represented by a Mobius value? If yes, what are the restrictions that the corresponding sharing system satisfies? And more generally, can Theorem 7 be extended to the case of nonconvex games with a nonempty core?

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Appendix A

It is useful to introduce the following three related concepts: a set function f, the Möbius inverse of f, denoted m, and the commonality function (Shafer, 1976) or co-Möbius inverse (Grabisch *et al.*, 2000) of f, denoted $\widehat{m}$. Then, the following four expressions simultaneously hold: for all $Y \in 2^Z_{-\emptyset}$,

$$\begin{cases} \widehat{m}(Y) = \sum_{X \supset Y} m(X) \\ m(Y) = \sum_{X \supset Y} (-1)^{x-y} \widehat{m}(X) \\ \widehat{m}(Y) = \sum_{X \supset \overline{Y}} (-1)^{z-x} f(X) \\ f(Y) = \sum_{X \supset \overline{Y}} (-1)^{x} \widehat{m}(X). \end{cases}$$

Appendix B

Proof of Theorem 2: The uniform Möbius value $\varphi(\nu, \mathcal{U})$ is defined for each nonempty coalition $X \subset Z$ by

$$\varphi_X(\nu, \mathcal{U}) = \sum_{\substack{Y \in 2^Z_{-\emptyset} \\ Y \supset X}} \Gamma_\nu(Y) \, u_Y(X) \tag{17}$$

where

$$u_Y(X) = \frac{x}{y},$$

x and y being the cardinalities of X and Y, respectively. Hence, by definition of the **PCC**, for each player $i \in Z$, (17) becomes

$$\begin{aligned} \varphi_i(\nu, \mathcal{U}) &= \sum_{Y \in 2^Z_{-\emptyset}} \Gamma_\nu(Y) \, u_Y(i) \\ &= \sum_{Y \in 2^Z_{-\emptyset}} \Gamma_\nu(Y) \frac{1}{y} \\ &= \sum_{\substack{Y \subset Z \\ i \in Y}} \sum_{X \subset Y} \frac{(-1)^{y-x} \nu(X)}{y} \\ &= \sum_{X \subset Z} \sum_{\substack{Y \subset Z \\ X_{+i} \subset Y}} \frac{(-1)^{y-x}}{y} \nu(X). \end{aligned} \tag{18}$$

Set

$$\lambda(i, X) \equiv \sum_{\substack{Y \subset Z \\ X_{+i} \subset Y}} \frac{(-1)^{y-x}}{y}.$$

When the player $i \in X$, there are $\binom{n-x}{y-x}$ coalitions Y such that $X \subset Y$. Consequently, we have:

$$\begin{aligned}
\lambda(i,X) &= \sum_{\substack{Y\subset Z\\ X_{+i}\subset Y}} \frac{(-1)^{y-x}}{y} \qquad (19)\\
&= \sum_{y=x}^{n} (-1)^{y-x} \binom{n-x}{y-x} \frac{1}{y}\\
&= \sum_{y=x}^{n} (-1)^{y-x} \binom{n-x}{y-x} \int_0^1 t^{y-1} dt\\
&= \int_0^1 t^{x-1} \sum_{y=x}^{n} (-1)^{y-x} \binom{n-x}{y-x} t^{y-x} dt\\
&= \int_0^1 t^{x-1} (1-t)^{n-x} dt.
\end{aligned}$$

It is well known that

$$\int_0^1 t^{x-1} (1-t)^{n-x} dt = \frac{(x-1)!\,(n-x)!}{n!} = \lambda(i,X). \qquad (20)$$

Note that, in (18), if the player $i \in X$, then $\lambda(i, X_{-i}) = -\lambda(i,X)$. Hence, (18) may be rewritten as follows:

$$\varphi_i(\nu, \mathcal{U}) = \sum_{\substack{X\subset Z\\ i\in X}} \lambda(i,X)\,(\nu(X) - \nu(X_{-i})). \qquad (21)$$

Using (20) and (21), we then get the desired expression, i.e.

$$\varphi_i(\nu, \mathcal{U}) = \frac{1}{n!} \sum_{\substack{X\subset Z\\ i\in X}} (x-1)!\,(n-x)!\,(\nu(X) - \nu(X_{-i})) = S_i(\nu).$$

□

Appendix C

Proof of Theorem 6: (Sufficiency) If the TU-game (Z,ν) is convex, then we must show that $\sum_{i\in Y} \varphi_i^p(\nu) \geq \nu(Y)$ for all nonempty coalitions $Y \subset Z$, i.e. $\varphi_Y^p(\nu) \geq \nu(Y)$.

By (6), we know that:

$$\begin{aligned}
&\sum_{X\subset Y_{-\emptyset}} \sum_{T\subset \overline{Y}} \Gamma_\nu(X\cup T)\, p_{X\cup T}(X) \qquad (22)\\
&= \sum_{X\subset Y_{-\emptyset}} \sum_{T\subset \overline{Y}} \sum_{S\subset T} (-1)^{t-s} \sum_{W\subset X} (-1)^{x-w} \nu(W\cup S)\, p_{X\cup T}(X)
\end{aligned}$$

whereas, by definition of a **PCC**,

$$\begin{aligned}
&\sum_{X\subset Y_{-\emptyset}} \sum_{T\subset Z\setminus Y} \sum_{S\subset T} (-1)^{t-s} \sum_{W\subset X} (-1)^{x-w} \nu(W)\, p_{X\cup T}(X)\\
&= \sum_{X\subset Y_{-\emptyset}} \Gamma_\nu(X)\, p_X(X) = \sum_{X\subset Y_{-\emptyset}} \Gamma_\nu(X) = \nu(Y). \qquad (23)
\end{aligned}$$

Hence, from (22) and (23), we obtain:

$$\begin{aligned}
\varphi_Y^p(\nu)-\nu(Y) &= \sum_{X\subset Y_{-\emptyset}}\sum_{T\subset\overline{Y}}\sum_{S\subset T}(-1)^{t-s}\sum_{W\subset X}(-1)^{x-w} \\
&\quad\times[\nu(W\cup S)-\nu(W)]\,p_{X\cup T}(X) \\
&= \sum_{X\subset Y_{-\emptyset}}\sum_{S\subset\overline{Y}}\sum_{W\subset X}(-1)^{x-w}[\nu(W\cup S)-\nu(W)] \\
&\quad\times\sum_{S\subset T\subset\overline{Y}}(-1)^{t-s}p_{X\cup T}(X) \\
&= \sum_{S\subset\overline{Y}}\sum_{R\subset\overline{Y}\setminus S}(-1)^{r}\sum_{X\subset Y_{-\emptyset}}\sum_{W\subset X}(-1)^{x-w} \\
&\quad\times[\nu(W\cup S)-\nu(W)]\,p_{X\cup S\cup R}(X) \\
&= \sum_{S\subset\overline{Y}}\sum_{R\subset\overline{Y}\setminus S}(-1)^{r} \\
&\quad\sum_{i\in Y}\underbrace{\sum_{\substack{X\subset Y\\ i\in X}}p_{X\cup S\cup R}(i)\times\sum_{W\subset X}(-1)^{x-w}[\nu(W\cup S)-\nu(W)]}_{(A)}.
\end{aligned} \tag{24}$$

We may rewrite (A) as follows:

$$\begin{aligned}
&\sum_{X\subset Y_{-i}}\left\{\sum_{W\subset X_{+i}}(-1)^{(x+1)-w}[\nu(W\cup S)-\nu(W)]\right\} \\
&\times p_{X_{+i}\cup S\cup R}(i).
\end{aligned}$$

Hence, (A) is equivalent to:

$$\begin{aligned}
&\sum_{X\subset Y_{-i}}\left\{\sum_{V\subset X}\sum_{W\subset V_{+i}}(-1)^{(v+1)-w}[\nu(W\cup S)-\nu(W)]\right\} \\
&\times\left\{\sum_{U\subset(Y_{-i})\setminus X}(-1)^{u}p_{U\cup X_{+i}\cup S\cup R}(i)\right\} \\
=&\sum_{\substack{X\subset Y\\ i\in X}}\underbrace{\sum_{\substack{V\subset X\\ i\in V}}\sum_{W\subset V}(-1)^{v-w}[\nu(W\cup S)-\nu(W)]}_{(B)} \\
&\times\left\{\sum_{U\subset Y\setminus X}(-1)^{u}p_{U\cup X\cup S\cup R}(i)\right\}.
\end{aligned} \tag{25}$$

By interchanging the summations, (B) becomes

$$\begin{aligned}
&\sum_{\substack{W\subset X\\ i\in W}}\left[\sum_{W\subset V\subset X}(-1)^{v-w}\right][\nu(W\cup S)-\nu(W)] \\
&+\sum_{W\subset X_{-i}}\left[\sum_{W\subset V\subset X_{-i}}(-1)^{v+1-w}\right][\nu(W\cup S)-\nu(W)] \\
=&\ \nu(X\cup S)-\nu(X)-\nu(X_{-i}\cup S)+\nu(X_{-i}).
\end{aligned}$$

First, set

$$\sigma(i,X,S) \equiv [\nu(X\cup S)-\nu(X)]-[\nu(X_{-i}\cup S)-\nu(X_{-i})].$$

Since $X\cap S=\emptyset$, the convexity of (Z,ν) implies that $\sigma(i,X,S)\geq 0$. Second, setting $W\equiv U\cup R$, we have

$$\rho(i,X,S) \equiv \sum_{W\subset Z\setminus(X\cup S)} (-1)^{w} p_{X\cup S\cup W}(i).$$

Using the same argument as for (**??**), we obtain $\rho(i,X,S)\geq 0$.

Therefore, using (24) leads to

$$\varphi_Y^p(\nu)-\nu(Y)=\sum_{S\subset\overline{Y}}\sum_{i\in Y}\sum_{X\subset Y}\sigma(i,X,S)\times\rho(i,X,S)\geq 0.$$

(Necessity) The proof is by contradiction. Assume the TU-game (Z,ν) is not convex and show that there exists a Möbius value that does not belong to the core. First, applying Proposition 4 of Chateauneuf and Jaffray (1989) allows one to say that the **PCC** Γ_ν of ν satisfies:

$$\sum_{\{i,j\}\subset X\subset Y}\Gamma_\nu(X)\geq 0$$

for all pair of players $\{i,j\}$ belonging to each coalition $Y\subset Z$ if and only if the TU-game (Z,ν) is convex. Then, since our game is not convex, there exists a coalition $Y\subset Z$ and a pair of players $i,j\in Y$ such that:

$$\sum_{\{i,j\}\subset X\subset Y}\Gamma_\nu(X)<0. \tag{26}$$

We now have to prove that there exists a Möbius value, $\varphi^p(\nu)$, which is not in the core, that is, $\varphi^p_{Y_{-i}}(\nu)-\nu(Y_{-i})<0$. Recall that $p_X(Y_{-i})=1$ when $X\subset Y_{-i}$. From (6), it follows that:

$$\begin{aligned}\varphi^p_{Y_{-i}}(\nu)-\nu(Y_{-i}) &= \sum_{\substack{X\subset Z\\ X\not\subset Y_{-i}}}\Gamma_\nu(X)\,p_X(Y_{-i}) \\ &= \sum_{\substack{X\subset Y\\ i\in X}}\Gamma_\nu(X)\,p_X(Y_{-i}) \\ &\quad+\sum_{\substack{X\subset Z\\ X\not\subset Y}}\Gamma_\nu(X)\,p_X(Y_{-i}).\end{aligned} \tag{27}$$

Two cases may then arise. In the first one, we have $Y=Z$. Then, replace $\varphi^p(\nu)$ in (27) by $\varphi^{p_\varepsilon}(\nu)$ associated with the probability $p=p_\varepsilon$ in which $p_\varepsilon(i)=p_\varepsilon(j)=(1-\varepsilon)/2$ and $p_\varepsilon(k)=\varepsilon/(n-2)$ for all player $k\in Z_{-i}$. Then:

$$\lim_{\varepsilon\to 0}\left[\varphi^p_{Y_{-i}}(\nu)-\nu(Y_{-i})\right]=\frac{1}{2}\sum_{\{i,j\}\subset X\subset Y}\Gamma_\nu(X), \tag{28}$$

which is negative by (26), i.e., there exists a positive ε such that $\varphi^p_{Y_{-i}}(\nu)-\nu(Y_{-i})<0$.

In the second case, we have $Y\subsetneq Z$. Then, replace $\varphi^p(\nu)$ in (27) by $\varphi^{p_\varepsilon}(\nu)$ associated with the probability $p=p_\varepsilon$ where $p_\varepsilon(i)=p_\varepsilon(j)=\varepsilon$, $p_\varepsilon(k)=\varepsilon^2$ for all player $k\in Y_{-i}$ and

$$p_\varepsilon(k)=\frac{1-p_\varepsilon(Y)}{n-y}$$

for all player $k \in Z \backslash Y$. Again, (28) holds , i.e. there exists a positive ε such that $\varphi^p_{Y_{-i}}(\nu) - \nu(Y_{-i}) < 0$. Hence, if the TU-game (Z, ν) is not convex, the constructed Möbius value $\varphi^p(\nu)$ does not belong to the core.

□

The RuBy Method for the Recommendation of a Best Choice From a Bipolar Valued Outranking Relation

Raymond Bisdorff[1], Patrick Meyer[1], and Marc Roubens[2]

[1] Service de Mathématiques Appliquées
Université du Luxembourg
1511 Luxembourg
{raymond.bisdorff|patrick.meyer}@uni.lu
[2] Faculté Polytechnique de Mons
7000 Mons, Belgium
roubens@belgacom.net

Abstract. This short note briefly presents a new and original way to solve the problem of a "best choice" recommendation in a multiple criteria decision aid framework. In particular it discusses how such a "best choice" can be constructed from a binary valued outranking relation defined on a finite set X of potential decision alternatives. The discussion is based on five natural principles.

1 Introduction

The goal of this extended abstract is to discuss how a "best choice"[3] recommendation may be rationally constructed from a binary valued outranking relation defined on a finite set X of potential decision alternatives. Such an outranking relation expresses the likelyhood of a global pairwise preference situation between the alternatives which combines an "at least as good" statement with the absence of any local veto. This decision aid problem is generally non trivial. In practise, most outranking relations result from a multiple criteria preference aggregation involving a majority concordance principle. In general such an aggregation doesn't produce a complete or transitive relation.

From a pragmatic point of view, the BC problematics is turned towards the selection of a unique ultimate "best" alternative. In practise, this kind of decision aid consists in the elicitation of a subset of "good" alternatives which is as restricted as possible. It is meant to help the decision maker to get as close as possible to the selection of a unique "best" alternative. In case this recommendation consists of several candidates, the decision aid process may be restarted with new and more detailed information in order to help selecting the final "best" alternative.

Apart from the European multiple criteria decision aid community [Roy85], this specific BC problematics has attracted quite low attention by the Operational Research field. Seminal work on it goes back to the first articles of Roy on the Electre I methods [Roy68,Roy69]. After Kitainik [Kit93], interest in solving the BC problem differently from the classical optimisation paradigm has reappeared. The recent work of Bisdorff and Roubens on valued kernels [BR96] has resulted in new attempts to solve the BC problem directly from the valued outranking graph. After first positive results [Bis00], methodological difficulties appeared when applying the outranking kernel concept to highly non transitive and partial outranking relations.

In this short note we therefore propose to present the major ideas of a new proposal to the BC problem and to revisit the logical and pragmatic foundations of this problematics. The objective is to propose a new and innovative decision aid methodology in the tradition of the pioneering work of Roy and Bouyssou [RB93].

[3] "best choice" will be written BC in the sequel.

2 Some fundamental concepts

Our starting point is a valued outranking digraph, denoted $\widetilde{G}^{\mathcal{L}}(X,\widetilde{\mathrm{S}})$, where X is a finite set of decision alternatives and $\widetilde{\mathrm{S}} : X \times X \to \mathcal{L}$ is a bipolar valued characterisation of an outranking relation on X taking its values in a bipolar evaluation domain $\mathcal{L}$.

Commonly $\mathcal{L}$ consists of the rational unit interval expressing the more or less credibility or robustness of an outranking statement. Throughout this paper we shall however suppose, except if stated otherwise, that $\mathcal{L} = \{-m,\ldots,0,\ldots,+m\}$ is a finite ordinal scale with $2m+1$ $(m \geq 1)$ values expressing a degree of likelyhood or robustness. If x and y are two alternatives of X, $\widetilde{\mathrm{S}}(x,y) = m$ signifies that the assertion "x outranks y" is *certainly true*; $\widetilde{\mathrm{S}}(x,y) > 0$ signifies that the assertion "x outranks y" is *more true than false*; $\widetilde{\mathrm{S}}(x,y) = 0$ signifies that the assertion "x outranks y" is *logically undetermined*, i.e. *neither true nor false*; $\widetilde{\mathrm{S}}(x,y) < 0$ signifies that the assertion "x outranks y" is *more false than true*; $\widetilde{\mathrm{S}}(x,y) = -m$ signifies that the assertion "x outranks y" is *certainly false*.

To be short we say that "x outranks y" is *$\mathcal{L}$-true* (respectively *$\mathcal{L}$-false*) if $\widetilde{\mathrm{S}}(x,y) > 0$ (respectively $\widetilde{\mathrm{S}}(x,y) < 0$).

A non empty subset Y of X is called a *choice* in $\widetilde{G}^{\mathcal{L}}$. Such a choice Y is said to be *$\mathcal{L}$-outranking* if and only if either, $Y = X$, or $x \notin Y \Rightarrow \exists y \in Y : \widetilde{\mathrm{S}}(y,x) > 0$. Similarly, a choice Y is said to be *$\mathcal{L}$-outranked* if and only if either $Y = X$, or $x \notin Y \Rightarrow \exists y \in Y : \widetilde{\mathrm{S}}(x,y) > 0$.

A choice Y is said to be *$\mathcal{L}$-independent* if and only if either, Y is a singleton, or $\forall x,y \in Y : \widetilde{\mathrm{S}}(x,y) < 0$. One should notice here that the concept of independence is not based on the negation of the $\mathcal{L}$-true outrankings. Such a negation would also include the couples of alternatives (x,y) for which $\widetilde{\mathrm{S}}(x,y) = 0$ holds.

An *$\mathcal{L}$-outranking* (*$\mathcal{L}$-outranked*) *kernel* is an $\mathcal{L}$-outranking ($\mathcal{L}$-outranked) and $\mathcal{L}$-independent choice.

The goal of our research is to determine a choice Y of X which can be used as a BC recommendation.

3 New foundations for the BC problematics

It is shown in [BRM05] that classical approaches to the BC problem present flaws and weaknesses. For example, the optimisation problem requires that any two alternatives are comparable. The Electre IS method [RB93] requires modifications of the original outranking digraph in order to present a single BC to the decision maker. The concept of $\mathcal{L}$-outranking kernel is also insufficient for the BC problematics, as it may not exist in certain digraphs or be only a subset of possible interesting recommendations.

Therefore we estimate that a new vision of this problem must be adopted. We define a new set of fundamental principles for the BC problematics. The two classical principles defined by Roy [Roy85] are still be present in this set, but are completed by 3 other *natural* ones.

A BC recommendation is a set of alternatives which will be used for a future proposal of a unique best alternative. This definition shows an important characteristic of any BC procedure. It should be interactive and tend towards the proposal of a unique best alternative. This observation is concordant with Roy's definition of the BC problem (see [Roy85]) where this implicit objective is emphasised.

A BC recommendation Y should therefore verify these 5 principles

- $\mathcal{B}_1$ Each alternative which in not selected must be considered as worse as at least one alternative of Y;
- $\mathcal{B}_2$ The subset of retained alternatives Y must be as small as possible;

- $\mathcal{B}_3$ The subset of retained alternatives should not be simultaneously a "best" and a "worst" choice (effective outranking);
- $\mathcal{B}_4$ The BC recommendation cannot contain another smaller BC recommendation (BC-stability);
- $\mathcal{B}_5$ The BC recommendation must be robust (with respect to impreciseness in the data) (robustness).

The interested reader can refer to an extended version [BRM05] of this short note where we detail each of these principles and their consequences. Furthermore we justify the choice of these principle in view of the BC problematics.

4 Solving the BC problematics

As shown in [BRM05], the problems linked to classical approaches of the BC problematics, lead us to define a particular choice, namely the *hyper-kernel*. A choice Y in $\widetilde{G}^{\mathcal{L}}$ is said to be *$\mathcal{L}$-hyper-independent* if it consists of disjoint cordless $\mathcal{L}$-circuits C^p of odd order ($p = 1, 3, \ldots$)[4]. Consequently an $\mathcal{L}$-outranking ($\mathcal{L}$-outranked) *hyper-kernel* is an $\mathcal{L}$-hyper-independent $\mathcal{L}$-outranking ($\mathcal{L}$-outranked) choice. It is possible to show that these hyper-kernels verify the five principles introduced in Section 3. A *pre-hyper-kernel* is a hyper-kernel for which at least one cordless $\mathcal{L}$-circuits C^p of odd order is in an undetermined situation. This means that it is impossible to determine if it belongs or not to the choice. Nevertheless, a deeper analysis on the choice may exclude it or include it for good.

We now present the algorithm for the resolution of the BC problematics:

1. Detection of the odd cordless $\mathcal{L}$-circuits of $\widetilde{G}^{\mathcal{L}}$.
2. Search for the $\mathcal{L}$-outranking ($\mathcal{L}$-outranked) (pre-)hyper-kernels ($\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_4$).
3. Determination of the robust effective $\mathcal{L}$-outranking (pre-)hyper-kernels ($\mathcal{B}_3$ and $\mathcal{B}_5$).
4. Ranking of the robust effective $\mathcal{L}$-outranking (pre-)hyper-kernels according to decreasing *logical determination*.

The BC recommendation is then given by the $\mathcal{L}$-outranking hyper-kernel(s) with the highest degree of logical determination.

5 Concluding remarks

This short note describes the very basic ideas of the RUBY procedure for the determination of a BC in a valued outranking digraph. For further details the interested reader should refer to [BRM05]. The procedure is based on five natural principles and introduces the concept of hyper-kernel of a digraph in order to build a BC recommendation.

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A Conjoint Measurement View on Fuzzy Integrals

Denis Bouyssou

CNRS—LAMSADE, Université Paris Dauphine
75775 Paris Cedex 16, France
bouyssou@lamsade.dauphine.fr

The dominant model in the field of MCDM uses additive value functions. This model has received a thorough axiomatic treatment within the framework of conjoint measurement, following the works of Gérard Debreu and Duncan Luce. This model implies that criteria are mutually independent, which may not always be appropriate. Choquet and Sugeno integrals have recently attracted much interest in MCDM as convenient tools to model interactions between criteria. The main purpose of this paper is to review the existing literature on these two models from the point of view of conjoint measurement, i.e., within a framework in which the only primitive is a preference relation defined on a product set that does not have to be homogeneous.

Whereas the measurement-theoretic foundations of Choquet and Sugeno integrals have been well studied in the area of decision making under uncertainty, a comparable analysis is still lacking in the area of MCDM. Indeed, the very conception of these two tools implies a "commensurability" hypothesis between the scales of the various criteria that is not easy to formalize within the framework of conjoint measurement.

We shall first review the various attempts that have been made to axiomatize Choquet and Sugeno integrals within a classical conjoint measurement framework. We then concentrate on the Sugeno integral, showing that existing axiomatic analyses of this tool allow suggesting new and simple interpretations of the aggregation it promotes. This will lead to a novel interpretation of the Sugeno integral that emphasizes its ordinal character and links it with "noncompensatory" aggregation models.

This analysis is based on joint work with Thierry Marchant and Marc Pirlot [1] and uses recent work in the area by Salvatore Greco, Benedetto Matarazzo and Roman Slowinski [2, 3]

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Representing Comparative Aggregate Likelihoods

G. Busanello[1], G. Coletti[1], R. Scozzafava[2], and B. Vantaggi[2]

[1] Dept. Matematica e Informatica, University of Perugia
Perugia, Italy
{busanello|coletti}@dipmat.unipg.it
[2] Dept. Metodi e Modelli Matematici, University "La Sapienza"
Roma, Italy
{romscozz|vantaggi}@dmmm.uniroma1.it

Abstract. We deal with coherent conditional probability as a general approach able to encompass some existing theories of uncertainty, as e.g. fuzzy sets [3]. We focus on situations where the decision maker is not able to evaluate numerically membership functions, but only to compare the relevant conditional events $E|H$, for a given E (comparative aggregate likelihoods). In this context we give a representation of these relations in terms of aggregate likelihoods.

1 Introduction

Coherent conditional probability is a general framework for dealing with different existing theories of uncertainty [2], as for example fuzzy sets [3].

We refer to the state of information (at a given moment) of a real (or fictitious) person (for instance, a "randomly" chosen one) that will be denoted by "You". If X is a (not necessarily numerical) random quantity with range C_X, let A_x, for any x in the range, be the event $(X = x)$. Now, let φ be any property related to the random quantity X: You can refer to a suitable membership function of the fuzzy subset of "elements" of C_X with the property φ. For example, if X is a numerical quantity and φ is the property "small", for You the membership function may be put equal to 1 for values of X less than a given x_1, while it is put equal to 0 for values greater than x_2; then it is taken as decreasing from 1 to 0 in the interval from x_1 to x_2: this choice of the membership function implies that, for You, elements of C_X less than x_1 have the property φ, while those greater than x_2 do not. So the real problem is that You are uncertain on having or not the property φ those elements of C_X between x_1 and x_2. Then the interest is in fact directed toward conditional events such as $E|A_x$, where x ranges over the interval from x_1 to x_2, with $E =$"You claim the property φ".

It follows that You may assign a probability value $P(E|A_x)$ equal, e.g., to 0.2, while You do not assign a degree of belief of 0.8 to the event E under the assumption A_x^c, since an additivity rule with respect to the conditioning events does not hold. In other words, it is sensible to identify the values of the membership function with suitable conditional probabilities [3]. In particular, putting H_o = "the value of X is greater than x_2", H_1 = "the value of X is less than x_1", we may assume that E and H_o are incompatible and that H_1 implies E, so that, by the properties of a conditional probability, $P(E|H_o) = 0$ and $P(E|H_1) = 1$. Notice that the conditional probability $P(E|A_x)$ has been directly introduced as a function on the set of conditional events, and this is possible since A_x are pairwise incompatible [2]. However, it is interesting to extend this evaluation to other conditional events; for example, it is possible to assign the (conditional) probability that "You claim the number x is small" knowing that its value is between $\underline{x}$ and $\bar{x}$. This aspect is very crucial, since in our approach an essential role is played by conditioning: in fact the very concept of conditional probability is deeper than the usual restrictive view emphasizing $P(E|H)$ only as a probability for each E given H (looked on as a given *fact*). Regarding instead the conditional probability as function only of the conditioning event

$P(E|\cdot)$, in [1] it is shown that coherence implies that the extension on $E|H$ are obtained as weighted averages (weights equal to zero or one are allowed) of the values $P(E|A_x)$ on the atoms. Possibility measures can be seen as borderline case [2].

In this talk we are dealing with situations where the decision maker is not able to give a numerical membership $P(E|A_x)$, but he is able only to compare the conditional events $E|H$ and $E|K$, for a given event E. Then, we focus on these relations and we study the representability by a sort of aggregate likelihoods (i.e. coherent conditional probabilities on $E|H$, for a given E). As particular case we obtain comparative possibilities.

2 Comparative relations

Let $\preceq$ be a binary relation on a set of conditional events $\mathcal{F} = \{E_i|H_i, F_i|K_i\}_{i\in I}$ expressing the intuitive idea of being "no more believable than". The symbols $\sim$ and $\prec$ represent, respectively, the symmetric and asymmetric parts of $\preceq$: $E|H \sim F|K$ means (roughly speaking) that $E|H$ is judged "equally believable" as $F|K$, while $E|H \prec F|K$ means that $F|K$ is "more believable" than $E|H$.

The relation $\preceq$ expresses a qualitative judgement and it is necessary to set up a system of rules assuring the consistency of the relation with some numerical model. More precisely, given a numerical framework of reference (singled-out by a numerical measure of uncertainty), it is necessary to find the conditions which are necessary and sufficient for the existence of a numerical assessment on the events representing a given ordinal relation.

We recall that a function f from $\mathcal{F}$ to $\boldsymbol{R}^+$ *represents* the relation $\preceq$ iff

$$E|H \preceq F|K \Longrightarrow f(E|H) \leq f(F|K),$$
$$E|H \prec F|K \Longrightarrow f(E|H) < f(F|K).$$

In [4] a condition – called (ccp) – assuring the representability of $\preceq$, defined on an arbitrary family $\mathcal{F}$ of conditional events, by means of a coherent conditional probability, and an interpretation in terms of coherent bets have been given:

(ccp) for every $E_i|H_i \preceq F_i|K_i \in \mathcal{F}$ there exist $\alpha_i, \beta_i \in [0,1]$ with $\alpha_i \leq \beta_i$ with $\alpha_i < \beta_i$ for $E_i|H_i \prec F_i|K_i$, such that, for every $n \in \boldsymbol{N}$ and for every $E_i|H_i \preceq F_i|K_i$, $\lambda_i, \lambda'_i, \geq 0, (i = 1, ..., n)$, one has ($I_A$ is the indicator of event A):

$$\sup_{H^o} \left\{ \sum_i \left[\lambda'_i (I_{F_i \wedge K_i} - \beta_i I_{K_i}) + \lambda_i (\alpha_i I_{H_i} - I_{E_i \wedge H_i}) \right] \right\} \geq 0$$

where H^o is the union of the conditioning events whose corresponding λ_i (or λ'_i) is positive.

Now, we focus on relations defined in a more specific setting, that one refereing to the above fuzzy context. Given an additive set $\mathcal{H}$, put

$$\mathcal{E} = \{E|H, \emptyset|H, H|H \,:\, H \in \mathcal{H}\}.$$

We consider a relation $\preceq$ defined on $\mathcal{E}$, and we give some necessary conditions for the existence of a coherent conditional probability on $\mathcal{E}$ representing $\preceq$:

(a) for any $E|H \in \mathcal{E}$ one has $\emptyset|H \preceq E|H \preceq H|H$;
(b) for any $H, K \in \mathcal{H}$, one has $\emptyset|H \prec H|H$, $H|H \sim K|K$ and $\emptyset|H \sim \emptyset|K$;
(c) $H \in \mathcal{H}$, $E \wedge H = \emptyset \Rightarrow E|H \sim \emptyset|H$, and $E \supseteq H \Rightarrow E|H \sim H|H$;
(d) $H, K \in \mathcal{H}$, with $H \wedge K = \emptyset$, $E|K \preceq E|H \Rightarrow E|K \preceq E|(H \vee K) \preceq E|H$;

(e) $H, K \in \mathcal{H}$ with $H \wedge K = \emptyset$, $E|H \sim E|(H \vee K)$ and $E|K \not\sim E|H$ imply $E|G \sim E|(G \vee K')$ with $K' \subseteq K$, for any event $G \in \mathcal{H}$ such that $G^c \wedge H \in \mathcal{H}$ and either $E|(G^c \wedge H) \not\sim E|H$ or $E|H \not\sim E|(G \wedge H^c \vee H)$;

(f) $H_1, H_2, H_1 \wedge H_2, H_1^c \wedge K, H_2^c \wedge K, K \in \mathcal{H}$ and $(H_1 \vee H_2) \subseteq K$,

$$E|(H_1^c \wedge K) \not\sim E|K \not\sim E|(H_2^c \wedge K) \Rightarrow E|(H_1 \wedge H_2) \sim E|K.$$

The proof that the above conditions holds for any relation induced by a coherent conditional probability is based on the characterization theorem given in [2], moreover these conditions can be deduced from (ccp). However the above conditions are not sufficient to assure the representability of $\preceq$ by means of a coherent conditional probability, it is enough to take the partition $\{H_1, ..., H_4\}$ and consider the following ordinal relation

$\emptyset|H_1 \sim ... \sim \emptyset|\Omega \prec E|H_1 \prec E|(H_1 \vee H_2) \prec E|H_2 \sim E|(H_1 \vee H_3) \sim$
$\sim E|(H_1 \vee H_2 \vee H_3) \sim E|(H_1 \vee H_4) \sim E|(H_1 \vee H_2 \vee H_4) \prec E|(H_2 \vee H_3) \prec E|(H_2 \vee H_4) \prec E|H_3 \sim$
$E|H_4 \sim E|(H_3 \vee H_4) \prec \ldots$.

For lack of space we omit the proof that $\preceq$ satisfies the conditions (a)–(f), but there is no coherent conditional probability representing it. Then, the above example shows that also in this simpler situation we cannot avoid a condition based on a betting scheme criterion.

Definition 1. *Let $\mathcal{H}$ be an additive class and $H^o = \bigvee_{H \in \mathcal{H}} H$. A complete relation $\preceq$ on $\mathcal{E} = \{E|H, \emptyset|H, H|H : H \in \mathcal{H}\}$ is a comparative aggregate likelihood if it satisfies conditions (a), (b), (c) and, for any choice of $\lambda_i, \mu_j, \delta_k, \xi_r \geq 0$ such that $\sum_J \mu_j > 0$,*

$$\sup_{H^o} \sum_I \lambda_i \left[k_i I_{H_i} - h_i I_{K_i})\right] + \sum_J \mu_j \left[k_j I_{H_j} - h_j I_{K_j})\right] - \sum_K \delta_k I_{H_k} + \sum_R \xi_r (I_{H_r} - 1) > 0$$

where k_i is the number of atoms contained in K_i (analogously h_i) and $i \in I$, $j \in J, k \in K, r \in R$ such that $E|H_i \preceq E|K_i$, $E|H_j \prec E|K_j$, $E|H_k \sim \emptyset|H^k$, $E|H_k \sim H^r|H^k$.

For comparative comparative aggregate likelihood we have the following result:

Theorem 1. *Let $\mathcal{H}$ be an additive class generated by n atoms $\{A_1, ..., A_n\}$, and $\preceq$ be a relation on $\mathcal{E} = \{E|H, \emptyset|H, H|H : H \in \mathcal{H}\}$ The following two statements are equivalent:*
(i) $\preceq$ is a comparative aggregate likelihood;
(ii) there exists a coherent conditional probability on $\mathcal{E}$ representing $\preceq$ with extensions such that $P(A_r|H^o) = \frac{1}{n}$ for any atom A_r in $\mathcal{H}$.

The proof is based essentially on a classic alternative theorem and on the characterization theorem for coherent conditional probability (see [2]).

Conditions (d), (e) and (f) are useful to split the problem into subproblems, by detecting the conditioning events belonging to different zero-layers [2], as the following result shows:

Theorem 2. *Let $\preceq$ be an ordinal relation on $\mathcal{E}$ satisfying the conditions (a–f). Let $H^o = \bigvee_{H \in \mathcal{H}} H$, $H^i = \bigvee_K \{K \in \mathcal{H} : K \subseteq H^{i-1}, E|K \not\sim E|(K^c \wedge H^{i-1}) \sim E|H^{i-1}\}$. If the restriction of $\preceq$ on $\mathcal{E}_j = \{E|H : H \in \mathcal{H}; H \subseteq H^j, H \not\subseteq H^{j+1}\}$ for any $j = 0, ..., k$, is representable by a coherent conditional probability, then $\preceq$ also is representable by a coherent conditional probability.*

From the above result it comes out that a relation satisfying conditions (a–f) and that can be decomposed in comparative aggregate likelihoods on $\mathcal{E}_j$ (with $j = 0, ..., n$), is representable by a coherent conditional probability on $\mathcal{E}$.

In the talk we show how to get as particular case (when $H^i \wedge H^{i+1}$ contains only one suitable atoms for $i = 0, ..., k$) comparative possibilities.

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On Equilibrated Strategies in Non-cooperative Games with Fuzzy Information

Dan Butnariu

Department of Mathematics, University of Haifa
31905 Haifa, Israel
dbutnaru@math.haifa.ac.il

We define the concept of two-persons (noncooperative) game as follows. Each player $i \in \{1,2\}$ has a set of alternatives Y_i and has to choose one of those alternatives based on more or less plausible information he has about what the other player intends to do. The information player i has about player's $j \neq i$ intentions is "fuzzy" and represented by fuzzy subsets A_i of Y_j. A *strategy of player* i is defined as a pair (A_i, y^i), where A_i is a fuzzy subset of Y_j and $y^i \in Y_i$. The meaning of such a strategy is that player i, based on the information described by the fuzzy set A_i, chooses alternative y^i. The degree of membership $A_i(y^j)$ of $y^j \in Y_j$ in the fuzzy subset A_i of Y_j represents the certainty of player i (determined by his evaluation of some information he has on player j) that player j will choose alternative y^j. The *expected payoff of player* i in the case that strategies $s_1 = (A_1, y^1)$ and $s_2 = (A_2, y^2)$ are played is a number $E_i(s_1, s_2) \in [0,1]$. A pair of strategies $s_i^* = (A_i^*, y_*^i)$, $i = 1,2$ is called *equilibrated* if

$$E_1(s_1^*, s_2^*) \geq E_1(s_1, s_2^*) \text{ and } E_2(s_1^*, s_2^*) \geq E_2(s_1^*, s_2),$$

for any other pair of strategies $s_i = (A_i, y^i)$, $i = 1,2$. The question is whether, and under which conditions, equilibrated strategies exist and, if affirmative, how to determine them. We will show that existence of equilibrated strategies is, essentially, a fixed point problem for a special type of fuzzy mapping. However, effectively computing such strategies is difficult in most practical cases. The special case where the rules of the game exclude any other simultaneous choices of strategies than pairs of the form $s_1 = (\{y^2\}, y^1)$ and $s_2 = (\{y^1\}, y^2)$ is exactly the situation when the players know for sure that in exchange for choosing alternative y^i the other player will choose y^j. This is the classical model of non-cooperative game in which each strategy can be identified with an alternative and in which equilibrated strategies are Nash equilibria. In such cases finding equilibrated strategies is still a computationally complex problem but, under certain conditions, it can be numerically solved (we will sketch an algorithm for doing that using a basic principle that seems to be new). The problem of finding equilibrated strategies in nonclassical games seems to be much more difficult. One reason for that is that players have to deal with uncertain information. The considerations above are still true when more than two players are involved in non-cooperative games where the behavior of each individual is conditioned by fuzzy information he has on the behavior of the other players.

A Machine Learning Approach to Multi-Criteria Aggregation

Bernard De Baets

Department of Applied Mathematics, Biometrics and Process Control
Ghent University
9000 Gent, Belgium
bernard.debaets@ugent.be

We present a rigorous framework for dealing with a specific type of supervised classification called *supervised ranking*, or, more precisely, *supervised ordered sorting*. The following major differences need to be distinguished:

(i) objects are assigned labels belonging to a *totally ordered set of labels*;
(ii) objects are not described in terms of attributes, but in terms of *(true) criteria*;
(iii) the labels are assigned in a *monotone way*: objects with equal or higher scores on all criteria do not receive a lower overall score, or, in other words, are not assigned a lower label.

The purpose of supervised ranking is then to produce such a monotone classifier on the basis of a learning sample. Existing approaches comprise instance-based methods as well as model-based methods such as various adaptations of decision trees, rough set methods, aggregation models such as TOMASO, etc.

Real-world data sets of this kind are usually pervaded with two undesirable phenomena: *doubt*, i.e. objects with identical scores but carrying a different label, and *reversed preference*, i.e. objects with better scores but carrying a lower label. We prefer to adopt a non-invasive approach by transforming contamination of the second type into the first type.

In this talk, we will focus on distribution classifiers, i.e. classifiers that do not necessarily assign a unique class label, but a probability distribution over the set of labels. In the present context, the monotonicity constraint naturally leads to the notion of stochastic dominance. We will confine ourselves to an ordinal setting and present a general framework from which several instance-based supervised ranking algorithms can be derived, such as the *Ordinal Stochastic Dominance Learner*.

The occurrence of reversed preference is indeed one of the limiting factors for the development of what could be called *ranking trees*. We will explain by means of some examples that ranking trees require non-trivial adaptations of the principles of growing, splitting and pruning of the usual classification trees. Of course, ranking trees are more appealing than OSDL because of their interpretability.

This talk is based on joint work with Kim Cao-Van and Stijn Lievens.

On Possibilistic Dominance

Bernard De Baets[1], Hans De Meyer[2], and Marc Roubens[3]

[1] Department of Applied Mathematics, Biometrics and Process Control
Ghent University
9000 Gent, Belgium
Bernard.DeBaets@ugent.be
[2] Department of Applied Mathematics and Computer Sciences
Ghent University
9000 Gent, Belgium
hans.demeyer@ugent.be
[3] Service de mathématique et de recherche opérationnelle
Faculté Polytechnique de Mons
7000 Mons, Belgium
Roubens@belgacom.net

We consider a possibility distribution f_X associated to a variable X, $f_X(x) = Po(X=x)$, for all x belonging to R, and we associate to this possibility distribution, two *shadow distributions* F_X^+ and F_X^-:

$$F_X^+(x) = Po(X \geq x) = \max_{u \geq x} Po(X = u) \; ; \quad F_X^-(x) = Po(X \leq x) = \max_{u \geq x} Po(X = u) .$$

Starting with two variables and their corresponding marginal possibility distributions f_X and f_Y, we define *the degree of possibility of X over* Y according to the extension principle [4] :

$$Po(X \geq Y) = \max_{u \geq v} C(f_X(u), f_Y(v)) \tag{1}$$

where C is a *conjunctor* operator (also called semicopula, see [1]).
C: $[0,1]^2 \to [0,1]$ satisfies two basic properties
(i) : monotonic (C is non decreasing in both arguments)
(ii) : limit conditions (zero is a neutral element : C(1,x) = C(y,1) = 1, $\forall x, y \in [0,1]$)

Obviously C(x, y) $\leq$ min(x, y)

We prove that

$$\max(Po(X \geq Y), Po(Y \geq X)) = 1, \text{ for any pair X and Y} \tag{2}$$

$$Po(X \geq Y) = \max_{x} C(Po(X \geq x), Po(Y \leq x)) \text{ for any pair X and Y} \tag{3}$$

$$Po(X \geq Z) \leq \max(Po(X \geq Y), Po(Y \geq Z)), \text{ for any triple X, Y,Z} \tag{4}$$

(2) and (4) indicate that $Po(X \geq Y)$ is a possibility relation that presents the property of *negative transitivity* (or min-max transitivity). The necessity relation

$Ne(X \geq Y) = 1 - Po(Y \geq X)$ is a *transitive relation* (in the max-min sense).

If one takes as conjunctor C the t-norm “min” that is usually considered when the extension principle is introduced (see Zadeh [4-5]) indicating that the bidimensional possibility distribution f_{XY} corresponds to non-interactive variables , it has been proved by Roubens and Vincke [4] that the fuzzy possibility relation $Po(X \geq Y)$ is a Ferrers relation, i.e.

$$\min(Po(X \geq Y), Po(Z \geq T)) \leq \max(Po(X \geq T), Po(Z \geq Y))$$

Finally if the bidimensional possibility distribution f_{XY} is used to define the possibility relation $Po(X \geq Y)$:

$$Po(X \geq Y) = \max_{u \geq v} f_{XY}(u, v)$$

the negative transitivity property does not necessarily hold.

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Stochastic Dominance Revisited

Hans De Meyer[1] and Bernard De Baets[2]

[1] Department of Applied Mathematics and Computer Science
Ghent University
9000 Gent, Belgium
hans.demeyer@ugent.be
[2] Department of Applied Mathematics, Biometrics and Process Control
Ghent University
9000 Gent, Belgium
bernard.debaets@ugent.be

We establish a pairwise comparison method for random variables. This comparison results in a probabilistic relation on a given set of random variables. The transitivity of this probabilistic relation is investigated in the case of independent random variables, as well as when these random variables are pairwisely coupled by means of a copula, more in particular the minimum operator or the Łukasiewicz t-norm. A deeper understanding of this transitivity, which can be captured only in the framework of cycle-transitivity, allows to identify appropriate strict or weak thresholds, depending upon the copula involved, turning the probabilistic relation into a strict order relation. Using $1/2$ as a fixed weak threshold does not guarantee to yield an acyclic relation, but is always one-way compatible with the classical concept of stochastic dominance. The proposed method can therefore also be seen as a way of generating graded as well as non-graded variants of that popular concept.

Closing the Study on the Decomposition of the Transitivity of Large Preference Relations

Susana Díaz[1], Susana Montes[2], and Bernard De Baets[3]

[1] Dept. of Statistics and O. R.
University of Oviedo
33007 Oviedo, Spain
diazsusana@uniovi.es
[2] Dept. of Statistics and O. R.
University of Oviedo
33271 Gijón, Spain
montes@uniovi.es
[3] Dept. of Applied Mathematics, Biometrics and Process Control
Ghent University
9000 Gent, Belgium
Bernard.DeBaets@UGent.be

Transitivity is an essential property in an ordering process and therefore a basic element in preference modeling. Our presentation is focused on the study of the propagation of transitivity from a large preference relation to its symmetric and asymmetric components.

This contribution is structured in two parts. In the first one we briefly recall some bounds found for the decomposition of transitivity from a large preference relation to its symmetric and asymmetric components. In the second part we present results that close the problem. We characterize the transitivity that any symmetric and asymmetric parts of a transitive large preference relation satisfy.

A large preference relation is just a reflexive relation denoted by R. The relation R connects two alternatives a and b (aRb) if the alternative a is *at least as good as* the alternative b.

In the crisp setting, a large preference relation admits a unique decomposition into a symmetric part I (indifference relation) and an asymmetric part P (strict preference relation). It is well known that in the classical setting the transitivity of I and P follow from the transitivity of R. It is also well known that if R is complete (R connects every pair of alternatives), the transitivity of R is characterized by the transitivity of I and P (see [1]).

In the fuzzy set context, there is neither a unique symmetric component nor a unique asymmetric component. Any generator, *i.e.* any commutative binary operator bounded between the Łukasiewicz t-norm and the minimum operator i leads to a decomposition of a large fuzzy preference relation [5]. We recall that given a generator i, we can decompose R as follows

$$I = i(R, R^t)$$
$$P = R - i(R, R^t)$$

where R^t is the transpose of R defined by $R^t(a,b) = R(b,a)$.
It is also well known that there is no unique definition of transitivity for fuzzy relations. The most common definition is associated to t-norms. The first studies on the propagation of transitivity (see [2–4]) concern only complete large preference relations. These works are focused on some types of transitivity related to the most important t-norms and they study which of those few types of transitivity I and P inherit from R. We have considered a more general approach in which T-transitivity for T a t-norm turns out to be a too restrictive notion. Given the transitivity of R we first bounded the transitivity of the generated I and P in the setting of f-transitivity for f a conjunctor [7]. Some of the most important bounds we found are presented in the first part of our contribution.

In the second part of this presentation we go much further. We characterize the transitivity that any indifference relation I obtained from an f-transitive R by means of any generator i satisfies. We also characterize the transitivity that the strict preference relation P satisfies.

We close several years of research on the problem of the decomposition of the transitivity of large preference relations since we also prove that the characterizations obtained are upper bounds for the transitivity of I and P. We present examples showing that those bounds cannot be surpassed.

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On Preferences Related to Aggregative Operators and Their Transitivity

J. Dombi, Zs. Gera, and N. Vincze

Department of Computer Algorithms and Artificial Intelligence
University of Szeged
6721 Szeged, Hungary
{dombi|gera}@inf.u-szeged.hu, vincze@szef.u-szeged.hu

1 Introduction

Several definitions of preference transitivities were introduced in the past, see e.g. [2]. In this paper we deal with two new transitivity definitions, namely the additive and the multiplicative transitivity.

Definition 1. *Let $p(x,y)$ be a preference function. It fulfills the additive transitivity if*

$$\left(p(x,y)-\frac{1}{2}\right)+\left(p(y,z)-\frac{1}{2}\right)=p(x,z)-\frac{1}{2}, \qquad \forall x,y,z \in [0,1].$$

Definition 2. *Let $p(x,y)$ be a preference function. It fulfills the multiplicative transitivity if*

$$\frac{p(y,x)}{p(x,y)}\frac{p(z,y)}{p(y,z)}=\frac{p(z,x)}{p(x,z)}, \qquad \forall x,y,z \in [0,1].$$

These definition correspond respectively to the nilpotent and the strict classes of fuzzy operators. We give two new preference functions which fulfill these transitivities. Both of them originate from the concept of aggregative operators [1]. The pseudo-associative additive aggregative operator is the mean operator

$$m(x,y)=f_m^{-1}\left(\frac{f_m(x)+f_m(y)}{2}\right),$$

where $f_m:[0,1]\to[0,1]$ is a nilpotent generator function. The associative multiplicative aggregative operator is

$$a(x,y)=f_a^{-1}\left(f_a(x)f_a(y)\right),$$

where $f_a:[0,1]\to[0,\infty]$ is a strict generator function. We define a preference function to be

$$p(x,y)=o(n(x),y),$$

where $o(x,y)$ is either an additive or a multiplicative aggregative operator and $n(x)$ is its corresponding strong negation. We prove that the (additive and multiplicative) preferences are strongly related to the Łukasiewicz and the Dombi operators, respectively. We show that the generator functions of the preference functions can only be the generators of these operators.

Theorem 1. *The preference function*

$$p(x,y)=f^{-1}\left(\frac{1}{2}\left(f(y)-f(x)\right)+f(\nu)\right)$$

has additive transitivity if and only if its generator function is $f(x)=cx$ and so

$$p(x,y)=\frac{y-x+1}{2}.$$

Theorem 2. *The preference function*

$$p(x,y) = f^{-1}\left(f(\nu)\frac{f(y)}{f(x)}\right)$$

has multiplicative transitivity if and only if its generator function is $f(x) = \frac{1-x}{x}$ *and so*

$$p(x,y) = \frac{1}{1+\frac{1-y}{y}\frac{x}{1-x}}.$$

We also show some interesting properties of these preferences and their corresponding aggregative and negation operators.

Proposition 1. *The following identities hold:*

- $p(x,y) = n(p(y,x))$
- $p(x,y) = p(n(y),n(x))$
- $p(x,y) = n(p(n(y),n(x)))$
- $p(o(x_1,\dots,x_n),o(y_1,\dots,y_n)) = o(p(x_1,y_1),\dots,p(x_n,y_n))$

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Refining Discrete Sugeno Integrals by Means Oo Choquet Integrals

Didier Dubois and Hélène Fargier

IRIT
31062 Toulouse Cedex, France
{dubois|fargier}@irit.fr

1 Introduction

Decision rules generalizing maximin and maximax criteria can be defined on an ordinal scale in the form of a Sugeno integral. An axiomatic approach to these qualitative criteria for decision under uncertainty in the style of Savage is in [1]. But these criteria suffer from a lack of discrimination. In order to cope with this limitation, extensions have been defined when the likelihood of events is measured by a possibility or a necessity measure— see [2]. Following this approach, a refined ranking of acts can be defined that is qualitative (it relies on the use of nested leximin and leximax procedures), and satisfies all the properties of expected utility rankings. And it can indeed be represented as a discrete expected utility or weighted average, where the utility function and the probability measures are big-stepped, i.e. form super-increasing (or decreasing) sequences.

Sugeno integral as a qualitative decision criterion suffers from the same defects. Here, we refine this criterion using similar leximin and leximax ingredients. The refinement of weighted maximin or maximax possibilistic criteria by expected utility made sense because these criteria do not strongly violate the sure thing principle: only a blurring effect is observed, which causes the lack of discrimination. However due to the strong violation of the Sure Thing Principe by Sugeno integral, the latter cannot be refined by means of an expected utility criterion. In fact, due to the role of comonotonic acts in the representation of Sugeno integrals, the natural numerical criterion refining them is *Choquet integral*. It makes sense to try and refine a Sugeno integral-based ordering by means of a Choquet integral. Indeed the expression of a Sugeno integral and of a discrete Choquet integral are similar. Moreover while Choquet integrals are additive for comonotonic acts, Sugeno integrals are both maxitive and minitive for comonotonic acts. Intuitively, restricting to acts that rank states in a prescribed order of consequence utilities, Choquet integral behaves like a regular integral and Sugeno integral behaves like a possibilistic criterion. So refining a Sugeno integral by means of a Choquet integral looks like the right way to go, relying on the possibility of refining possibilistic criteria by expected utility.

There are two approaches one might think of for achieving this program.

- Applying the possibilistic criteria transformation directly on the original definition of Sugeno integral, preserving the nature of the original capacity. However, this approach does not address the potential lack of discrimination of the set-function. The latter can be refined in turn if needed.
- Applying the possibilistic criteria transformation on the power set of the state space and the ordinal Moebius transform of the capacity, so as to get a big-stepped random set on the numerical side. The questionable feature of this method is that the nature of the capacity changes during the transformation.

2 Capacity-preserving refinements

Let f be a mapping (an act) from a finite set S (states) to a finite set X (consequences), and γ be an L-capacity on S for a finite chain L. In the standard expression of Sugeno integral, $S_{\gamma,u}(f) =$

$\max_{\lambda_i \in L} \min(\lambda_i, \gamma(F_{\lambda_i}))$ (where $F_{\lambda_i} = \{s \in S, f(s) \geq \lambda_i\}$), the two operators max and min are monotonic but not strictly, hence two nested drowning effects. The simplest idea to refine Sugeno integral is to consider a refinement of its maxmin expression using a leximin criterion embedded within a leximax criterion [2]. To make this generalization clear, let us simply consider that leximin and leximax orderings are defined on sets of tuples whose components belong to a totally ordered set $(\Omega, \trianglerighteq)$, say $Leximin(\trianglerighteq)$ and $Leximax(\trianglerighteq)$. Suppose $(\Omega, \trianglerighteq) = (L^l, Leximin)$ or $(\Omega, \trianglerighteq) = (L^l, Leximax)$, with any $l \in \mathbb{N}$. Then, nested lexicographic ordering relations can be recursively defined, in order to compare L-valued matrices.

Consider for instance the relation $\succeq_{lmax(\succeq lmin)}$ obtained by the procedure $Leximax(Leximin(\geq))$. It applies to matrices $[a]$ of dimensions $p \times q$ with coefficients a_{ij} in $(L, \geq)$. These matrices can be totally ordered in a very refined way by this relation. Denote by $a_{i\cdot}$ row i of $[a]$. Let $[a^\star]$ and $[b^\star]$ be rearranged matrices $[a]$ and $[b]$ such that terms in each row are reordered increasingly and rows are arranged lexicographically top-down in decreasing order. $[a] \succ_{lmax(\succeq lmin)} [b]$ is defined as follows : $\exists k \leq p$ s.t. $\forall i < k, a^\star_{i\cdot} =_{Leximin} b^\star_{i\cdot}$ and $a^\star_{k\cdot} >_{Leximin} b^\star_{k\cdot}$.

Relation $\succeq_{lmax(\succeq lmin)}$ is a complete preorder. $[a] \simeq_{lmax(\succeq lmin)} [b]$ if and only if both matrices have the same coefficients up to the above described rearrangement. Moreover, $\succeq_{lmax(\succeq lmin)}$ refines the ranking obtained by the optimistic criterion: $\max_i \min_j a_{ij} > \max_i \min_j b_{ij}$ implies $[a] \succ_{lmax(\succeq lmin)} [b]$.

Especially, if $[a]$ Pareto-dominates $[b]$ in the strict sense ($\forall i, j, a_{ij} \geq b_{ij}$ and $\exists i^*, j^*$ such that $a_{i^* j^*} > b_{i^* j^*}$), then $[a] \succ_{lmax(\succeq lmin)} [b]$.This leads to the following decision rule:

$$f \succeq^{lsug} g \Leftrightarrow [f]_{\gamma,u} \succeq_{lmax(\succeq lmin)} [g]_{\gamma,u} \tag{1}$$

where $\forall f \in X^S, [f]_{\gamma,u}$ is a $(m+1) \times 2$ matrix $[f]$ on $(L, \leq)$ with coefficients $f_{i1} = \lambda_i$ and $f_{i2} = \gamma(F_{\lambda_i})$, $i = 0, m$. The properties of the $\succeq_{lmax(\succeq lmin)}$ are thus inherited: $\succeq^{lsug}$ is a complete and transitive relation. It refines the ranking of events provided by $S_{\gamma,u}$. Moreover, f is indifferent to g ($f \sim^{lsug} g$) iff $\forall \lambda, \gamma(F_\lambda) = \gamma(G_\lambda)$.

Being a leximax(leximin) procedure, $\succeq^{lsug}$ can be encoded by a sum of products. We can for instance use a "big-stepping" function χ^*, built with respect to the number of levels in L :$\chi^*(\lambda_m) = 0$; and $\chi(\lambda_i) = \frac{K}{N^{2^i}}, i = 0, m-1$, where $N = m$ and K can be any normalization factor. Let us set it so that $\chi(\gamma(S)) = 1$. We can now immediately derive:

$$f \succeq^{lsug} g \Leftrightarrow \sum_{\lambda \in L} \chi^*(\lambda) \cdot \chi^*(\gamma(F_\lambda)) \geq \sum_{\lambda \in L} \chi^*(\lambda) \cdot \chi^*(\gamma(G_\lambda))$$

So, we defined a new evaluation function $EU^{lex} = \sum_{\lambda \in L} \chi^*(\lambda) \cdot \chi^*(\gamma(F_\lambda))$, that refines the ranking provided by $S_{\gamma,u}$. It should be noticed that $EU^{lex}(1_L A 0_L) = \chi^*(\gamma(A))$ i.e. the comparison of events in the sense of EU^{lex} is perfectly equivalent to the one in terms of γ — that is why we say that this refinement preserves the capacity. More generally, the procedure is perfectly unbiased in the sense that the original information, i.e. the ordinal evaluation of the likelihood of the events on L and the one of the utility degrees of the consequence on the same scale is preserved.

It can be shown that $\succeq^{lsug}$ satisfies Savage axioms P1, and P3 to P5 – but obviously not $P2$, since γ can be a non additive capacity. Unsurprisingly, $\succeq^{lsug}$ satisfies the comonotonic Sure Thing Principle — and is ordinally equivalent to a Choquet integral, namely the one based on the utility $u' = \chi^* \circ u$ and the capacity $\nu = \chi^* \circ \gamma$: $f \succeq^{lsug} g \Leftrightarrow Ch_{\chi^* \circ \gamma, \chi^* \circ u}(f) \geq Ch_{\chi^* \circ \gamma, \chi^* \circ u}(g)$.

The intuition behind this result is that the ranking of acts is not modified when replacing $\chi^*(\gamma(\lambda_i)$ by $\chi^*(\gamma(\lambda_i)) - \chi^*(\gamma(\lambda_{i+1}))$ in the definition of EU^{lex} since $\gamma(\lambda_{i+1})$ is negligible with respect to $\gamma(\lambda_i)$. We thus get the Choquet integral. It should be noticed that, when the capacity is a possibility measure Π (resp. a necessity measure N), one does not recover the ranking of acts provided by the refinements of possibilistic decision criteria into expected utility as done in [2]. $\succeq^{lsug}$ preserves the capacity while in this reference a probability measure is used that refines it.

2.1 Refinement using Moebius transforms

Another approach to the same problem may start from the expression of Sugeno integral involving all subsets of S:

$$S_\gamma(f) = \max_{A\subseteq S} \min(\gamma^\#(A), u_A(f))$$

where $u_A(f) = \min_{s\in A} u(f(s))$ and $\gamma^\#(A) = \gamma(A)$ if $\gamma(A) > max_{B\subsetneq A}\gamma(B)$, and 0_L otherwise. We look for an expression of the form $\sum_{A\subseteq S} m_\nu(A) \times u_A(f)$ which is a Choquet integral in terms of the Moebius transform m_ν of a numerical capacity ν.

The above expression of the Sugeno integral has the standard maxmin form w.r.t. a possibility distribution (over the power set). The increasing transformation χ^* that changes a maxmin form into a sum-product encoding of its leximax(leximin) refinement, yields $EU^{lex\#}(f) = \sum_{A\in 2^S} \chi^*(u_A(f)).\chi^*((\gamma^\#(A)))$.

Notice that here the referential is not S nor L, but 2^S; so, in the definition of χ^*, we set $N = 2^{Card(S)}$. We normalize the transformation is such a way that $\sum_{A\in 2^S} \chi^*(\gamma^\#(A)) = 1$. So, the function $m_* : 2^S \mapsto [0,1]$: $m_*(A) = \chi^*(\gamma^\#(A))$ is a mass assignment. Note that m_* is a big-stepped mass function in the sense that: $m_*(A) > 0 \Longrightarrow m_*(A) > \sum_{B\subseteq S,\ \text{s. t.} m_*(B)<m_*(A)} m_*(B)$.

It is easy to show that $\chi^*(u_A(f)) = \chi^*(\min_{s\in A} u(f(s))) = \min_{s\in A} \chi^*(u(f(s)))$. Then:

$$EU^{lex\#}(f) = \sum_{A\subseteq S} m_*(A)\ .\ \min_{s\in A} \chi^*(u(f(s)))$$

is a Choquet integral w.r.t. a belief function which refines the original Sugeno integral. This shows that any Sugeno integral can be refined by a Choquet integral *w.r.t a belief function*.

Contrary to what happens with the Choquet integral obtained in the previous section, the capacity γ is generally not preserved under this second transformation. The resulting Choquet integral is always pessimistic, and sometimes not more discriminant than the original criterion, sometimes more discriminant than the previous refinement. Two particular cases are interesting to consider:

- If γ is a possibility measure Π, then $\gamma^\#(A)$ is positive on singletons of positive possibility only. In other words, $\gamma^\#$ coincides with the possibility distribution of Π and the Moebius expression of the Sugeno integral coincides with the expression of the optimistic possibilistic criterion. So m_* is a regular big-stepped probability and the Choquet integral collapses on a regular expected utility. We retrieve the maximal expected utility refinement proposed in [2].
- On the contrary if γ is a necessity measure N, Ch_{Bel_*,u^*} does not collapse at all with an expected utility. Indeed, Bel_* is a necessity measure ordinally equivalent to the original one. In this case, the resulting Choquet integral is one with respect to a necessity measure. Only the "max-min" framing of the Sugeno integral has been turned into a "sum-product" framing: the transformation has preserved the nature of the original capacity and the refinement identified in Section 2 is retrieved.

When utilities are of the zero-one type, the above refinements are generally void since Sugeno integral coincide with $\gamma(A)$ for some event A. Moreover this is also true when the capacities are uninformative (for instance a necessity function induced by a uniform possibility distribution). The full-fledged refinement of a Sugeno integral should then involve a third step : refining the capacity itself. Some preliminary definitions and results will be presented to this aim.

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Nonstrict Bisymmetric Aggregation Functions Revisited

János FODOR*

Institute of Intelligent Engineering Systems, Budapest Tech
1034 Budapest, Hungary
fodor@bmf.hu

Among the oldest tools both in mathematics and in applications are the *means* of two or more numbers: arithmetic, geometric, harmonic means for example. Such means are special cases of the so-called regular *quasi-arithmetic mean M*:

$$M(x,y) = f^{-1}\left(\frac{f(x)+f(y)}{2}\right) \qquad (x,y \in [a,b]), \tag{1}$$

where f is some strictly monotonic and continuous function on the interval $[a,b]$.

Aczél [1] proved the nice result that a function M of two variables defined on $[a,b]^2$ is a regular quasi-arithmetic mean (i.e., can be represented as in (1)) if and only if M is continuous, symmetric, strictly increasing in each argument, idempotent and fulfils the *bisymmetry equation* for all $x_{11}, x_{12}, x_{21}, x_{22} \in [a,b]$:

$$M[(M(x_{11},x_{12}),M(x_{21},x_{22})] = M[M(x_{11},x_{21}),M(x_{12},x_{22})]. \tag{2}$$

In [2] we studied means satisfying the conditions of Aczél's theorem, except strict monotonicity. That is, we completely described the family of continuous, symmetric, (non-strictly) increasing, idempotent, bisymmetric functions.

The aim of the present talk is to extend some of the results in [2]. The extension is twofold: on one hand, we can prove essentially the same results as we did in [2], but under weaker conditions. On the other hand, some interesting ordinal-sum-like new constructions of continuous bisymmetric aggregations can be derived.

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Capacities and Games on Lattices: A Survey of Results

Michel Grabisch

Université Paris I - Panthéon-Sorbonne
Michel.Grabisch@lip6.fr

Abstract. We provide a survey of recent developments about capacities (or fuzzy measures) and ccooperative games in characteristic form, when they are defined on more general structures than the usual power set of the universal set, namely lattices. In a first part, we give various possible interpretations and applications of these general concepts, and then we elaborate about the possible definitions of usual tools in these theories, such as the Choquet integral, the Möbius transform, and the Shapley value.

1 Introduction

Among the recent advances in capacity (or fuzzy measures) and cooperative game theory, a notable fact is the emergence of new notions of capacities and games which are defined on more general structures than the usual Boolean lattice of the subsets of the universal set. Apart of the mathematical interest brought by such works, the main motivation lies in an attempt to model the real world in a more accurate way.

As it is often the case with generalizations, the main difficulty is to find the right definitions for the usual tools and concepts used in the theory. Concerning capacity theory, fundamental concepts are the Choquet integral and the Möbius transform, while for cooperative game theory, the Shapley value and the core are important notions.

Our aim is to provide a survey of recent advances along these lines. We will see that, although the generalization of the Möbius transform and Choquet integral do not raise particular difficulties, a proper definition of the Shapley value is much more a topic of discussion. We will address also the case of bipolar structures, and show that these structures cannot be reduced to a classical lattice structure, although they can be isomorphically mapped to lattices.

In all our discussion, we consider the universal set to be finite. We denote it by N, and $|N| = n$.

2 Capacities, fuzzy measures, games and the like

Definition 1. *A* capacity *on N is a set function $\mu : 2^N \to \mathbb{R}_+$ such that $\mu(\emptyset) = 0$, and $A \subseteq B \subseteq N$ implies $\mu(A) \leq \mu(B)$. This last property is called* monotonicity.

Capacities have been introduced by Choquet in 1953 [9]. A capacity is *normalized* if $\mu(N) = 1$. In 1974, Sugeno proposed a similar notion (up to some condition of continuity), which he called *fuzzy measures* [47]. Other names which are commonly used are *nonadditive measures* (Denneberg [12]), and *monotonic measures*.

Definition 2. *A* transferable utility game in characteristic form *or for simplicity* game, *is a set function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$.*

The above definition is the central concept of *cooperative game theory* (see, e.g., [13, 10, 3, 43]). The only difference between games and capacities is that monotonicity is dropped for the former. Hence monotonic games coincide with capacities, and non monotonic fuzzy measures, a term which is sometimes used, coincide with games. In the sequel, notation v implicitely designates a game, while μ refers to a capacity.

Given a subset $A \subseteq N$, the precise meaning of the quantity $\mu(A)$ or $v(A)$ depends on the kind of the intended application or domain, in fact essentially what the universal set N is supposed to represent.

- N **is the set of states of nature.** Then $A \subseteq N$ is an *event*, and $\mu(A)$ is the degree of certainty, belief, etc., that A contains the true state of the world. We are here in decision under uncertainty or under risk.
- N **is the set of criteria, or attributes, or sources**. Then $A \subseteq N$ is a group of criteria (or attributes, sources), and $\mu(A)$ is the degree of importance of A for making decision. Corresponding domains are multicriteria decision making, multiattribute utility theory, multiattribute classification, data fusion, etc. In the framework of multicriteria decision making, it is possible to give a more precise definition for $\mu(A)$: it is the overall score of an alternative having score = 1 (maximal) for all criteria in A, and 0 (minimal) for other criteria [29, 38]. This kind of interpretation can be carried on other domains as well.
- N **= set of voters**. Then $A \subseteq N$ is called a *coalition*, and $v(A) = 1$ iff the bill passes when coalition A votes in favor of the bill, and $v(A) = 0$ else.
- N **= set of political parties**. Then $A \subseteq N$ is called a *coalition*, and $v(A) = 1$ iff the coalition of parties wins the election, and $v(A) = 0$ else. These two last examples are a subdomain of cooperative game theory, called *voting games*.
- N **= set of players, agents, companies, etc.** Then $A \subseteq N$ is also called a coalition, and $v(A)$ is the worth (payoff, income, etc.) won by A if all players in A agree to cooperate, and the other ones do not. The concerned domain is cooperative game theory.

3 Generalizations of games and capacities

3.1 Motivations

A first question is:

Why do we need generalizations of classical games and capacities?

The answer to this question is simply that we need them in order to model reality in a more accurate way. Let us elaborate on this, and distinguish several cases.

- A first situation is that some subsets of N may be not meaningful, so that the structure is no more the Boolean lattice 2^N of all subsets of N, but a subcollection of it. More specifically, when N is the set of states of nature, some events may be not observable or not meaningful. Note that in probability theory, it is the usage to define probabilities on algebras (families of subsets closed under unions and complement), not on the whole power set. In the case of political parties, it means that some coalitions of parties are unlikely to occur, or even impossible (coalitions mixing left and right parties). When N is the set of voters, it means that some voting situations (i.e., the set of voters voting in favor) are unlikely to occur or impossible. Lastly, when N is the set of players in a general sense, it may happen that some coalitions are infeasible, for some reasons depending on the precise meaning attached to players (e.g., competitive companies for which it is impossible to cooperate).
- A second possibility is that subsets of N may be not "black and white", which means that the membership of an element to N may be not simply resume to a matter of member or nonmember. This is the case with multicriteria decision making when underlying scales are bipolar, i.e., a

central value exists on each scale, which is a demarcation between values considered as "good", and values considered as "bad", the central value being neutral. When building the model, we must then distinguish for a given alternative criteria which have a good value, from those which have a bad value (or a neutral one). In voting games, it is convenient to consider that players may also abstain, hence each voter has three possibilities, so that giving only the set of voters voting in favor is not enough to describe the voting situation (ternary voting games). When N is the set of players, one may consider that each player can play at different levels of participation, ranging from no participation to full participation. If there is a finite number of such participation levels, it corresponds to multichoice games, when a degree of participation is defined on $[0,1]$, it corresponds to fuzzy games.

- A last possibility is that, after all, elements of interest may be not subsets of N. Global games work on partitions of players, not on coalitions, while games in partition function form and global coalitional games work on the set of partitions and coalitions together.

3.2 Examples of generalized games

Let us introduce main examples of games defined on more general structures.

Games on convex geometries (Bilbao 1998) [2, 4, 3]: a vvvvvvvcollection $\mathcal{L}$ of subsets of N is a *convex geometry* if it contains the empty set, is closed under intersection, and $S \in \mathcal{L}$, $S \neq N$ implies that it exists $j \in N \setminus S$ such that $S \cup j \in \mathcal{L}$. Then, $v : \mathcal{L} \to \mathcal{R}$ is a game on convex geometry $\mathcal{L}$ if $v(\emptyset) = 0$. Convex geometries are dual of antimatroids (see, e.g., [37]), and Bilbao studied also games defined on matroids [3], which are an abstraction of independent systems (see again [37]).

Games with precedence constraints (Faigle 1989) [15, 16]: N being the set of players, let us define a partially ordered set $P := (N, \leq)$, where $\leq$ is a relation of *precedence* among players: $i \leq j$ if the presence of j enforces the presence of i in any coalition $S \subseteq N$. Hence, a *valid coalition* of P is a subset S of N such that $i \in S$ and $j \leq i$ entails $j \in S$.

Ternary voting games (Felsenthal and Machover 1997) [17]: a *ternary voting game* is a voting game where each voter $i \in N$ may vote in favor, against or abstain. Hence, a voting situation is denoted (A,B), where A is the set of voters voting in favor, and B those voting against. Introducing the notation

$$\mathcal{Q}(N) := \{(A,B) \mid A, B \subseteq N, A \cap B = \emptyset\} \tag{1}$$

which represents the set of all voting situations, a ternary voting game is a function $v : \mathcal{Q}(N) \to \{-1, 1\}$. $v(A,B) = 1$ iff the bill passes in voting situation (A,B), $v(A,B) = -1$ iff the bill is rejected.

Another way of denoting a situation (A,B) is to use a vector notation $x \in \{-1,0,1\}^n$ defined as follows:

$$(A,B) \in \mathcal{Q}(N) \cong x \in \{-1,0,1\}^n, \text{ with } x_i = \begin{cases} 1, & \text{if } i \in A \\ -1, & \text{if } i \in B \\ 0, & \text{else.} \end{cases} \tag{2}$$

Hence $\mathcal{Q}(N) \cong \{-1,0,1\}^n \cong 3^N$.

Bi-cooperative games (Bilbao 2000) [3]: they can be seen as a generalization of ternary voting games, like voting games are generalized to (classical) cooperative games. In such games, each player $i \in N$ may play as a defender, a defeater, or does not participate. A *bi-coalition* $(A,B) \in \mathcal{Q}(N)$ represents a situation where A is the defending coalition, and B the opponent coalition. A *bi-cooperatice game* is a function $v : \mathcal{Q}(N) \to \mathbb{R}$ such that $v(\emptyset,\emptyset) = 0$. $v(A,B)$ is the payoff of the game in situation (A,B).

Multichoice games (Hsiao and Raghavan 1993) [34]: each player $i \in N$ has at disposal a totally ordered set of levels of participation labelled $0, 1, \dots, m$, where 0 indicates no participation, and m full participation. A coalition is replaced by a *profile of participation* $x \in \{0, 1, \dots, m\}^n$, where x_i is the level of participation of player i. A *multichoice game* is a function $v : \{0, 1, \dots, m\}^n \to \mathbb{R}$ such that $v(0, \dots, 0) = 0$. The quantity $v(x)$ is the payoff of the game for profile x.

Fuzzy games (Aubin, 1981) [1]: each player has a membership degree in a coalition, considered as a fuzzy set. It can be seen as a multichoice game with a continuum of level of participations. Fuzzy games have been studied by Butnariu and Klement [7], and more recently by Branzei and Tijs [6, 49].

Global games (Gilboa and Lehrer 1991) [20]: let us consider the set of partitions of N, which we denote by $\Pi(N)$. When endowed with the relation of coarseness (i.e., a partition $\mathcal{P}$ is *coarser* than a partition $\mathcal{P}'$ if any set of $\mathcal{P}$ is a superset of some set of $\mathcal{P}'$), the set of partition is a lattice (nondistributive, but geometric). Figure 3.2 shows the lattice of partitions of $\{1, 2, 3, 4\}$. A *global game* is a function $v : \Pi(N) \to \mathbb{R}$.

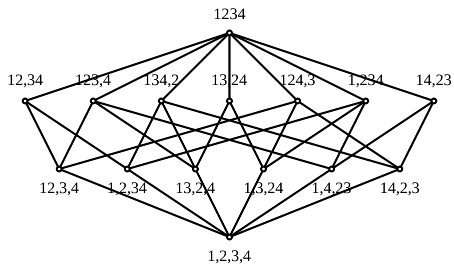


Fig. 1. The lattice of partitions of $\{1, 2, 3, 4\}$

Games in partition function form (Thrall and Lucas 1963) [48]: in these games, the worth of a coalition A depends on the other coalitions which are formed, supposing the set of formed coalitions is a partition of N. For a given partition $\mathcal{P} \in \Pi(N)$, a quantity $v(S, \mathcal{P})$ is defined for any $S \in \mathcal{P}$.

3.3 Examples of generalized capacities

There is much less examples in this category. Here are the few examples we are aware of.

Bi-capacities (Grabisch and Labreuche 2002) [24, 24]: they have been introduced in the field of multicriteria decision making. Let N be the set of criteria. Each criterion $i \in N$ is defined on a *bipolar scale*: a *neutral level* exists (most often the value 0 is taken as neutral level), such that values above it are felt as “good”, and values below it are felt as “bad” by the decision maker. Hence, 3 reference levels are needed to describe the DM’s preferences: the *satisfactory level* (usually the value 1), the neutral level (0), and the *inacceptable level* (usually taken as the value -1). Any combination of the 3 levels is called a *ternary alternative*, denoted by (A, B): A is the set of satisfied criteria, and B the set of unsatisfied criteria. Hence, $\mathcal{Q}(N)$ is the set of ternary alternatives.

A *bi-capacity* is a function $v : \mathcal{Q}(N) \to \mathbb{R}$ such that $v(\emptyset, \emptyset) = 0$, and $A \subseteq B$ implies $v(A, \cdot) \leq v(B, \cdot)$, $v(\cdot, A) \geq v(\cdot, B)$. If normalization applies then $v(N, \emptyset) = 1$ and $v(\emptyset, N) = -1$.

Although bi-cooperative games and bi-capacities were proposed independently and in different domains, formally bi-capacities are monotonic normalized bi-cooperative games.

k-ary capacities (Grabisch and Labreuche 2003) [23]: instead of considering 3 reference levels as for bi-capacities, $k+1$ reference levels are considered on each criterion, their meaning depending on the application considered. *k*-ary capacities correspond in fact to monotonic multichoice games.

4 Games and capacities on lattices

All previous examples of games and capacities are particular cases of games and capacities on lattices.

Definition 3. *Let L be a set and $\leq$ a partial order (antisymmetric and transitive) on L. $(L, \leq)$ is said to be a* lattice *if for any $x, y \in L$, the least upper bound $x \vee y$ and the greatest lower bound $x \wedge y$ always exist. $\top$ and $\bot$ are the greatest and least elements of L, if they exist.*

Definition 4. *Let $(L, \leq)$ be a lattice.*

(i) $v : L \to \mathbb{R}$ *is a* game on lattice L *if* $v(\bot) = 0$.
(ii) $\mu : L \to \mathbb{R}_+$ *is a* capacity on lattice L *if it is a monotonic game, i.e. $x \leq y$ implies $v(x) \leq v(y)$ (*isotone *or* order-preserving *mapping from $(L, \leq)$ to $(\mathbb{R}, \leq)$).*

We denote by $\mathcal{G}(L)$ the set of games on L.

4.1 Some useful facts on lattices

We give in this section some basic results and definitions on lattices which are useful for the sequel (for a good introduction to the topic, see [11]).

For $x, y \in L$, we say that x *covers* y (denoted $x \succ y$) if $x > y$ and there is no z such that $x > z > y$. The lattice is *distributive* if $\vee, \wedge$ obey distributivity.

An element $j \in L$ is *join-irreducible* if it is not the bottom element and it cannot be expressed as a supremum of other elements. Equivalently j is join-irreducible if it covers only one element. Join-irreducible elements covering $\bot$ are called *atoms*, and the lattice is *atomistic* if all join-irreducible elements are atoms. The set of all join-irreducible elements of L is denoted $\mathcal{J}(L)$.

An important property is that in a distributive lattice, any element x can be written as an irredundant supremum of join-irreducible elements in a unique way (Birkhoff theorem):

$$x = \bigvee_{i \in J} i, \quad \text{for some } J \subseteq \mathcal{J}(L) \tag{3}$$

$P \subseteq L$ is a *downset* or *ideal* if $y \leq x$ and $x \in P$ imply $y \in P$. Remarking that in a distributive lattice one can always write $x = \bigvee_{i \in \mathcal{J}(L) | i \leq x} i$, Birkhoff's theorem can be rephrased as follows: any distributive lattice is isomorphic to the lattice of all downsets of $\mathcal{J}(L)$.

In a finite setting, *Boolean lattices* are of the type 2^N for some set N, i.e. they are isomorphic to the lattice of subsets of some set, ordered by inclusion. Boolean lattices are atomistic, and atoms corresponds to singletons. A *linear lattice* is such that $\leq$ is a total order. All elements are join-irreducible, except $\bot$.

Given lattices $(L_1, \leq_1), \dots, (L_n, \leq_n)$, the *product lattice* $L = L_1 \times \cdots \times L_n$ is endowed with the product order $\leq$ of $\leq_1, \dots, \leq_n$ in the usual sense. Elements of x can be written in their vector form $(x_1, \dots, x_n)$. We use the notation (x_A, y_{-A}) to indicate a vector z such that $z_i = x_i$ if $i \in A$, and $z_i = y_i$ otherwise. Similarly L_{-i} denotes $\prod_{j \neq i} L_j$, while $L_K := \prod_{j \in K} L_j$. All join-irreducible elements of L are of the form $(\bot_1, \dots, \bot_{j-1}, i_0, \bot_{j+1}, \dots, \bot_n)$, for some j and some join-irreducible element i_0 of L_j.

A *vertex* of L is any element whose components are either top or bottom. We denote $\Gamma(L)$ the set of vertices of L. Note that $\Gamma(L) = L$ iff L is Boolean.

4.2 Games on product lattices

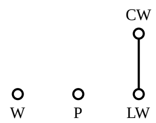
We focus from now on on a specific type of game on lattice, where the lattice is a product of distributive lattices. The motivation for such an approach will be given below.

We consider $L := L_1 \times \cdots \times L_n$, where $L_1, \ldots, L_n$ are finite distributive lattices. Each lattice L_i represents the (partially) ordered set of actions, choices, levels of participation of player i to the game. Each lattice may be different.

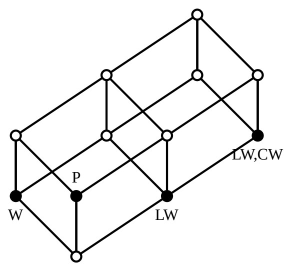
Let us show that most of previous examples can be casted into this framework. If $L_i := \{\bot, \top\}, \forall i \in N$, then we get classical games on 2^N. If $L_i = \{0,1,2\}, \forall i \in N$ we obtain bi-cooperative games (however, see Sec. 10), and ternary voting games on 3^N. If $L_i = \{0,1,\ldots,m\}, \forall i \in N$ we obtain multichoices games on $(m+1)^N$.

One may wonder how the set L_i of all possible actions of player i can be obtained, and why it should be distributive. We consider that each player $i \in N$ has at his disposal a set of *elementary* or *pure* actions $j_1, \ldots, j_{n_i}$. These elementary actions form a partially ordered set $(\mathcal{J}_i, \leq)$, but not necessarily a lattice. Then the set $(\mathcal{O}(\mathcal{J}_i), \subseteq)$ (i.e. the set of downsets) is a distributive lattice denoted L_i, whose join-irreducible elements precisely correspond to the elementary actions, by Birkhoff's theorem.

For example, assume that players are gardeners who take care of some garden or park. Elementary actions are watering (W), light weeding (LW), careful weeding (CW), and pruning (P). All these actions are benefic for the garden and clearly LW<CW, but otherwise actions seem to be incomparable. They form the following partially ordered set:



which in turn form the following lattice of possible actions:



Let us give now an equivalent view of games on lattices, which is due to Faigle and Kern [16], namely games with precedence constraints. We recall that a *valid coalition* of P is a subset S of N such that $i \in S$ and $j \leq i$ entails $j \in S$. Hence, the collection $\mathcal{C}(P)$ of all valid coalitions of P is the collection of all downsets (ideals) of P. It is known that the collection of downsets of a poset is a distributive lattice. Take for example $N = \{a,b,c,d\}$, and $a \leq b, c \leq b, c \leq d$ as a precedence order (Fig. 4.2). Let us show that we can recover our situation, considering that N is the set of players, and for each $i \in N$, let $\mathcal{J}_i := \{j_1, \ldots, j_{n_i}\}$ be the set of elementary actions of player i. We know from the above that $L_i = \mathcal{O}(\mathcal{J}_i)$ for all i. We introduce now the set of virtual players

$$N' := \bigcup_{i \in N} \mathcal{J}_i \tag{4}$$

equipped with the partial order $\leq$ induced by the partial orders on each $\mathcal{J}_i$. Then valid coalitions of $(N', \leq)$ in the sense of Faigle and Kern correspond bijectively to elements of $\mathcal{O}(\mathcal{J}_1) \times \cdots \times \mathcal{O}(\mathcal{J}_n) = L_1 \times \cdots \times L_n$.

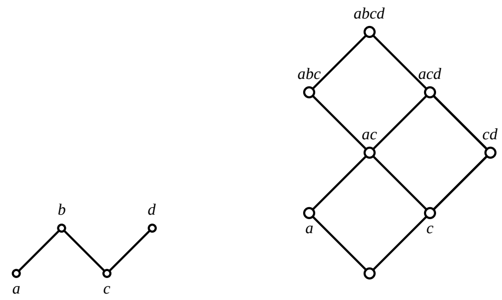


Fig. 2. Exemple of precedence order (left) and the corresponding set of valid coalitions (right)

4.3 Roadmap

This general framework being established, we should re-build usual tools from game theory and capacity theory for the general case of lattice structure. The following concepts lie among the most useful ones:

- the Choquet integral (capacity theory)
- the Möbius transform (capacity theory), otherwise called dividends (game theory); unanimity games
- the Shapley value (game theory, capacity theory)
- the core (game theory, capacity theory)
- the entropy (probability theory, hence capacity theory).

In the sequel we provide a survey of results on these topics.

5 The Choquet integral for bi-capacities

Let $f : N \to [0,1]$ and a capacity μ. We denote for simplicity $f(i)$ by f_i, $i \in N$. The *Choquet integral* [9] of f w.r.t. μ is defined by:

$$\int f \, d\mu := \sum_{i=1}^{n} [f_{(i)} - f_{(i-1)}]\mu(A_{(i)}) \quad (5)$$

with $0 =: f_{(0)} \leq f_{(1)} \leq \cdots \leq f_{(n)}$ and $A_{(i)} := \{(i), \ldots, (n)\}$, i.e., we have applied a permutation on N such that f becomes non decreasing. The *canonical polyhedra* of $[0,1]^n$ are defined by $\{x \in [0,1]^n \mid x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)}\}$, for some permutation σ on N. Clearly, these canonical polyhedra partition the set of functions from N to $[0,1]$ into simplices where the same "weights" $\mu()$ are used in (5).

We recall the following result.

Proposition 1. *Let F be a function on $[0,1]^n$, which is known only on the vertices of the hypercube, and let us find a simplest linear interpolation to determine F entirely:*

$$F(x) = \sum_{A \subseteq N \mid (1_A, 0_{A^c}) \in \mathcal{V}(x)} \Big[\alpha_0(A) + \sum_{i=1}^{n} \alpha_i(A) x_i\Big] F(1_A, 0_{A^c}),$$

where $\mathcal{V}(x)$ is the set of vertices used for the linear interpolation of x, and $\alpha_i(A) \in \mathbb{R}$, $i = 0, \ldots, n$, $\forall A \in \mathcal{V}(x)$. Moreover, we impose that $\mathrm{conv}(\mathcal{V}(x))$ *contains x, and any $x \in [0,1]^n$ should belong to a unique polyhedron (except for common facets), with continuity ensured (triangulation of $[0,1]^n$).*

Then the unique linear interpolation with no constant terms is the Choquet integral, and the triangulation is obtained by the canonical polyhedra.

Lovász [39], considering the problem of extending the domain of pseudo-Boolean functions to $[0,1]^n$ in a linear way (for this extension problem, see also Singer [46]), incidentally discovered the formula of the Choquet integral. The fact that the so-called *Lovász extension* was the Choquet integral was remarked by Marichal [40]. The above result of uniqueness can be found in [27].

Let us apply the same interpolative approach to the case of bi-capacities. The main idea is that for a given point $x \in [-1,1]^n$, it suffices to go back into the positive quadrant $[0,1]^n$ by taking the absolute value $|x|$, and there to apply the interpolation formula (classical Choquet integral), but using vertices of the original quadrant containing x. This leads to the following definition.

Definition 5. *Let v be a bi-capacity and f be a real-valued function on N. The* (general) Choquet integral *of f w.r.t v is given by*

$$\int f\,dv := \int |f|\,d\nu_{N_f^+} \tag{6}$$

where $\nu_{N_f^+}$ is a game on N defined by

$$\nu_{N_f^+}(C) := v(C \cap N_f^+, C \cap N_f^-), \tag{7}$$

and $N_f^+ := \{i \in N \,|\, f_i \geq 0\}$, $N_f^- = N \setminus N_f^+$.

A similar construction can be done for k-ary capacities [23].

6 The Möbius transform

Following the general definition of Rota [44] (see also [5, p. 102]), we have readily a definition for any game defined on any lattice, or even for games defined on any partially ordered set, provided it is locally finite (i.e., any interval is finite) and with a bottom element. Let v be a game on $(L,\leq)$, the *Möbius transform* of v is the function $m : L \longrightarrow \mathbb{R}$ solution of the equation:

$$v(x) = \sum_{y \leq x} m(y). \tag{8}$$

The expression of m is obtained through the Möbius function μ by:

$$m(x) = \sum_{y \leq x} \mu(y,x)\,f(y) \tag{9}$$

where μ is a function on L^2 defined inductively by

$$\mu(x,y) = \begin{cases} 1, & \text{if } x = y \\ -\sum_{x \leq t < y} \mu(x,t), & \text{if } x < y \\ 0, & \text{otherwise}. \end{cases} \tag{10}$$

Note that μ depends only on the structure of $(L,\leq)$.

7 The Shapley value

7.1 The classical case

The Shapley value, or more generally the notion of value, is one of the most important concept in cooperative game theory. A *value* or *solution concept* is any function $\phi : \mathcal{G}(2^N) \longrightarrow \mathbb{R}^N$, which represents an imputation of income to each player, supposing that all players will join the grand coalition N, so that the amount $v(N)$ has to be shared among players. The value is *efficient* if $\sum_{i\in N} \phi_i(v) = v(N)$. Among other properties or axioms values should satisfy, the following ones are classical.

- **linearity (l)**: ϕ is linear over $\mathcal{G}(2^N)$.
- **dummy axiom (d)**: if i is dummy for v, then $\phi^v(i) = v(i)$.
- **null axiom (n)**: if i is null for v, then $\phi^v(i) = 0$.
- **symmetry (s)**: ϕ does not depend on the labelling of the players.

A player i is *dummy* if $v(S \cup i) = v(S) + v(i)$ for any $S \subseteq N \setminus i$. A player is *null* if $v(S \cup i) = v(S)$ for any $S \subseteq N \setminus i$. Remark that a null player is such that $v(i) = 0$, hence it is also a dummy player. Note also that the dummy axiom is stronger than the null axiom. The *Shapley value* [45] is the unique value satisfying axioms **l**, **n**, **s** and efficiency, and is given by

$$\phi^v(i) := \sum_{S \subseteq N \setminus i} \frac{(n-s-1)!s!}{n!} [v(S \cup i) - v(S)], \tag{11}$$

where $s := |S|$.

An equivalent definition can be obtained in a combinatorial way as an average of the contribution of player i over all maximal chains in 2^N:

$$\phi^v(i) = \frac{1}{n!} \sum_{\mathcal{C} \in \mathcal{M}(2^N)} [v(S^i_{\mathcal{C}}) - v(S^i_{\mathcal{C}} \setminus i)] \tag{12}$$

where $\mathcal{M}(2^N)$ is the set of all maximal chains in the lattice 2^N, and for each such chain $\mathcal{C}$, $S^i_{\mathcal{C}}$ is the first subset in the maximal chain containing i.

The Shapley value can be also obtained through unanimity games and linearity as follows. Unanimity games are closely linked to the Möbius transform, since any game v can be written as

$$v = \sum_{S \subseteq N} m(S) u_S \tag{13}$$

where m is the Möbius transform of v, and u_S is the *unanimity game centered on S*, defined by:

$$u_S(T) = \begin{cases} 1, & \text{if } T \supseteq S \\ 0, & \text{else.} \end{cases} \tag{14}$$

A natural axiom for the Shapley value of unanimity games is

$$\phi^{u_S}(i) = \begin{cases} \frac{1}{|S|}, & \text{if } i \in S \\ 0, & \text{else,} \end{cases} \tag{15}$$

since only players in S have a contribution to the game, and all players in S are symmetric (anonymous). By linearity of the Shapley value, we get

$$\phi^v(i) = \sum_{S \ni i} \frac{m(S)}{|S|}, \tag{16}$$

which is equivalent to (11).

7.2 The Shapley value for multichoice games

We shall examine in the sequel various propositions for a definition of the Shapley value for multichoice games. We recall that $L = L_1 \times \cdots \times L_n$, and all L_i's are linear lattices, denoted by $L_i := \{0, 1, 2, \dots, l_i\}$, where 0 means non participation. Elements x of L are called participation profiles, with x_i the level of participation of player i. $\widetilde{(0_{-i}, k_i)}$ is the profile where player i plays at level k, the other ones not participating. We often write $\widetilde{k_i}$ for $(0_{-i}, k_i)$, and $\top_i$ for l_i. Many different approaches exist, which do not coincide:

- approach of Faigle and Kern [16]
- approach of Hsiao and Raghavan [34]
- approach of Grabisch and Lange [21, 30]

Let us detail first the approach of Faigle and Kern. The basic idea is to axiomatize the Shapley value for unanimity games, and then to apply linearity (combinatorial approach). The expression is the following:

$$\phi^v_{\mathrm{FK}}(k_i) = \frac{1}{|\mathcal{M}(L)|} \sum_{C \in \mathcal{M}(L)} [v(x_{k_i}) - v(\underline{x}_{k_i})] \tag{17}$$

where $\mathcal{M}(L)$ is the set of maximal chains in L, and x_{k_i} is the first in the sequence C such that $x_{k_i} \geq \widetilde{k_i}$, and $\underline{x}_{k_i}$ is its predecessor. Although the expression is simple and appealing, let us remark that the number of maximal chains for the multichoice case is:

$$|\mathcal{M}(L)| = \frac{\left(\sum_{i \in N} l_i\right)!}{\prod_{i \in N}(l_i!)} = \binom{l}{l_1}\binom{l - l_1}{l_2}\binom{l - l_1 - l_2}{l_3} \cdots 1, \tag{18}$$

with $l := \prod_{i \in N} l_i$. For 5 players having each 3 actions, this already gives $(15)!/6^5 = 168{,}168{,}000$. Also, some of the axioms proposed by Faigle and Kern are not intuitive in a game theoretic sense (e.g., the hierarchical strength axiom).

The basic idea of the Shapley value of Hsiao and Raghavan is also to axiomatize the Shapley value for unanimity games, and then to apply linearity. The original feature is to put weights $w_1 < w_2 < \cdots < w_l$ on participation levels. The expression of the Shapley value for unanimity games they obtain is as follows:

$$\phi^{u_x}_{\mathrm{HR}}(k_i) = \begin{cases} \frac{w_k}{\sum_{i \in N} w_{x_i}}, & \text{if } k = x_i \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

The expression for any game is extremely complex and will not be reported here. It has been proved that for no set of weights $w_1, \dots, w_l$, the values of H-S and F-K always coincide [6]. Although the axioms are appealing, the resulting formula is almost inapplicable. Also, the role of the weights $w_1, \dots, w_n$ is not clear.

We present now our approach. The main idea is to follow as much as possible the original axioms of Shapley. We aim at defining a value $\Phi^v(k_i)$ representing the contribution of player i playing at level k vs. non participation of i. This contrasts with the two previous approaches, which represent the contribution of player i playing at level k compared to the situation where he plays at level $k-1$. For some $k \in L_i$, $k \neq 0$, player i is said to be *k-null* (or simply k_i *is null*) if $v(x, k_i) = v(x, 0_i)$, for any $x \in L_{-i}$. Similarly, for some $k \in L_i$, $k \neq 0$, player i is said to be *k-dummy* (or simply k_i *is dummy*) if $v(x, k_i) = v(x, 0_i) + v(\widetilde{k_i})$, $\forall x \in L_{-i}$. Based on these definitions, we propose the following axioms.

- **Linear axiom (L):** Φ^v is linear on the set of games $\mathcal{G}(L)$
- **Null axiom (N):** $\forall v \in \mathcal{G}(L)$, for all null k_i, $\Phi^v(k_i) = 0$.

– **Dummy axiom (D):** $\forall v \in \mathcal{G}(L)$, for all dummy k_i, $\Phi^v(k_i) = v(\widetilde{k_i})$.
– **Monotonicity axiom (M):** if v is monotone, then $\Phi^v(k_i) \geq 0$, for all $k > 0, i \in N$.

The next axiom is similar to the symmetry axiom. Since all lattices L_i may be different, a direct transposition of the classical symmetry axiom is not possible. Let $\Gamma(L) := \{0_1, \top_1\} \times \cdots \times \{0_n, \top_n\}$ be the set of vertices of L. We introduce a subspace of $\mathcal{G}(L)$:

$$\mathcal{G}_0(L) := \{v \in \mathcal{G}(L) \mid v(x) = 0, \forall x \notin \Gamma(L)\} \tag{20}$$

– **Symmetry axiom (S):** Let σ be a permutation on N. Then for any game $v \in \mathcal{G}_0(L)$,

$$\Phi^{v^{\sigma^{-1}}}(\top_i^{\sigma}) = \Phi^v(\top_i) \tag{21}$$

where for any $x \in \Gamma(L)$, $x^\sigma := (x_1^\sigma, \dots, x_n^\sigma)$, and

$$x_i^\sigma := \begin{cases} 0_i, & \text{if } x_{\sigma(i)} = 0_{\sigma(i)} \\ \top_i, & \text{if } x_{\sigma(i)} = \top_{\sigma(i)} \end{cases} \tag{22}$$

and for any $v \in \mathcal{G}_0(L)$, v^σ is a game in $\mathcal{G}_0(L)$ such that $v^\sigma(x) := v(x^\sigma)$, for any $x \in \Gamma(L)$.

For example, if $L := \{0,1,2\} \times \{0,1,2,3,4\} \times \{0,1,2,3\}$, and the permutation σ is defined by

i	1 2 3
$\sigma(i)$	2 3 1

then $(2,0,0)^\sigma = (0,0,3)$, $(2,0,3)^\sigma = (0,4,3)$.

Next axiom is not in the original set of axioms of Shapley, and concerns a kind of homogeneity of the structure of the L_i's.

– **Invariance axiom (I):** Let us consider v_1, v_2 on L such that for some $i \in N$,

$$\begin{aligned} v_1(x, k_i) &= v_2(x, (k-1)_i), \quad \forall x \in L_{-i}, \forall 1 < k \leq l_i \\ v_1(x, 0_i) &= v_2(x, 0_i), \quad \forall x \in L_{-i}. \end{aligned}$$

Then $\Phi^{v_1}(k_i) = \Phi^{v_2}((k-1)_i)$, $1 < k \leq l_i$.
– **Efficiency axiom (E):** $\sum_{i \in N} \Phi^v(\top_i) = v(\top)$.

Proposition 2. *Under axioms **(L)**, **(D)**, **(M)**, **(S)**, **(I)** and **(E)**,*

$$\Phi^v(k_i) = \sum_{x \in \Gamma(L_{-i})} \frac{(n - h(x) - 1)! h(x)!}{n!} [v(x, k_i) - v(x, 0_i)], \quad 1 \leq k \leq l_i, i \in N, \tag{23}$$

where $h(x) := |\{k \in N \setminus i \mid x_k = \top_k\}|$.

Remark that the result is very close to the classical formula of Shapley. For a formula on more general lattices (but without axiomatization) and for the interaction index, see [28].

8 The core

8.1 The classical case

In game theory, the *core* of v is another way to define rational imputations for players. Specifically, it is a set of imputations such that no subcoalition has interest to form:

$$\mathcal{C}(v) := \{\phi \in \mathbb{R}^n \mid \phi(N) = v(N) \text{ and } \phi(A) \geq v(A), \forall A \subseteq N\} \tag{24}$$

with $\phi(A) := \sum_{i \in A} \phi(i)$. Otherwise said, it is the set of additive games dominating v and coinciding on N. Whenever nonempty, the core is a convex set. It is reduced to the singleton $\{v\}$ if the game is additive.

The same concept exists also in capacity theory. It is seen as the set of probability measures dominating a given capacity (see properties of the core in [8]).

A related concept is the *Weber set*. It is the convex hull of the set $\mathcal{M}(v)$ of marginal worth vectors

$$\mathcal{W}(v) := \mathrm{co}(\mathcal{M}(v)), \tag{25}$$

where a marginal worth vector is defined as the increment of v along a maximal chain in the Boolean lattice 2^N. Specifically, to any permutation π on N, we associate a maximal chain

$$A_0^\pi := \emptyset \subset A_1^\pi := \{\pi(1)\} \subset A_2^\pi := \{\pi(1), \pi(2)\} \subset \cdots \subset A_n^\pi := N \tag{26}$$

with $A_i^\pi := \{\pi(1), \ldots, \pi(i)\}$. Then, the corresponding *marginal worth vector* $x^\pi(v)$ is defined by:

$$x^\pi_{\pi(i)}(v) := v(A_i^\pi) - v(A_{i-1}^\pi), \quad i = 1, \ldots, n. \tag{27}$$

The following proposition summarizes well-known results. We recall that a game is convex if $v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$ for any $A, B \subseteq N$.

Proposition 3. *Let v be a game on N. The following holds.*

(i) $\mathcal{C}(v) \subseteq \mathcal{W}(v)$.
(ii) $\mathcal{C}(v) \neq \emptyset$ *if v is convex.*
(iii) *v is convex iff $\mathcal{C}(v) = \mathcal{W}(v)$ (i.e., the set of marginal worth vectors is the set of vertices of the core).*

8.2 The case of multichoice games

Here also, several different approaches have been proposed, the first one being by Faigle [15], see also the works of Tijs et al. [50]. For a detailed comparison of these previous works with our approach, see [51, 31]. We give below the main elements of our approach, and the one of Faigle.

v being a multichoice game, we say that v is *convex* if $v(x \vee y) + v(x \wedge y) \geq v(x) + v(y)$, for all $x, y \in L$, and v is *additive* if for every $x, y \in L$ such that $x \wedge y = \bot$, it holds $v(x \vee y) = v(x) + v(y)$.

We denote by $\mathcal{A}(L)$ the set of additive games on L. The following definition is a direct transposition of the classical definition.

Definition 6. *The* precore *of a multichoice game v on L is defined by*

$$\mathcal{PC}(v) := \{\phi \in \mathcal{A}(L) \mid \phi(x) \geq v(x), \forall x \in L, \text{ and } \phi(\top) = v(\top)\}. \tag{28}$$

This is in fact the definition of Faigle. However, it is easy to see that the precore, although convex, is unbounded. Indeed, considering a 2-choice game with two players, hence $L := \{0,1,2\}^2$, the conditions on ϕ to be element of the precore write:

$$\begin{aligned}\phi(2,0)+\phi(0,2) &= v(2,2)\\ \phi(1,0) &\geq v(1,0)\\ \phi(0,1) &\geq v(0,1)\\ \phi(1,0)+\phi(0,1) &\geq v(1,1).\end{aligned}$$

Remark that $\phi(1,0)$ and $\phi(0,1)$ may be taken arbitrarily large. We denote by $\mathcal{PC}^F(v) := \mathrm{co}(\mathrm{Ext}(\mathcal{PC}(v)))$ the polytope of $\mathcal{PC}(v)$, where Ext() is the set of extreme points (vertices) of some convex set.

To avoid these drawbacks, we propose the next definition, where normalization occurs at each level.

Definition 7. *The* core *of a multichoice game v on N is defined as:*

$$\mathcal{C}(v) := \{\phi \in \mathcal{A}(L) \mid \phi(x) \geq v(x), \forall x \in L, \\ \text{and } \phi(k\wedge l_1,\dots,k\wedge l_n) = v(k\wedge l_1,\dots,k\wedge l_n), k=1,\dots,\max_j l_j\}.$$

As for the classical case, we introduce marginal worth vectors ψ^C as the vectors of increments along maximal chains C in the lattice L. Coordinates of ψ^C are denoted by $\psi^C_{k_j}$, for any player $j \in N$ and any level $k>0$ in L_j. To any marginal vector is associated an additive game ϕ^C defined by

$$\phi^C_{k_j} := \sum_{p=1}^{k} \psi^C_{p_j}. \tag{29}$$

The set of all such additive games is called $\mathcal{PM}(v)$, and the *pre-Weber set* $\mathcal{PW}(v)$ is defined as the convex hull of all additive games in $\mathcal{PM}(v)$. Considering only *restricted* maximal chains in L, i.e., those passing through all $(k\wedge l_1,\dots,k\wedge l_n)$, $k=1,\dots,\max_j l_j$, we define $\mathcal{M}(v)$, the set of all additive games ϕ^C corresponding to marginal worth vectors associated to all restricted maximal chains. Then the *Weber set* $\mathcal{W}$ is defined as the convex hull of all additive games in $\mathcal{M}(v)$.

The following has been shown, which generalizes the classical results of Prop. 3.

Proposition 4. *Let v be a multichoice games on L. The following holds.*

(i) $\mathcal{PC}^F(v) \subseteq \mathcal{PW}(v)$
(ii) $\mathcal{C}(v) \subseteq \mathcal{W}(v)$
(iii) If v is convex, then $\mathcal{C}(v) = \mathcal{W}(v)$
(iv) If v is convex, then $\mathcal{PC}^F(v) = \mathcal{PW}(v)$.

9 The entropy

The entropy is a central notion in probability and information theory. The first attempt to generalize the classical definition of Shannon to the case of capacities was done by Yager [52], by considering the Shannon entropy of the Shapley value of the capacity. A slightly different approach was taken by Marichal and Roubens [42, 41], which turned out to have better properties. In particular, it is strictly increasing towards the capacity which maximizes entropy. Its expression for some capacity μ is given below:

$$H_{\mathrm{MR}}(\mu) := \sum_{i=1}^{n} \sum_{S \subseteq N \setminus i} \frac{(n-s-1)!s!}{n!} h(\mu(S \cup i) - \mu(S)) \tag{30}$$

where $h(x) := -x \log x$, for $x > 0$, and $h(0) := 0$. An important result, due to Dukhovny [14], shows that the definition of Marichal and Roubens can be written as an average of classical entropy along maximal chains of the Boolean lattice 2^N, specifically:

$$H_{\mathrm{MR}}(\mu) = \frac{1}{n!} \sum_{\mathcal{C} \in \mathcal{M}(2^N)} H_{\mathrm{S}}(p^{\mu,\mathcal{C}}) \tag{31}$$

where H_{S} is the Shannon entropy, and $p^{\mu,\mathcal{C}}$ the probability induced by the maximal chain $\mathcal{C}$ and the capacity μ, i.e., using the same notations as for Eq. (12):

$$p^{\mu,\mathcal{C}}(\{i\}) = \mu(S^i_{\mathcal{C}}) - \mu(S^i_{\mathcal{C}} \setminus i), \quad i \in N \tag{32}$$

(identical to marginal worth vectors).

Honda and Grabisch have shown that the above definition could be generalized without losing its nice properties for capacities on particular set systems [33]. Let us consider $\mathcal{N}$ a subcollection of 2^N. Then we call $(N, \mathcal{N})$ (or simply $\mathcal{N}$ if no ambiguity occurs) a *set system* if $\mathcal{N}$ contains $\emptyset$ and N. A set system is a particular partially ordered set when endowed with inclusion, hence usual definitions apply, in particular the notion of maximal chain. We denote the set of all maximal chains of $\mathcal{N}$ by $\mathcal{M}(\mathcal{N})$. $(N, \mathcal{N})$ is a regular set system if for any $\mathcal{C} \in \mathcal{M}(\mathcal{N})$, the length of $\mathcal{C}$ is n, i.e. $|\mathcal{C}| = n+1$. Equivalently, $\mathcal{N}$ is a regular set system if and only if $|A \setminus B| = 1$ for any $A, B \in \mathcal{N}$ such that $A \succ B$. Let μ be a capacity on $(N, \mathcal{N})$. For any $\mathcal{C} \in \mathcal{M}(2^N)$, define $p^{\mu,\mathcal{C}}$ by (32) again. Then the entropy of μ on $(N, \mathcal{N})$ is given by:

$$H_{\mathrm{HG}}(\mu) := \frac{1}{|\mathcal{M}(\mathcal{N})|} \sum_{\mathcal{C} \in \mathcal{M}(\mathcal{N})} H_{\mathrm{S}}(p^{\mu,\mathcal{C}}). \tag{33}$$

H_{HG} is a continuous function of μ, and $0 \leq H_{\mathrm{HG}} \leq \log n$, with equality at left attained if and only if μ is a 0-1 valued capacity, and at right if and only if μ is the additive uniform capacity. Moreover, H_{HG} is strictly increasing towards the value of the additive uniform capacity.

The entropy for capacities has been axiomatized by Kojadinovic *et al.* [36] using a recursive axiom difficult to interpret. Honda and Grabisch have axiomatized H_{HG} in a different way [32], avoiding such an axiom, and following Faddeev's classical axiomatization of Shannon entropy.

10 The case of bipolar structures

Let us come back on bi-capacities and bi-cooperative games. First works on bi-capacities [22, 24, 25] have taken for granted that these were capacities defined on the lattice $(\mathcal{Q}(N), \sqsubseteq)$, with $(A, A') \sqsubseteq (B, B') \Leftrightarrow A \subseteq B$ and $A' \supseteq B'$. Doing so, bi-capacities are indeed monotonic mappings and match the general definition of capacities on lattices (see Def. 4).

There is nevertheless something discordant in the fact that doing so, since $(\mathcal{Q}(N), \sqsubseteq)$ is isomorphic to the lattice 3^n, bi-cooperative games become in some sense isomorphic to multichoice games with $m = 2$, a conclusion which may be surprising if one consider the interpretation behind them. Let us elaborate on this last point. We may say that for a 2-choice game, the underlying levels of participation are naturally labelled $0, 1, 2$, with 0 indicating non participation, 1 a mild participation, and 2 a full participation. For bi-cooperative games, keeping the same labelling leads to something rather odd, since 0 means (full) participation against, 1 non participation, and 2 (full) participation. Hence, a more natural labelling would be $-1,0,1$, the 0 value being central, and $-1,1$ being symmetric extremes. This suggests that:

(i) the point $(\emptyset,\emptyset)$ is central in the structure $\mathcal{Q}(N)$, although in 3^N, $(1,\dots,1)$ has no central role;
(ii) bi-cooperative games are not 2-choice games, but rather a symmetrization of classical cooperative games.

Looking at the definition of the Choquet integral for bi-capacities (Sec. 5), one can see that it already follows the above principle.

Consequently, the order $\sqsubseteq$ should be replaced by the product order $\subseteq$: $(A,A') \subseteq (B,B') \Leftrightarrow A \subseteq B$ and $A' \subseteq B'$. Interestingly enough, this was the first definition proposed by Bilbao [3] for the underlying structure of bi-cooperative games. Now, $(\mathcal{Q}(N),\subseteq)$ is no more a lattice, but an inf-semilattice.

Consequently, a proper definition of the Möbius transform is not the one proposed in [24], solution of the equation:

$$v(A,A') = \sum_{(B,B')\sqsubseteq(A,A')} m(B,B') \tag{34}$$

but it should be the solution of the equation:

$$v(A,A') = \sum_{(B,B')\subseteq(A,A')} m(B,B') \tag{35}$$

whose solution is:

$$m(A,A') = \sum_{\substack{B\subseteq A\\ B'\subseteq A'}} (-1)^{|A\setminus B|+|A'\setminus B'|} v(B,B'). \tag{36}$$

This function m, which could be called the *bipolar Möbius transform*, has been first proposed by Fujimoto [18, 19], in order to avoid the complicated expression of the Choquet integral in terms of the Möbius transform given in [25]. Indeed, using the (bipolar) Möbius transform, the Choquet integral simply writes:

$$\int f\,dv = \sum_{(A,A')\in\mathcal{Q}(N)} m(A,A') \Big[\bigwedge_{i\in A} f_i^+ \wedge \bigwedge_{j\in A'} f_j^- \Big]. \tag{37}$$

The definition of entropy for bi-capacities, as it is given by Kojadinovic and Marichal [35], follows in fact the same philosophy. It writes:

$$H_{\mathrm{KM}}(v) := \frac{1}{2^n} \sum_{N^+\subseteq N} \frac{1}{n!} \sum_{\pi\in\Pi_N} H_S(p^v_{\pi,N^+}) \tag{38}$$

where π is any permutation on N, and p^v_{π,N^+} is the probability distribution induced by v and the maximal chain induced by π in the sublattice $[(\emptyset,\emptyset),(N^+,N\setminus N^+)]$.

In summary, bi-capacities and bi-cooperative games should be considered as a particular symmetrization of capacities and cooperative games, as well as $\mathcal{Q}(N)$ should be considered as a symmetrization of $\mathcal{P}(N) = 2^N$. We call this particular symmetrization *bipolar extension*, and show now that this can be made fairly more general [27].

Definition 8. *Let $(L,\leq)$ be an inf-semilattice with bottom element $\perp$. We define its* bipolar extension *by*

$$\widetilde{L} := \{(x,y) \mid x,y \in L, x\wedge y = \perp\}, \tag{39}$$

which we endow with the product order $\leq$ on L^2.

Clearly, $\mathcal{Q}(N) = \widetilde{\mathcal{P}(N)}$. The following holds.

Proposition 5. *Let $(L, \leq)$ be an inf-semilattice.*

(i) *$(\widetilde{L}, \leq)$ is an inf-semilattice whose bottom element is $(\perp, \perp)$, where $\leq$ is the product order on L^2.*
(ii) *The set of join-irreducible elements of $\widetilde{L}$ is*

$$\mathcal{J}(\widetilde{L}) = \{(j, \perp) \mid j \in \mathcal{J}(L)\} \cup \{(\perp, j) \mid j \in \mathcal{J}(L)\}. \tag{40}$$

(iii) *The Möbius function on $\widetilde{L}$ is given by:*

$$\mu_{\widetilde{L}}((z,t),(x,y)) = \mu_L(z,x)\mu_L(t,y). \tag{41}$$

Bipolar extensions have been further investigated in [26], concerning the definition of the Choquet integral or other aggregation operators on such structures.

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The Minimal Dominant Set is a Non-Empty Core-Extension*

László Á. Kóczy[1] and Luc Lauwers[2]

[1] Department of Economics
University of Maastricht
6200MD Maastricht, The Netherlands
l.koczy@algec.unimaas.nl
[2] Center for Economic Studies, Catholic University Leuven
3000 Leuven, Belgium
luc.lauwers@econ.kuleuven.ac.be

For a characteristic function form game there are two fundamental and strongly linked problems: (i) what coalitions will form, and (ii) how will the members of these coalitions distribute their total coalitional worth. We attempt to answer these questions. Following Harsányi [2], we presuppose some bargaining process among the players. At first, one of the players proposes some outcome (a payoff vector augmented with a coalition structure). In case some coalition could gain by acting for themselves, it can reject this initial outcome and propose a second outcome. Of course, in order to be able to make a counter-proposal, the deviating coalition is a member of the new coalition structure and none of the players in the deviating coalition loses when moving towards the new outcome. We impose an additional condition that we call *outsider-independence*: a coalition C that belongs to the initial coalition structure and that does not contain a deviating player survives the deviation; the players in C stay together and keep their pre-deviation payoffs. This contrasts with, for example, the approach by Sengupta and Sengupta [4], and Shenoy [6, Section 5]. They tackle the same problem without incorporating such an outsider-independence condition: the deviating coalition is allowed to determine the payoffs and the structure of *all* players. This seems unrealistic to us. In contrast, our approach is based on the observation that outsiders' payoffs are unaffected by the formation of the deviating coalition and hence outsiders do not necessarily notice the deviation until the new coalition structure is announced.

Once such a counter-proposal has popped up, another coalition may reject this counter-proposal in favour of a third outcome, and so forth. This bargaining process generates a dominating chain of outcomes. In case the game has a non-empty coalition structure core [1], the bargaining process enters this core after a finite number of steps [3]. Conclusion: the coalition structure core, if non-empty, is accessible.

Similarly to the core, the coalition structure core has an important shortcoming: non-emptiness is far from being guaranteed. The present paper tackles games with an empty set of undominated outcomes.

We impose three conditions upon a solution concept. *First*, we insist on accessibility: from each outcome there is a dominating chain that enters the solution. *Second*, the solution is closed for domination: each outcome that dominates an outcome in the solution also belongs to the solution. The intuition behind this axiom is straightforward. In case there are no "undominated outcomes", there might exist "undominated sets" of outcomes. Such a set must be closed for outsider-independent domination. A collection of outcomes that combines accessibility and closedness is said to be a dominant set. And, *third*, from all the dominant sets, we only retain the minimal (with respect to inclusion) ones.

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The following observation provides a further argument in favour of these three conditions: in case the game generates undominated outcomes, then the accessibility of the coalition structure core implies that this core is the unique minimal dominant set. Uniqueness and non-emptiness extends to arbitrary games:

Theorem 1. *Each characteristic function form game has exactly one minimal dominant set. Moreover, this minimal dominant set is non-empty.*

In other words, the minimal dominant set is a non-empty coalition structure core extension. On the one hand, the three conditions we impose upon a solution concept are strong enough to filter out the coalition structure core (in case it is non-empty), and on the other hand these conditions are weak enough to return a non-empty set of outcomes in case the game has an empty coalition structure core. As a matter of fact, the minimal dominant set meets Zhou's [8] minimal qualifications for a solution concept: non-imposition with respect to the coalition structure[3] and non-emptiness.

We close the discussion on Theorem 1 with an example. Consider a three player game with an empty core: singletons have a zero value, pairs have a value equal to 8, and the grand coalition has a value 9. The payoff vector $(4,4,0)$ supported by the coalition structure $(\{1,2\},\{3\})$ belongs to the minimal dominant set. This outcome, however, is not efficient: the total payoff in this vector amounts to 8, where the value 9 is obtainable. On the other hand, the efficient outcome $(3,3,3;\{1,2,3\})$ does not belong to the minimal dominant set. Hence, the minimal dominant set might contain inefficient outcomes and at the same time there might be efficient outcomes outside the minimal dominant set. Where the core selects those outcomes that satisfy efficiency and stability, these two properties are not so well linked as soon the core is empty (Section 5 returns to this issue).

Along the proof of Theorem 1 we come across the following properties of the outsider-independent domination relation. First, the set of outcomes that indirectly dominate an (initial) outcome is closed in the Euclidean topology. And, second:

Theorem 2. *There exists a natural number $\tau = \tau(n)$ such that for each game with n players and for all outcomes a and b in this game, we have that a indirectly dominates b if and only if there exists a dominating chain from b to a of length at most τ.*

As a consequence, the accessibility axiom can be sharpened: for each game the minimal dominant set can be reached via τ subsequent counter-proposals. This number τ can be imposed as a time-limit for the completion of the bargaining process.

Theorem 2 dramatically improves previous results on the accessibility of the core. We mention two of them. First, Wu [7] has shown the existence of an infinite bargaining scheme that converges to the core and rephrased this result as "the core is globally stable". Second, Sengupta and Sengupta [5] construct for each imputation a sequence of dominating imputations that enters the core in finitely many steps. We extend these results to the coalition structure core and to the minimal dominant set. In addition, we provide an upper bound for the length of the dominating chains.

Finally, Theorem 2 implicitly provides directions on how to compute the minimal dominant set. The proof of Theorem 2 rests upon a stratification of the set of all imputations into a finite number of classes. Each class gathers imputations that we label *similar*. Apparently, the minimal dominant set coincides with the union of some of these classes. As such, the search for the minimal dominant set boils down to a finite problem. As an illustration, we retake the above three player game. Here,

[3] In the framework of endogenous coalition formation, a solution concept "is not a priori defined for payoff vectors of a particular coalition structure, and it does not always contain payoff vectors of every coalition structure," [8, p. 513].

the set of outcomes is partitioned into 29 classes. First, there are 19 (non-empty) classes of efficient outcomes:

$$(x,\{1,2,3\}) \text{ with } x_1+x_2+x_3=9,\ x_i+x_j \bowtie^1_{ij} 8,\ x_k+x_l \bowtie^2_{kl} 9,\ x_m \geq 0,$$

where the indices i, j, k, l, and m all run over the set $\{1,2,3\}$ and where $\bowtie$ stands for either $<$ or $\geq$. Additional labels are used to distinguish different instances –which may be different inequalities– from each other.

Next, there are 9 classes in which one player is standing alone: $(x;\{i,j\},\{k\})$ with $\{i,j,k\} = \{1,2,3\}$, $x_i+x_j=8$, $x_i \bowtie^1_i 8$, $x_j \bowtie^2_j 8$, and $x_k=0$. Finally, there is the zero-outcome: $(0;\{1\},\{2\},\{3\})$. The minimal dominant set collects 26 of these classes: the (large) class

$$(x,\{1,2,3\}) \text{ with } x_1+x_2+x_3=9,\ x_1+x_2<8,\ x_1+x_3<8, \text{ and } x_2+x_3<8,$$

and the zero-outcome are excluded from the minimal dominant set.

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Approximation and Identification of Capacities and Bi-Capacities Using a Minimum Distance Principle

Ivan Kojadinovic

École polytechnique de l'université de Nantes
LINA CNRS FRE 2729
44306 Nantes, France
Ivan.Kojadinovic@polytech.univ-nantes.fr

With outranking methods [Roy and Bouyssou, 1993], multi-attribute utility theory (MAUT) [Keeney and Raiffa, 1976] is probably the most frequently applied approach to multi-criteria decision aiding (MCDA) [Vincke, 1992]. Given a set $\mathcal{A} := \{a,b,c,\dots\}$ of *alternatives* and a set $N := \{1,\dots,n\}$ of *criteria*, the practical application of MAUT roughly consists in synthesizing, for each alternative, the n different points of view quantified by the criteria in order, typically, to help the decision maker choose a subset of alternatives that can be considered the best for him. More precisely, in such a context, each alternative $a \in \mathcal{A}$ is identified with its vector of *scores* $(a_1,\dots,a_n) \in \mathbb{R}^n$ where, for any $i \in N$, a_i represents the *utility* of a for the decision maker with respect to (w.r.t.) criterion i. The preferences of the decision maker over the alternatives, represented by a binary relation $\succeq_{\mathcal{A}}$ supposed transitive and complete in the considered context, are then to be modeled by means of a *global utility function* $U : \mathbb{R}^n \to \mathbb{R}$ such that

$$a \succeq_{\mathcal{A}} b \iff U(a_1,\dots,a_n) \geq U(b_1,\dots,b_n), \qquad \forall a,b \in \mathcal{A}.$$

For the above model to make sense, it is clearly necessary that the utilities be *commensurable*, i.e. $a_i = a_j$ if and only if, for the decision maker, the alternative a is satisfied to the same extent on criteria i and j [see e.g. Grabisch et al., 2003, for a more complete discussion on commensurability].

The form of the global utility function U depends on the hypotheses on which the MCDA problem is grounded. When *mutual preferential independence* [see e.g. Vincke, 1992] among criteria can be assumed, it is frequent to consider that the global utility function is additive and takes the form of a weighted arithmetic mean. This assumption is however rarely verified in practice. In order to be able to take interaction phenomena among criteria into account, it has been proposed to substitute a monotone set function on $N := \{1,\dots,n\}$, called *capacity* [Choquet, 1953] or *fuzzy measure* [Sugeno, 1974], to the weight vector involved in the calculation of weighted arithmetic means. Such an approach can be regarded as taking into account not only the importance of each criterion but also the importance of each subset of criteria. A natural extension of the weighted arithmetic in such a context is the *Choquet integral* w.r.t. the defined capacity [Grabisch, 1992, Marichal, 2000, Labreuche and Grabisch, 2003].

Grabisch and Labreuche [2005a,b] have however recently shown that even such a general aggregation function as the Choquet integral w.r.t. a capacity is not suited for situations where the utilities to be aggregated lie on a *bipolar* scale. Compared to a classical (*unipolar*) scale, a bipolar scale is characterized by the additional presence of a neutral value such that values above this neutral reference point are considered to be "good" or "positive" by the decision maker, whereas values below it are considered to be "bad" or "negative" [see Grabisch and Labreuche, 2005c, for a complete discussion on bipolarity]. In order to derive aggregation models taking into account the specificity of bipolar scales, Grabisch and Labreuche [2005a,b,c] have recently introduced the notion of *bi-capacity*, extending that of capacity, and have proposed a natural generalization of the Choquet integral in that context.

The use of a Choquet integral as a global utility function clearly requires the prior identification of the underlying capacity if the utility scale is unipolar, or of the underlying bi-capacity if the utility scale is bipolar. The learning data from which the capacity or the bi-capacity is to be determined consists of what Marchant [2003] calls the *initial preferences* of the decision maker : usually, a partial preorder over the set of alternatives, a partial preorder over the set of criteria, intuitions about the importance of the criteria, etc.

Generalizing the minimum variance approach to capacity identification recently put forward in [Kojadinovic, 2005] and following Marichal [1998, Chap. 7], we propose to use a minimum distance principle for capacity (resp. bi-capacity) identification grounded on natural distances between capacities (resp. bi-capacities). For practical purposes, we focus on quadratic distances between capacities and bi-capacities which enables us to implement the minimum distance principle under the form of a strictly convex quadratic program. Furthermore, as we shall see, the capacity (resp. bi-capacity) identification problem is closely related to the capacity approximation problem [Marichal, 1998, Chap. 7] (resp. bi-capacity approximation problem), which we are able to address as well using the proposed minimum distance principle. The derived methodology has been implemented within the `kappalab` package [Grabisch and Kojadinovic, 2005] for the GNU R statistical system [R Development Core Team, 2005], an application of which will be presented.

In the first part of our presentation, after defining the notions of game, capacity and Choquet integral in the context of aggregation, we shall study quadratic objective functions, and in particular we will focus on quadratic distances, that can be practically used within a quadratic program for capacity identification or approximation. In the second part of our presentation, the concepts presented in the first part shall be generalized : the notions of bi-cooperative game, bi-capacity and Choquet integral w.r.t. a bi-capacity will be introduced and the quadratic distances between capacities previously defined shall be extended to bi-capacities. The last part of the presentation will be devoted to two applications of the proposed minimum distance approach, one to capacity approximation, the other to capacity identification.

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Calculation and Control With Ordered Fuzzy Numbers

Witold Kosinski[1] and Piotr Prokopowicz[2]

[1] Polish-Japanese Institute of Information Technology
Warsaw, Poland
wkos@pjwstk.edu.pl
[2] Inst. for Environmental Mechanics and Applied Computer Science
Bydgoszcz University, Poland
piotrekp@ab-byd.edu.pl

Classical fuzzy sets are convenient as far as a simple interpretation in the set-theoretical language is concerned. However, we could ask: How can we imagine a fuzzy information, say X, in such a way that by adding it to the fuzzy information A the fuzzy information C will be obtained? We answer that question in terms of the so-called ordered fuzzy numbers (OFN). For constructing those numbers the concept of the membership function of a fuzzy set, introduced by L. Zadeh in 1965 as a fundamental concept of the fuzzy (multivalued) logic, has been weakened by requiring a mere *membership relation*; consequently a fuzzy number arises as an ordered pair of continuous real functions defined on the interval $[0,1]$. Four algebraic operations: addition, subtraction, multiplication and division of such fuzzy numbers are constructed in a way that renders them an algebra. Further, a normed topology is introduced which makes them a Banach space. Several drawbacks of the Zadeh's fuzzy calculation are then absent. Defuzzyfication operations on the algebra of ordered fuzzy number can be introduced with the help of the Banach-Kakutami-Riesz representation theorem in terms of pairs of Radon measures. Algebraic operations on OFN give a unique possibility to define new types of compositional rules of fuzzy inference which play a key role in approximate reasoning when conclusions from a set of fuzzy *If-Then* rules are to derive. The proposed operations have been implemented as the algebra in the form of a *fuzzy calculator* working under Windows and written as a component of operating system - Windows (9x/XP) as well as a fuzzy controler.

Contour Lines of Rotation-Invariant T-norms

Koen C. Maes and Bernard De Baets

Department of Applied Mathematics, Biometrics and Process Control
Ghent University
9000 Gent, Belgium
{Koen.Maes|Bernard.DeBaets}@UGent.be

Abstract. Expressing the characteristic properties of a left-continuous t-norm in terms of contour lines, provides us more insight into its geometrical structure. In particular, we focus on the decomposition and construction of t-norms that are rotation invariant w.r.t. one of their contour lines.

1 Introduction

The contour lines of a left-continuous increasing $[0,1]^2 \to [0,1]$ mapping T are defined as follows:

$$C_a : [0,1] \to [0,1] : x \mapsto \sup\{t \in [0,1] \mid T(x,t) \leq a\} ,$$

with $a \in [0,1]$. It will be clear from the context which mapping T we are considering. In particular, whenever T is a t-norm, it holds that $C_a(x) = I_T(x,a)$, where I_T denotes the *residual implicator* of T. Hence, I_T determines the contour lines of T and can be used to interpret $C_a(x)$ as a $[0,1]^2 \to [0,1]$ mapping (variables x and a). The contour lines of a continuous t-norm T are also called *level functions* [9]. Based on the contour lines of a left-continuous t-norm T, Jenei [8] provides sufficient conditions for T to be the Łukasiewicz t-norm $T_{\mathbf{L}}$, resp. the product t-norm $T_{\mathbf{P}}$. In this paper, we provide new insights into the geometrical structure of a rotation-invariant t-norm by examining its contour lines. Rotation-invariant t-norms play a profound role in various fields such as fuzzy preference modelling [1] and fuzzy logic [3].

2 T-norm properties

Starting from a left-continuous increasing $[0,1]^2 \to [0,1]$ mapping T, we can easily redefine left-continuous t-norms by means of contour lines. The neutral element, commutativity and associativity are easily translated to conditions on contour lines.

Theorem 1. [11] *A left-continuous increasing $[0,1]^2 \to [0,1]$ mapping T has neutral element 1 if and only if the equivalence $C_a(x) = 1 \Leftrightarrow x \leq a$ holds for every $(x,a) \in [0,1]^2$.*

The commutativity of a left-continuous increasing $[0,1]^2 \to [0,1]$ mapping T satisfying $T(0,1) = T(1,0) = 0$ does not always ensure the symmetry of its contour lines. However, we can illustrate that the commutativity of T shows up through the *orthosymmetry* of its contour lines [10, 11]. A contour line C_a of T is orthosymmetrical if and only if $C_a(x) = \inf\{t \in [0,1] \mid C_a(t) < x\}$.

Theorem 2. [11] *A left-continuous increasing $[0,1]^2 \to [0,1]$ mapping T that satisfies $T(0,1) = T(1,0) = 0$ is commutative if and only if all its contour lines C_a, with $a \in [0,1]$, are orthosymmetrical.*

We can also use contour lines to express the associativity of T.

Theorem 3. [11] *A left-continuous increasing $[0,1]^2 \to [0,1]$ mapping T that satisfies $T(1,0)=0$ is associative if and only if $C_a(T(x,y)) = C_{C_a(x)}(y)$ holds for every $(x,y,a) \in [0,1]^3$.*

For a left-continuous t-norm T, taking into account the correspondence between its residual implicator I_T and its contour lines C_a, the equality in the previous theorem coincides with the *portation law* [4]: $I_T(x, I_T(y,z)) = I_T(T(x,y),z)$, for every $(x,y,z) \in [0,1]^3$.

3 Rotation-invariant t-norms

Consider a non-constant decreasing $[0,1] \to [0,1]$ mapping M that is involutive on $[1^M,1]$. Note that we use an exponential notation x^M to denote the immage of x under M. A t-norm T is said to be *rotation invariant w.r.t. M* if

$$T(x,y) \leq z \;\Leftrightarrow\; T(y,z^M) \leq x^M \,,$$

for every $(x,y,z) \in [1^M,1]^3$. If $1^M = 0$, we obtain the classical definition of rotation invariance [2,4]. Take $a \in [0,1[$ and consider an arbitrary increasing $[a,1] \to [0,1]$ bijection σ. The following lemma is indispensable to lay bare the tight relationship between our definition of rotation invariance and the classical one.

Lemma 1. *For every t-norm T, the $[0,1]^2 \to [0,1]$ mapping T^a, defined by*

$$T^a(x,y) = \sigma(\max(a, T(\sigma^{-1}(x), \sigma^{-1}(y))))$$

is a t-norm. For every $b \in [0,1]$ it then holds that $C^a_b = \sigma \circ C_{\sigma^{-1}(b)} \circ \sigma^{-1}$ where C^a_b denotes the corresponding contour line of T^a.

Given a strict negator N and $a \in [0,1[$, define the mapping $N_{\underline{a}} : [0,1] \to [a,1]$ by

$$x^{N_{\underline{a}}} = \begin{cases} 1 & \text{, if } x \leq a\,, \\ \sigma^{-1}(\sigma(x)^N) & \text{, elsewhere}\,. \end{cases}$$

In particular $N_{\underline{0}} = N$ and $x^{N_{\underline{1}}} = 1$, for every $x \in [0,1]$.

Theorem 4. *Let M be a non-constant decreasing $[0,1] \to [0,1]$ mapping that is involutive on $[a,1]$, with $a := 1^M$. Then there exists an involutive negator N such that $M = N_a$. A t-norm T is rotation invariant w.r.t. M if and only if T^a is rotation invariant w.r.t. N.*

Every t-norm T that is rotation invariant w.r.t. an involutive negator N is necessarily left-continuous and $N = C_0$ [4]. Taking into account Lemma 1 and Theorem 4 we immediately obtain the following result.

Theorem 5. *Consider an involutive negator N and $a \in [0,1[$. If a t-norm T is rotation invariant w.r.t. $N_{\underline{a}}$, then it is left-continuous on $\mathcal{D} = \{(x,y) \in [0,1]^2 \mid x^{N_{\underline{a}}} < y\}$ and $N_{\underline{a}} = C_a$.*

Furthermore, it can be shown that the rotation invariance of a t-norm is equivalent to the continuity of one of its contour lines.

Theorem 6. [11] *A t-norm T is rotation invariant w.r.t. a contour line C_a, $a \in [0,1[$, if and only if C_a is continuous.*

4 Constructing rotation-invariant t-norms

Dealing with rotation-invariant t-norms it suffices to only consider those t-norms for which the contour line C_0 is continuous (Theorem 4). Assuming some additional continuity conditions, we will now attempt to reconstruct a rotation-invariant t-norm T in the area $\mathcal{D}$ strictly above C_0. First, partition $\mathcal{D}$ into four parts:

$$\begin{aligned}
\mathcal{D}_{\mathrm{I}} &= \{(x,y) \in]\beta,1]^2 \mid y > C_\beta(x)\}\,,\\
\mathcal{D}_{\mathrm{II}} &= \{(x,y) \in]a,\beta] \times]\beta,1] \mid y > C_a(x)\}\,,\\
\mathcal{D}_{\mathrm{III}} &= \{(x,y) \in]\beta,1] \times]a,\beta] \mid y > C_a(x)\}\,,\\
\mathcal{D}_{\mathrm{IV}} &= \{(x,y) \in]\beta,1]^2 \mid y \leq C_\beta(x)\}\,.
\end{aligned}$$

Let β denote the unique fixpoint of C_0 (i.e. $C_0(\beta) = \beta$). As will become clear, area $\mathcal{D}_{\mathrm{I}}$ is crucial in the construction and decomposition of rotation-invariant t-norms.

Theorem 7. *Consider a t-norm T that is rotation invariant w.r.t. its contour line C_0. Let σ be an arbitrary increasing $[\beta,1] \to [0,1]$ bijection, with β the unique fixpoint of C_0. Then there exists a t-norm $\tilde{T}$ with contour lines $\tilde{C}_b$ such that*

$$T(x,y) = \begin{cases} \sigma^{-1}\left[\tilde{T}(\sigma(x),\sigma(y))\right] & , \text{ if } (x,y) \in \mathcal{D}_{\mathrm{I}}\,,\\ C_0\left(\sigma^{-1}\left[\tilde{C}_{\sigma(C_0(x))}(\sigma(y))\right]\right) & , \text{ if } (x,y) \in \mathcal{D}_{\mathrm{II}}\,,\\ C_0\left(\sigma^{-1}\left[\tilde{C}_{\sigma(C_0(y))}(\sigma(x))\right]\right) & , \text{ if } (x,y) \in \mathcal{D}_{\mathrm{III}}\,. \end{cases} \tag{1}$$

Conversely, we wonder when an arbitrary left-continuous t-norm $\tilde{T}$ without zero divisors ensures that $T|_{\mathcal{D}}$ fulfills the properties of a t-norm.

Theorem 8. *Consider an involutive negator N and its unique fixpoint β. Let $\tilde{T}$ be a left-continuous t-norm without zero divisors and with contour lines $\tilde{C}_b$. Take an arbitrary increasing $[\beta,1] \to [0,1]$ bijection σ. Define $C_0 := N$ and $C_\beta := (\tilde{C}_0)_{\underline{\beta}}$. Then the $[0,1]^2 \to [0,1]$ mapping T, defined by Eq.* (1) *and satisfying $T(x,y) = 0$, whenver $(x,y) \notin \mathcal{D}$, is a t-norm that is rotation invariant w.r.t. its contour line C_0.*

Our approach in the previous theorem amounts to the rotation construction of Jenei [5, 7]. On the other hand, if $\tilde{T}$ has zero divisors and we still want that $T|_{\mathcal{D}_{\mathrm{I}} \cup \mathcal{D}_{\mathrm{IV}}}$ is just a rescaling of $\tilde{T}$, then $\mathcal{D}_{\mathrm{IV}}$ must be a square (see also [5, 7]). For t-norms T that are continuous on $\mathcal{D}$ we will show that $T|_{\mathcal{D}}$ is totally fixed by $T|_{\mathcal{D}_{\mathrm{I}}}$.

Theorem 9. *Consider a t-norm T that is rotation invariant w.r.t. its contour line C_0 and that is continuous on $\mathcal{D}$. Let σ be an arbitrary increasing $[\beta,1] \to [0,1]$ bijection, with β the unique fixpoint of C_0. Then there exists a t-norm $\tilde{T}$ with contour lines $\tilde{C}_b$ such that*

$$T(x,y) = \begin{cases} \sigma^{-1}\left[\tilde{T}(\sigma(x),\sigma(y))\right] & , \text{ if } (x,y) \in \mathcal{D}_{\mathrm{I}}\,,\\ C_0\left(\sigma^{-1}\left[\tilde{C}_{\sigma(C_0(x))}(\sigma(y))\right]\right) & , \text{ if } (x,y) \in \mathcal{D}_{\mathrm{II}}\,,\\ C_0\left(\sigma^{-1}\left[\tilde{C}_{\sigma(C_0(y))}(\sigma(x))\right]\right) & , \text{ if } (x,y) \in \mathcal{D}_{\mathrm{III}}\,,\\ C_0\left(\sigma^{-1}\left[\tilde{T}\left(\sigma(C_\beta(x)),\sigma(C_\beta(y))\right)\right]\right) & , \text{ if } (x,y) \in \mathcal{D}_{\mathrm{IV}}\,. \end{cases} \tag{2}$$

Conversely, every continuous t-norm $\tilde{T}$ can be used to construct $T|_{\mathcal{D}}$.

Theorem 10. *Consider an involutive negator N and its unique fixpoint β. Let $\tilde{T}$ be a continuous t-norm with contour lines $\tilde{C}_b$. Take an arbitrary increasing $[\beta,1] \to [0,1]$ bijection σ. Define $C_0 := N$ and $C_\beta := (\tilde{C}_0)_{\underline{\beta}}$. Then the $[0,1]^2 \to [0,1]$ mapping T, defined by Eq. (2) and satisfying $T(x,y) = 0$, whenver $(x,y) \notin \mathcal{D}$, is a t-norm that is rotation invariant w.r.t. its contour line C_0.*

We only presented here a selection of our results. We are able to interpret Jenei's rotation-annihilation construction [6, 7] into our framework and we can uniquely determine $T|_{\mathcal{D}_{\mathrm{IV}}}$ under some additional conditions. Examining numerous examples, we know that $T|_{\mathcal{D}_{\mathrm{IV}}}$ is not always uniquely fixed by the behaviour of $T|_{\mathcal{D}_{\mathrm{I}}}$.

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A Measurement-Theoretic Axiomatization of Trapezoidal Membership Functions Defined on Extensive Structures

Thierry Marchant

Ghent University
9000 Gent, Belgium
thierry.marchant@ugent.be

1 Introduction

Some authors (e.g. [9, 4, 2, 1, 3, 6, 7]) have characterized different measurement techniques that permit us to represent the membership of some objects by a function that is unique up to some transformations (strictly increasing, positive affine or linear) thereby showing how we can try to measure the membership on some kind of scale (ordinal, interval, ratio).

In these papers, the set X of objects for which we want to measure the membership has no special structure (for example, $X = \{$ Ahmed, Bob, Chan $\}$) and the membership function directly maps X in $[0,1]$, as illustrated in fig. 1. So, even if we measure the membership of these objects (or people) in the fuzzy set 'tall' on an interval scale, we cannot obtain a parametric membership function like the trapezoidal one.

But in many applications, contrary to what is done in these theoretical papers, the membership of an object in a fuzzy set is not defined directly: the set X is first mapped into $\mathbb{R}$ (often using a physical instrument). For example, the height of Ahmed is represented by the real number $f(\text{Ahmed})$, in meters. Then another mapping—the membership function μ_{tall}—maps each real number (in some range) on a membership degree. For example, $f(\text{Ahmed})$ is mapped on $\mu_{\text{tall}}(f(\text{Ahmed}))$. This is illustrated in fig. 1.

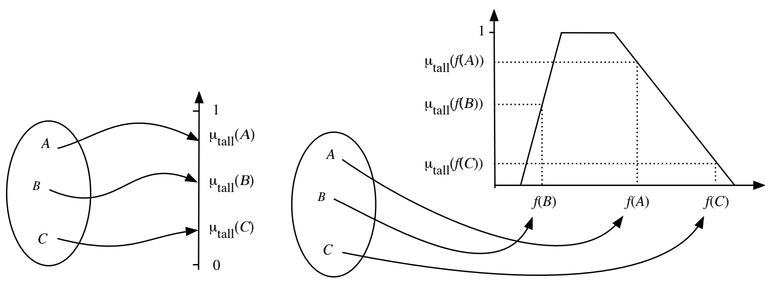

Fig. 1. Left: Direct representation of the membership. Right: Indirect representation of the membership

In Section 2, We will try to analyze, from a measurement-theoretic viewpoint, the indirect approach. We will not suppose that f is given. Instead, we will start with a structure allowing to construct the representation f (for example the height measured in meters) with some nice uniqueness properties. Then we will introduce another structure allowing to construct the representation μ of the membership in some fuzzy set. Finally, we will present some conditions, linking both structures, that permit to indirectly measure the membership and that yield trapezoidal membership functions.

Note that, in order to obtain a trapezoidal membership function, f must be unique up to some transformations and the set of transformations must be a subset of the positive affine transformations.

Otherwise, when varying the representation f, we cannot guarantee that the membership function will remain trapezoidal. This means that f must be an interval or ratio scale. There are different measurement techniques leading to interval scales: extensive measurement, bisection, conjoint measurement, difference measurement, ... In this paper, only extensive measurement will used.

2 Main result

In this paper, the primitives are

- $X = \{x, y, \ldots\}$, the universal set (uncountable),
- $\succsim_A^*$, a binary relation defined on $X \times X$ and representing differences of membership in a fuzzy set A, as perceived by an expert. For instance, $xy \succsim_A^* wz$ means the difference of membership between x and y is at least as large as the difference of membership between w and z,
- $\circ$, a binary operation from $X \times X$ into X. It is a 'concatenation' operation. For instance, if x and y are rods, then $x \circ y$ represents the composite rod obtained by laying x and y end to end in a straight line,
- $\succsim$, a binary relation on X.

The primitives are empirically observable. They are not explained nor defined by the theory. In particular, the fuzzy set A has no mathematical structure or property. It is just an expression in ordinary language (e.g. 'tall') that can be seen as a fuzzy set.

The conditions guaranteeing the existence of a numerical representation μ_A of the relation $\succsim_A^*$, unique up to positive affine transformations are well known. These conditions are those characterizing algebraic difference structures [5].

Similarly, the conditions that guarantee the existence of a numerical representation f of the relation $\succsim$, unique up to positive linear transformations are also standard in the literature. These conditions are those characterizing closed extensive structures [5].

We now introduce some new notation and two new conditions that will make it possible to construct a trapezoidal membership function.

Let $\succsim_A$ be a binary relation on X defined by $x \succsim_A y$ iff $xy \succsim_A^* xx$. If $\langle X, \succsim_A^* \rangle$ is an algebraic difference structure, then $\succsim_A$ is a weak order. Let $T(A) = \{x \in X : x \succsim_A y \ \forall y \in X\}$ (the set of elements with maximal membership in A) and $B(A) = \{x \in X : y \succsim_A x \ \forall y \in X\}$ (the set of elements with minimal membership in A). These two sets may be empty. Let $L_A = \{x \in X : z \precsim x \precsim y \ \forall y \in T(A)$ and $\forall z \in B(A)$ with $z \prec y\}$ and $R_A = \{x \in X : z \precsim x \precsim y \ \forall z \in T(A)$ and $\forall y \in B(A)$ with $z \prec y\}$. The elements in L_A correspond to the increasing part of the membership function while those in R_A correspond to the decreasing part.

A 1 Quasi-Convexity. *There are $x_1, x_2, x_3, x_4 \in X$, with $x_1 \prec x_2 \prec x_3 \prec x_4$, such that:*

- $x \in B(A)$ *iff* $x \precsim x_1$ *or* $x \succsim x_4$*;*
- $x \in T(A)$ *iff* $x_2 \precsim x \precsim x_3$*;*
- $x_1 \prec x \precsim y \prec x_2$ *implies* $y \succsim_A x$*;*
- $x_3 \prec x \precsim y \prec x_4$ *implies* $x \succsim_A y$*.*

Remark that Quasi-Convexity implies that $T(A)$ and $B(A)$ are not empty.

Quasi-Convexity is a very mild condition. It just says that, when moving from small to large elements (w.r.t. $\succsim$), the membership is first minimal then increases, reaches a maximum, decreases and reaches again the same minimum. The next condition is much stronger: it imposes a very strict consistency or compatibility between the closed extensive structure (often measured with a physical instrument) and the algebraic difference structure (based on the knowledge of the expert).

A 2 Consistency. *For all x, y in L_A, if there is z such that $z \circ z \sim x \circ y$, then $xz \sim_A^* zy$. The same holds for all x, y in R_A.*

We denote by $X/\sim$ the set of equivalence classes on X under $\sim$.

Theorem 1. *Let the structures $\langle X, \succsim_A^* \rangle$ and $\langle X, \succsim, \circ \rangle$ be, respectively, an algebraic difference structure and a closed extensive structure with $X/\sim$ uncountable. If, in addition, $\langle X, \succsim_A^*, \succsim, \circ \rangle$ satisfies Quasi-Convexity (A1) and Consistency (A2), then there exist $f_A : X \mapsto \mathbb{R}_0^+$ and $\mu_A : X \mapsto [0, 1]$ such that*

$$\begin{array}{c} \mu_A(x) - \mu_A(y) \geq \mu_A(z) - \mu_A(w) \\ \Updownarrow \\ xy \succsim_A^* zw, \forall x, y, z, w \in X, \end{array} \tag{1}$$

$$\mu_A(x) = 0 \ \forall x \in B(A), \tag{2}$$

$$\mu_A(x) = 1 \ \forall x \in T(A), \tag{3}$$

$$f(x) \geq f(y) \Leftrightarrow x \succsim y, \ \forall x, y \in X, \tag{4}$$

$$f(x \circ y) = f(x) + f(y) \ \ \forall x, y \in X, \tag{5}$$

$$\mu_A(x) = a_A^L f(x) + b_A^L \ \ \forall x, y \in L_A. \tag{6}$$

and

$$\mu_A(x) = a_A^R f(x) + b_A^R \ \ \forall x, y \in R_A, \tag{7}$$

with $a_A^L > 0$ and $a_A^R < 0$.

The function μ_A is unique. The functions μ_A and f' also satisfy (2–7) iff there is a real numbers $p > 0$ such that $f' = pf$. We then have $a_L' = a_L/p$, $b_L' = b_L(1 - a_L/p)$, $a_R' = a_R/p$ and $b_R' = b_R(1 - a_R/p)$.

The proof strategy is simple. Using classical results about algebraic difference structures and extensive structures, we construct two representations: one of $\succsim_A^*$ and one of $\succsim$. By Quasi-Convexity, one of these representations must be an increasing transformations ϕ of the other one on L_A and on R_A. Using Consistency allows us to write a functional equation involving ϕ. This functional equation has only one solution: ϕ must be affine. A detailed proof of a similar result on bisymmetric structures can be found in [8].

3 Conclusion

Theorem 1 does not justify or legitimize the use of trapezoidal membership functions; it presents conditions under which such membership functions can represent the knowledge of an expert. Some experimental research is necessary to determine whether these conditions are met in practice.

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Cumulative Distribution Functions and Moments of Weighted Lattice Polynomials

Jean-Luc Marichal

Applied Mathematics Unit, University of Luxembourg
1511 Luxembourg, Luxembourg
jean-luc.marichal@uni.lu

Abstract. We give the cumulative distribution functions, the expected values, and the moments of weighted lattice polynomials when regarded as real functions. Since weighted lattice polynomial functions include Sugeno integrals, lattice polynomial functions, and order statistics, our results encompass the corresponding formulas for these particular functions.

1 Introduction

The cumulative distribution functions (c.d.f.'s) and the moments of order statistics have been discovered and studied for many years (see e.g. [4]). Their generalizations to lattice polynomial functions, which are nonsymmetric extensions of order statistics, were investigated very recently by Marichal [7] for independent variables and then by Dukhovny [5] for dependent variables.

Roughly speaking, an n-ary *lattice polynomial* is any well-formed expression involving n real variables $x_1,\ldots,x_n$ linked by the lattice operations $\wedge = \min$ and $\vee = \max$ in an arbitrary combination of parentheses. In turn, such an expression naturally defines an n-ary *lattice polynomial function*. For instance,

$$p(x_1,x_2,x_3) = (x_1 \wedge x_2) \vee x_3$$

is a 3-ary lattice polynomial function.

Lattice polynomial functions can be generalized by regarding certain variables as parameters, like in the 2-ary polynomial

$$p(x_1,x_2) = (c \wedge x_1) \vee x_2,$$

where c is a real constant. Such "parameterized" lattice polynomial functions, called *weighted lattice polynomial* functions [8, 11], are very often considered in the area of nonlinear aggregation functions as they include the whole class of discrete Sugeno integrals [12, 13].

In this paper we give a closed-form formula for the c.d.f. of any weighted lattice polynomial function in terms of the c.d.f.'s of its input variables. More precisely, considering an n-ary weighted lattice polynomial function p and n independent random variables $X_1,\ldots,X_n$, X_i $(i=1,\ldots,n)$ having c.d.f. $F_i(x)$, we give a formula for the c.d.f. of $Y_p := p(X_1,\ldots,X_n)$. We also yield a formula for the expected value $\mathbb{E}[g(Y_p)]$, where g is any measurable function. The special cases $g(x) = x$, x^r, $[x-\mathbb{E}(Y_p)]^r$, and e^{tx} give, respectively, the expected value, the raw moments, the central moments, and the moment-generating function of Y_p.

This paper is organized as follows. In Section 2 we recall the basic material related to lattice polynomial functions and their weighted versions. In Section 3 we provide the announced results. In Section 4 we investigate the particular case where the input random variables are uniformly distributed over the unit interval. Finally, in Section 5 we provide an application of our results to the reliability analysis of coherent systems.

Weighted lattice polynomial functions play an important role in the areas of nonlinear aggregation and integration. Indeed, as we mentioned above, they include all the discrete Sugeno integrals, which are very useful aggregation functions in many areas. More details about the Sugeno integrals and their applications can be found in the remarkable edited book [6].

2 Weighted lattice polynomials

In this section we give some definitions and properties related to weighted lattice polynomial functions. More details and proofs can be found in [8].

As we are concerned with weighted lattice polynomial functions of random variables, we do not consider weighted lattice polynomial functions on a general lattice, but simply on an interval $L := [a,b]$ of the extended real number system $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. Clearly, such an interval is a bounded distributive lattice, with a and b as bottom and top elements. The lattice operations $\wedge$ and $\vee$ then represent the minimum and maximum operations, respectively. To simplify the notation, we also set $[n] := \{1, \ldots, n\}$ for any integer $n \geqslant 1$.

Let us first recall the definition of a lattice polynomial (with real variables); see e.g. Birkhoff [2, §II.5].

Definition 1. *Given a finite collection of variables $x_1, \ldots, x_n \in L$, a* lattice polynomial *in the variables $x_1, \ldots, x_n$ is defined as follows:*

1. *the variables $x_1, \ldots, x_n$ are lattice polynomials in $x_1, \ldots, x_n$;*
2. *if p and q are lattice polynomials in $x_1, \ldots, x_n$, then $p \wedge q$ and $p \vee q$ are lattice polynomials in $x_1, \ldots, x_n$;*
3. *every lattice polynomial is formed by finitely many applications of the rules 1 and 2.*

When two different lattice polynomials p and q in the variables $x_1, \ldots, x_n$ represent the same function from L^n to L, we say that p and q are equivalent and we write $p = q$. For instance, $x_1 \vee (x_1 \wedge x_2)$ and x_1 are equivalent.

The weighted lattice polynomial functions are defined as follows.

Definition 2. *A function $p : L^n \to L$ is an n-ary* weighted lattice polynomial function *if there exists an integer $m \geqslant 0$, parameters $c_1, \ldots, c_m \in L$, and a lattice polynomial function $q : L^{n+m} \to L$ such that*

$$p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n, c_1, \ldots, c_m) \qquad (x_1, \ldots, x_n \in L).$$

Because L is a distributive lattice, any weighted lattice polynomial function can be written in *disjunctive* and *conjunctive* forms as follows.

Proposition 1. *Let $p : L^n \to L$ be any weighted lattice polynomial function. Then there are set functions $\alpha : 2^{[n]} \to L$ and $\beta : 2^{[n]} \to L$ such that*

$$p(x) = \bigvee_{S \subseteq [n]} \Big[\alpha(S) \wedge \bigwedge_{i \in S} x_i\Big] = \bigwedge_{S \subseteq [n]} \Big[\beta(S) \vee \bigvee_{i \in S} x_i\Big].$$

Proposition 1 naturally includes the classical lattice polynomial functions. To see it, it suffices to consider nonconstant set functions $\alpha : 2^{[n]} \to \{a,b\}$ and $\beta : 2^{[n]} \to \{a,b\}$, with $\alpha(\varnothing) = a$ and $\beta(\varnothing) = b$.

The set functions α and β that disjunctively and conjunctively generate the polynomial function p in Proposition 1 are not unique. For example, as we have already observed above, we have

$$x_1 \vee (x_1 \wedge x_2) = x_1 = x_1 \wedge (x_1 \vee x_2).$$

However, it can be shown that, from among all the possible set functions that disjunctively generate a given weighted lattice polynomial function, only one is nondecreasing. Similarly, from among all the possible set functions that conjunctively generate a given weighted lattice polynomial function, only one is nonincreasing. These particular set functions are given by

$$\alpha(S) = p(\mathbf{e}_S) \quad \text{and} \quad \beta(S) = p(\mathbf{e}_{[n]\setminus S}),$$

where, for any $S \subseteq [n]$, $\mathbf{e}_S$ denotes the characteristic vector of S in $\{a,b\}^n$, i.e., the n-dimensional vector whose ith component is a, if $i \in S$, and b, otherwise. Thus, an n-ary weighted lattice polynomial function can always be written as

$$p(x) = \bigvee_{S\subseteq[n]} \Big[p(\mathbf{e}_S) \wedge \bigwedge_{i\in S} x_i\Big] = \bigwedge_{S\subseteq[n]} \Big[p(\mathbf{e}_{[n]\setminus S}) \vee \bigvee_{i\in S} x_i\Big].$$

The best known instances of weighted lattice polynomial functions are given by the discrete *Sugeno integrals*, which consist of a nonlinear discrete integration with respect to a *fuzzy measure*.

Definition 3. *An L-valued fuzzy measure on $[n]$ is a nondecreasing set function $\mu : 2^{[n]} \to L$ such that $\mu(\varnothing) = a$ and $\mu([n]) = b$.*

The Sugeno integrals can be presented in various equivalent forms. The next definition introduce them in one of their simplest forms (see [12]).

Definition 4. *Let μ be an L-valued fuzzy measure on $[n]$. The Sugeno integral of a function $x : [n] \to L$ with respect to μ is defined by*

$$\mathcal{S}_\mu(x) := \bigvee_{S\subseteq[n]} \Big[\mu(S) \wedge \bigwedge_{i\in S} x_i\Big].$$

Thus, any function $f : L^n \to L$ is an n-ary Sugeno integral if and only if it is a weighted lattice polynomial function fulfilling $f(\mathbf{e}_\varnothing) = a$ and $f(\mathbf{e}_{[n]}) = b$.

3 Cumulative distribution functions and moments

Consider n independent random variables $X_1, \ldots, X_n$, X_i $(i \in [n])$ having c.d.f. $F_i(x)$, and set $Y_p := p(X_1, \ldots, X_n)$, where $p : L^n \to L$ is any weighted lattice polynomial function. Let $H : \overline{\mathbb{R}} \to \{0,1\}$ be the Heaviside step function defined by $H(x) = 1$, if $x \geqslant 0$, and 0, otherwise. For any $c \in \overline{\mathbb{R}}$, we also define the function $H_c(x) = H(x-c)$.

The c.d.f. of Y_p is given in the next theorem.

Theorem 1. *Let $p : L^n \to L$ be a weighted lattice polynomial function. Then, the c.d.f. of Y_p is given by*

$$F_p(y) = 1 - \sum_{S\subseteq[n]} \big[1 - H_{p(\mathbf{e}_S)}(y)\big] \prod_{i\in[n]\setminus S} F_i(y) \prod_{i\in S} [1 - F_i(y)].$$

As a corollary, we retrieve the c.d.f. of any lattice polynomial function; see [7].

Corollary 1. *Let $p : L^n \to L$ be a lattice polynomial function. Then, the c.d.f. of Y_p is given by*

$$F_p(y) = 1 - \sum_{\substack{S\subseteq[n] \\ p(\mathbf{e}_S)=b}} \prod_{i\in[n]\setminus S} F_i(y) \prod_{i\in S} [1 - F_i(y)].$$

Let us now consider the expected value $\mathbb{E}[g(Y_p)]$, where $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is any measurable function. From its expression we can compute the expected value and the moments of Y_p.

By definition, we simply have

$$\mathbb{E}[g(Y_p)] = \int_{-\infty}^{\infty} g(y)\, \mathrm{d}F_p(y).$$

Using integration by parts, we can derive an alternative expression of $\mathbb{E}[g(Y_p)]$. We then have the following result.

Theorem 2. *Let $p : L^n \to L$ by any weighted lattice polynomial function. For any measurable function $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ such that*

$$\lim_{y\to\infty} g(y)[1 - F_i(y)] = 0 \quad (i \in [n]),$$

then

$$\mathbb{E}[g(Y_p)] = \lim_{y\to-\infty} g(y) + \sum_{S\subseteq[n]} \int_{-\infty}^{p(\mathbf{e}_S)} \prod_{i\in[n]\setminus S} F_i(y) \prod_{i\in S}[1 - F_i(y)]\, \mathrm{d}g(y).$$

4 The case of uniformly distributed variables on the unit interval

We now examine the case where the random variables $X_1,\ldots,X_n$ are uniformly distributed on $[0,1]$. We also assume $L = [0,1]$.

Recall that the *incomplete Beta function* is defined, for any $u, v > 0$, by

$$B_z(u,v) := \int_0^z t^{u-1}(1-t)^{v-1}\, \mathrm{d}t \qquad (z \in \mathbb{R}),$$

and the *Beta function* is defined, for any $u, v > 0$, by $B(u,v) := B_1(u,v)$.

According to Theorem 2, for any weighted lattice polynomial function $p : [0,1]^n \to [0,1]$ and any measurable function $g : [0,1] \to \overline{\mathbb{R}}$, we have

$$\mathbb{E}[g(Y_p)] = g(0) + \sum_{S\subseteq[n]} \int_0^{p(\mathbf{e}_S)} y^{n-|S|}(1-y)^{|S|}\, \mathrm{d}g(y).$$

Let us now examine the case of the Sugeno integrals. As these integrals are usually considered over the domain $[0,1]^n$, we naturally calculate their expected values when their input variables are uniformly distributed over $[0,1]^n$. Since any Sugeno integral is a particular weighted lattice polynomial, its expected value then writes

$$\int_{[0,1]^n} \mathcal{S}_\mu(x)\, \mathrm{d}x = \sum_{S\subseteq[n]} B_{\mu(S)}(n - |S| + 1, |S| + 1).$$

Surprisingly, this expression is very close to that of the expected value of the Choquet integral with respect to the same fuzzy measure.

Let us recall the definition of the Choquet integrals (see [3]). Just as for the Sugeno integrals, the Choquet integrals can be expressed in various equivalent forms. We present them in one of their simplest forms; see e.g. [9].

Definition 5. *Let μ be an $[0,1]$-valued fuzzy measure on $[n]$. The Choquet integral of a function $x : [n] \to [0,1]$ with respect to μ is defined by*

$$\mathcal{C}_\mu(x) := \sum_{S\subseteq[n]} \mu(S)\Big[\sum_{T\supseteq S}(-1)^{|T|-|S|}\bigwedge_{i\in T} x_i\Big].$$

For comparison purposes, the expected value of $\mathcal{C}_\mu$ is given by (see e.g. [10])

$$\int_{[0,1]^n} \mathcal{C}_\mu(x)\,\mathrm{d}x = \sum_{S\subseteq[n]} \mu(S)\,B(n-|S|+1,|S|+1).$$

5 Application to reliability theory

In this final section we show how the results derived here can be applied to the reliability analysis of certain coherent systems. For a reference on reliability theory, see e.g. [1].

Consider a system made up of n independent components, each component C_i ($i\in[n]$) having a lifetime X_i and a reliability $r_i(t) := \Pr[X_i > t]$ at time $t > 0$. Additional components, with constant lifetimes, may also be considered.

We assume that, when components are connected in series, the lifetime of the subsystem they form is simply given by the minimum of the component lifetimes. Likewise, for a parallel connection, the subsystem lifetime is the maximum of the component lifetimes.

It follows immediately that, for a system mixing series and parallel connections, the system lifetime is given by a weighted lattice polynomial function

$$Y_p = p(X_1,\ldots,X_n)$$

of the component lifetimes. We then have explicit formulas for the c.d.f., the expected value, and the moments of the system lifetime.

For example, the system reliability at time $t > 0$ is given by

$$R_p(t) := \Pr[Y_p > t] = \sum_{S\subseteq[n]} \big[1-H_{p(\mathbf{e}_S)}(y)\big]\prod_{i\in S} r_i(t)\prod_{i\in[n]\setminus S}[1-r_i(t)].$$

Moreover, for any measurable function $g : [0,\infty] \to \overline{\mathbb{R}}$ such that

$$\lim_{t\to\infty} g(t)r_i(t) = 0 \quad (i\in[n]),$$

we have, by Theorem 2,

$$\mathbb{E}[g(Y_p)] = g(0) + \int_0^\infty R_p(t)\,\mathrm{d}g(t).$$

Example 1. If $r_i(t) = e^{-\lambda_i t}$ ($i\in[n]$), we can show that

$$\mathbb{E}[Y_p] = p(\mathbf{e}_\varnothing) + \sum_{\substack{S\subseteq[n]\\ S\neq\varnothing}}\sum_{T\subseteq S}(-1)^{|S|-|T|}\,\frac{1-e^{-\lambda(S)\,p(\mathbf{e}_T)}}{\lambda(S)},$$

where $\lambda(S) := \sum_{i\in S}\lambda_i$.

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Decision Making When One is Facing Doubt Regarding the Consequences of Their Actions

Benoît Menoni

CREST-LFA & EUREQua
F-92245 Malakoff Cedex, France
menoni@ensae.fr

Since the seminal analysis of decision making under risk carried out by von Neumann and Morgenstern[12] and the pioneering study of decision making under uncertainty fulfilled by Savage[10], various contributions have lead decision making theorists to develop a wide range of models that allow them either to advice people when the latter face a choice or to explain why they picked out a given alternative. The so-called expected utility approach is dominantly used by economists when they study decision making. It specifies that the decision maker (DM), when they want to value an action, uses some probability distribution over the contingencies they are dealing with and a utility function that converts the various consequences into monetary payoffs; the evaluation of an action is nothing but the expected value of the utility of its consequences with respect to that probability distribution. In order to deal with some inconsistencies pointed out, among others, by Allais[1], Ellsberg[5] and Mossin[9], several contributions have been later developed. Schmeidler[11] shows that a DM whose preferences conform to the axioms he considers evaluates their actions in a expected-utility fashion, using a capacity instead of a probability distribution. Gilboa and Schmeidler[6] consider a situation such that the DM have in mind a list of possible scenarios; each of them corresponds to a probability distribution on the set of the states of the world. Thus, facing a random variable, the DM can compute as many expected values as scenarios. The authors provide a rationale for them to evaluate the random variable as the smallest of its expected values. Ghirardato[7] extends the framework studied by Savage[10] to situations where the DM does not know precisely the consequences of their actions in a given state of the world and is merely able to make out a list of potential consequences. They know the consequence will lay somewhere in the list but cannot specify which one of those will eventually occur. The author shows that the axioms studied by Savage[10] plus two additional ones that are specific to his setting lead the DM to evaluate their actions again through a Choquet integral. Lastly, Jaffray and Jeleva[8] study situations where the implications of a given action are completely known and understood – *analyzed* in their terminology – on an particular event and more vague and imprecise on the complement of that event. The authors show that if the DM's preferences obey some rules then the valuation attributed to a given action should only depend on the analyzed event itself, the expected value of the utility of the consequences provided by the action on the analyzed event and the worst and the best consequences of the action on the non analyzed event.

Most of these contributions consist in a relaxing of the properties imposed to the preference relation among acts the DM is endowed with. Besides, all of them consider actions – acts in Savage[10]'s terminology – as mappings from $\mathcal{S}$, the set of states of the world, to some set $\mathcal{X}$ that can be the set of consequences $\mathcal{C}$ itself or some other set derived from the latter such as the power set of $\mathcal{C}$ or the set of all simple probability distributions on set of $\mathcal{C}$. We address two comments to such a formalism. Firstly, it is possible that the DM understands the course of action they have to implement to realize a given action yet they cannot specify the outcome that will result from their action. For example, the DM may understands what they have to do if they want to invest some of their money in company A. A much more difficult task for them is to know what will be the precise value of their portfolio fourteen months from now. Secondly, it is possible that the DM thinks that the list of states of the world they have in mind is somewhat coarse. However, for lack of time, of ressource, of intellectual abilities, they cannot refine these contingencies. In the previous example, the DM may have in mind a few economic indicators that can determine the value of their portfolio yet will it be enough for them to make accurate anticipations? These two remarks may prevent one from modeling the actions as acts *à la* Savage[10]. Indeed, this requires to assign to any action in any state of the world a unique consequence. Furthermore, attributing a list of consequences as suggested by Ghirardato[7] means that none of the consequences considered as possible plays a particular role, that they are all on an equal footing. However, even if the quality of the information acquired by the DM is low, it can nonetheless helps the latter to sort the elements of the list. The aim of the paper is to take into account those two remarks; it suggests to consider an act as a mapping from $\mathcal{S}$, the set of the states of the world, to Δ, the set of the *possibility distributions* over the set of outcomes $\mathcal{C}$ with the following interpretation : given (1) their knowledge, their understanding of the implications of a given action and (2) the occurrence of a given state of the world, the DM attributes to each outcome a degree of *possibility* that varies from *total impossibility* to *total possibility*. In other words, an act is supposed to induce, in any state of the world $s \in \mathcal{S}$, a ranking over the various outcomes $c \in \mathcal{C}$. In a way, this amounts to give an ordinal structure to Ghirardato[7]'s lists. Anscombe and Aumann[2] suggest this idea yet these authors require the consequence of an act, in any state of the world, to be a probability distribution – a *lottery ticket* in their terminology – over $\mathcal{C}$. Dealing with possibility distributions rather than probability distributions is, to our mind, less demanding for it does not require the weight attributed to every consequence to be a real number lying between 0 and 1 nor the sum of those weights to be equal to 1. Moreover possibility seems to describe human reasoning better than probability. The axioms used in the paper are closed to the ones developed by Dubois *et al.*[3] and Dubois *et al.*[4]; they allow us to derive from preferences over acts a valuation of the induced possibility distributions in an expected-utility fashion. We then show how to aggregate these state-wise evaluations in a consistent way; this constitutes the main result of the paper.

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Inclusion-Exclusion Families With Respect to Set Functions⋆

Toshiaki Murofushi

Department of Computational Intelligence and Systems Sciences
Tokyo Institute of Technology
Yokohama 226-8502, Japan
murofusi@dis.titech.ac.jp

This abstract gives basic properties of inclusion-exclusion families, or interadditive families, with respect to set functions over a nonempty finite set X; for example, the collection of all possible inclusion-exclusion families with respect to set functions over X is isomorphic to the free bounded distributive lattice generated by X.

Throughout the abstract, X is assumed to be a nonempty finite set.

1 Families of sets

This section gives a summary of existing results on a lattice structure of families of subsets equipped with a certain partial order (e.g., [1], [2]).

Definition 1. *1. A family $\mathcal{A}$ of sets is called an* antichain *if $\{A, A'\} \subseteq \mathcal{A}$ and $A \subseteq A'$ together imply $A = A'$.*

2. A family $\mathcal{H}$ of sets is said to be hereditary *if $H' \subseteq H \in \mathcal{H}$ implies $H' \in \mathcal{H}$.*

Let $\mathbb{A}(X) \stackrel{\text{def}}{=} \{\mathcal{A} \subseteq 2^X \mid \mathcal{A} \text{ is an antichain}\}$ and $\mathbb{H}(X) \stackrel{\text{def}}{=} \{\mathcal{H} \subseteq 2^X \mid \mathcal{H} \text{ is hereditary}\}$.

Definition 2. *For $\mathcal{S} \subseteq 2^X$, we define $\mathrm{Max}\mathcal{S} \in \mathbb{A}(X)$ and $\mathrm{Her}\mathcal{S} \in \mathbb{H}(X)$ by*

$$\mathrm{Max}\mathcal{S} \stackrel{\text{def}}{=} \{A \mid A \text{ is maximal in } \mathcal{S} \text{ with respect to set inclusion } \subseteq\},$$
$$\mathrm{Her}\mathcal{S} \stackrel{\text{def}}{=} \{H \mid H \subseteq S \text{ for some } S \in \mathcal{S}\}.$$

Definition 3. *For $\mathcal{S}, \mathcal{T} \subseteq 2^X$.*

$$\mathcal{S} \sqsubseteq \mathcal{T} \stackrel{\text{def}}{\iff} \mathcal{S} \subseteq \mathrm{Her}\mathcal{T}, \qquad \mathcal{S} \equiv \mathcal{T} \stackrel{\text{def}}{\iff} \mathcal{S} \sqsubseteq \mathcal{T} \text{ and } \mathcal{T} \sqsubseteq \mathcal{S}.$$

Proposition 1. *Let $\mathcal{S}, \mathcal{T} \subseteq 2^X$.*

1. $\mathcal{S} \equiv \mathrm{Max}\mathcal{S} \equiv \mathrm{Her}\mathcal{S}$.
2. $\mathcal{S} \equiv \mathcal{T} \iff \mathrm{Max}\mathcal{S} = \mathrm{Max}\mathcal{T} \iff \mathrm{Her}\mathcal{S} = \mathrm{Her}\mathcal{T}$.

Obviously, $\sqsubseteq$ is a preorder on $2^{(2^X)}$, i.e., it is reflexive and transitive, and $\equiv$ is an equivalence relation on $2^{(2^X)}$. We denote by $[\mathcal{S}]$ the equivalence class of $\mathcal{S} \in 2^{(2^X)}$ with respect to $\equiv$. Let $\sqsubseteq_\equiv$ be the partial order on the quotient $2^{(2^X)}/\equiv$ induced by $\sqsubseteq$, i.e.,

$$[\mathcal{S}] \sqsubseteq_\equiv [\mathcal{T}] \stackrel{\text{def}}{\iff} \mathcal{S} \sqsubseteq \mathcal{T} \quad \text{for } \mathcal{S}, \mathcal{T} \subseteq 2^X.$$

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Let $\mathfrak{L}(X)$ be the set of lattice polynomials of elements of X defined by

$$\mathfrak{L}(X) \stackrel{\text{def}}{=} \left\{ \bigvee_{S \in \mathcal{S}} \bigwedge_{x \in S} x \;\middle|\; \mathcal{S} \in 2^{(2^X)} \right\},$$

where $\bigvee \emptyset = 0$ and $\bigwedge \emptyset = 1$. Then $(\mathfrak{L}(X), \wedge, \vee, 0, 1)$ is the free bounded distributive lattice $(\mathfrak{L}(X), \wedge, \vee, 0, 1)$ generated by X, where a bounded lattice is a lattice with the greatest element 1 and the least element 0.

Proposition 2. *Each of* $(2^{(2^X)}/\equiv, \sqsubseteq_\equiv)$, $(\mathbb{A}(X), \sqsubseteq)$, $(\mathbb{H}(X), \sqsubseteq)$ *is isomorphic to the free bounded distributive lattice* $(\mathfrak{L}(X), \wedge, \vee, 0, 1)$ *generated by X. Especially,* $(\mathbb{H}(X), \sqsubseteq)$ *is the lattice* $(\mathbb{H}(X), \cap, \cup, \emptyset, 2^X)$ *of sets. The isomorphism* $\varphi : \mathfrak{L}(X) \to 2^{(2^X)}/\equiv$ *is given as*

$$\varphi\left(\bigvee_{S \in \mathcal{S}} \bigwedge_{x \in S} x \right) = [\{X \setminus S \mid S \in \mathcal{S}\}]. \qquad (1)$$

2 Set functions and the Choquet integral

The contents of this section are a few modification of existing results (e.g., [3]).

Definition 4. *A function* $\mu : 2^X \to \mathbb{R}$ *is called a* set function *(*with intercept*) over X. A set function* μ *is said to be* without intercept *if* $\mu(\emptyset) = 0$*. The* essential part *of a set function* μ *is the set function* $\mu_\emptyset$ *defined by* $\mu_\emptyset(E) = \mu(E) - \mu(\emptyset)$ *for* $E \subseteq X$*. A set function* μ *is said to be* modular *if* $\mu(E \cup F) + \mu(E \cap F) = \mu(E) + \mu(F)$ *for every pair E and F of subsets of X. A modular set function without intercept is said to be* additive*. Let*

$$\mathbb{SF}(X) \stackrel{\text{def}}{=} \{\mu \mid \mu \text{ is a set function over } X\}, \qquad \mathbb{SF}_\emptyset(X) \stackrel{\text{def}}{=} \{\mu \in \mathbb{SF}(X) \mid \mu(\emptyset) = 0\}.$$

Hereinafter μ is assumed to be a set function over X, i.e., $\mu \in \mathbb{SF}(X)$.

Definition 5. *(cf. [4]) The* Choquet integral $(\mathrm{C})\int f(x)\,d\mu(x)$ *of a function* $f : X \to \mathbb{R}$ *with respect to* μ *is defined by*

$$(\mathrm{C})\int f\,d\mu \stackrel{\text{def}}{=} \mu(\emptyset) + \sum_{i=1}^{|X|} [f(x_i) - f(x_{i-1})]\,[\mu(A_i) - \mu(\emptyset)],$$

where $x_1, x_2, \ldots, x_{|X|}$ *is a permutation of the elements of X satisfying the condition* $f(x_1) \le f(x_2) \le \cdots \le f(x_{|X|})$*,* $f(x_0) \stackrel{\text{def}}{=} 0$*, and* $A_i \stackrel{\text{def}}{=} \{x_i, x_{i+1}, \ldots, x_{|X|}\}$ *(*$i = 1, 2, \ldots, |X|$*).*

Obviously it holds that

$$(\mathrm{C})\int f\,d\mu = \mu(\emptyset) + (\mathrm{C})\int f\,d\mu_\emptyset.$$

There are two distinct definitions of the Choquet integral over $E \subseteq X$:

$$(\mathrm{C})\int_E f\,d\mu \stackrel{\text{def}}{=} (\mathrm{C})\int (f \upharpoonright E)\,d(\mu \upharpoonright 2^E), \qquad (\mathrm{C})\int_E f\,d\mu \stackrel{\text{def}}{=} (\mathrm{C})\int f \cdot 1_E\,d\mu,$$

where 1_E is the indicator of E. In this abstract, however, we may adopt whichever one.

Definition 6. *The* Möbius transform μ^{M} *of* μ *is a set function over* X *defined by*

$$\mu^{\mathrm{M}}(E) \stackrel{\text{def}}{=} \sum_{F \subseteq E} (-1)^{|E \setminus F|} \mu(F).$$

By definition, $\mu^{\mathrm{M}}(\emptyset) = \mu(\emptyset)$. In addition, $(\mu_{\emptyset})^{\mathrm{M}}(E) = \mu^{\mathrm{M}}(E)$ for every $E \in 2^X \setminus \{\emptyset\}$.

Definition 7. *A subset* F *of* X *is called a* focus, *or a* focal element, *of* μ *if* $\mu^{\mathrm{M}}(F) \neq 0$. *The family of foci of* μ *is denoted by* $\mathcal{F}(\mu)$; *that is,* $\mathcal{F}(\mu) \stackrel{\text{def}}{=} \{F \subseteq X \mid \mu^{\mathrm{M}}(F) \neq 0\}$.

Obviously $\mathcal{F}(\mu_{\emptyset}) = \mathcal{F}(\mu) \setminus \{\emptyset\}$.

Definition 8. μ *is said to be* k-modular *if* $k = \max\{|F| \mid F \in \mathcal{F}(\mu)\}$, *where* $\max \emptyset \stackrel{\text{def}}{=} 0$. *A* k*-modular set function without intercept is said to be* k-additive.

The following proposition includes the definition of null set.

Proposition 3. *Let* $N \subseteq X$. *The following conditions are equivalent to each other.*

(a) N *is a* null set *with respect to* μ.
(b) $\mu(E \setminus N) = \mu(E)$ *whenever* $E \subseteq X$.
(c) $N \subseteq X \setminus \bigcup \mathcal{F}(\mu)$.
(d) *For every* $f : X \to \mathbb{R}$,

$$(\mathrm{C}) \int_X f \, d\mu \;=\; (\mathrm{C}) \int_{X \setminus N} f \, d\mu.$$

$X \setminus \bigcup \mathcal{F}(\mu)$ is the greatest null set. If N is a null set, then $\mu(N) = \mu(\emptyset)$. The family of null sets with respect to $\mu_{\emptyset}$ coincides with the family of null sets with respect to μ.

3 Inclusion-exclusion families

The contents of this section are a few modification of existing results (e.g., [3]). The following theorem includes the definition of inclusion-exclusion family.

Theorem 1. *Let* μ *be a set function over* X *and* $\mathcal{S}$ *a family of subsets of* X. *The following conditions are equivalent to each other.*

(a) $\mathcal{S}$ *is an* inclusion-exclusion family, *or an* interadditive family, *with respect to* μ.
(b) *For every* $E \subseteq X$

$$\mu(E) \;=\; \sum_{\mathcal{T} \subseteq \mathcal{S},\; \mathcal{T} \neq \emptyset} (-1)^{|\mathcal{T}|+1} \mu\left(\bigcap \mathcal{T} \cap E \right). \tag{2}$$

(c) *Eq.* (2) *holds for every* $E \in 2^X \setminus \mathrm{Her}\mathcal{S}$.
(d) $\mathcal{F}(\mu) \sqsubseteq \mathcal{S}$, *or equivalently* $\mu^{\mathrm{M}}(E) = 0$ *for every* $E \in 2^X \setminus \mathrm{Her}\mathcal{S}$.
(e) *There exists a collection* $\{\mu_S\}_{S \in \mathcal{S}}$ *of set functions, each* μ_S *of which is defined on* 2^S, *such that for every* $E \subseteq X$

$$\mu(E) \;=\; \sum_{S \in \mathcal{S}} \mu_S(E \cap S).$$

(f) *There exists a collection* $\{\mu_S\}_{S \in \mathcal{S}}$ *of set functions, each* μ_S *of which is defined on* 2^S, *such that for every function* $f : X \to \mathbb{R}$

$$(\mathrm{C}) \int_X f \, d\mu \;=\; \sum_{S \in \mathcal{S}} (\mathrm{C}) \int_S f \, d\mu_S.$$

By the theorem above, $\mathcal{F}(\mu)$ itself is an inclusion-exclusion family and one of the least ones with respect to $\sqsubseteq$. Hence $\mathrm{Max}\mathcal{F}(\mu)$ is the least antichain inclusion-exclusion family and $\mathrm{Her}\mathcal{F}(\mu)$ is the least hereditary inclusion-exclusion family. A family $\mathcal{S}$ of subsets of X is an inclusion-exclusion family with respect to $\mu_\emptyset$ iff $\mathcal{S} \cup \{\emptyset\}$ is an inclusion-exclusion family with respect to μ.

Proposition 4. *Let $\mathcal{S}$ be an inclusion-exclusion family.*

1. *$X \setminus \bigcup \mathcal{S}$ is a null set.*
2. *If N is a null set, then $\{S \setminus N \mid S \in \mathcal{S}\}$ also is an inclusion-exclusion family.*

Proposition 5. *Let k be a nonnegative integer less than or equal to $|X|$. μ is at most k-modular iff $\binom{X}{k}$ is an inclusion-exclusion family.*

If we consider only set functions without intercept, since there is no $\mu \in \mathbb{SF}_\emptyset(X)$ such that $\mathcal{F}(\mu) = \{\emptyset\}$, we may exclude $\{\emptyset\}$ from consideration. Then the collection of all possible inclusion-exclusion families with respect to set functions without intercept over X is isomorphic to the free upper-bounded distributive lattice generated by X, where an upper-bounded lattice is a lattice with the greatest element 1. Since $\emptyset \notin \mathcal{F}(\mu)$ for all $\mu \in \mathbb{SF}_\emptyset(X)$ and $\mathcal{S} \setminus \{\emptyset\} \equiv \mathcal{S}$ for all $\mathcal{S} \subseteq 2^X$ except $\mathcal{S} = \{\emptyset\}$, instead of excluding $\{\emptyset\}$ from the collection of families of subsets, we can exclude $\emptyset$ from families of subsets. Let $\mathbb{A}_{\setminus\emptyset}(X) \stackrel{\text{def}}{=} \{\mathcal{A} \setminus \{\emptyset\} \mid \mathcal{A} \in \mathbb{A}(X)\}$ and $\mathbb{H}_{\setminus\emptyset}(X) \stackrel{\text{def}}{=} \{\mathcal{H} \setminus \{\emptyset\} \mid \mathcal{H} \in \mathbb{H}(X)\}$. Note that $2^{(2^X \setminus \{\emptyset\})} = \{\mathcal{F}(\mu) \mid \mu \in \mathbb{SF}_\emptyset(X)\}$, $\mathbb{A}_{\setminus\emptyset}(X) = \{\mathrm{Max}\mathcal{F}(\mu) \mid \mu \in \mathbb{SF}_\emptyset(X)\} = \mathbb{A}(X) \setminus \{\{\emptyset\}\}$, and $\mathbb{H}_{\setminus\emptyset}(X) = \{(\mathrm{Her}\mathcal{F}(\mu)) \setminus \{\emptyset\} \mid \mu \in \mathbb{SF}_\emptyset(X)\}$. In addition, a set function without intercept with an inclusion-exclusion family $\mathcal{S}$ is determined by its values on $(\mathrm{Her}\mathcal{S}) \setminus \{\emptyset\}$.

Proposition 6. *Each of $((2^{(2^X)} \setminus \{\{\emptyset\}\})/\equiv, \sqsubseteq_\equiv)$, $(2^{(2^X \setminus \{\emptyset\})}/\equiv, \sqsubseteq_\equiv)$, $(\mathbb{A}_{\setminus\emptyset}(X), \sqsubseteq)$, $(\mathbb{H}(X) \setminus \{\{\emptyset\}\}, \sqsubseteq)$, $(\mathbb{H}_{\setminus\emptyset}(X), \sqsubseteq)$ is isomorphic to the free upper-bounded distributive lattice $(\mathfrak{L}(X) \setminus \{0\}, \wedge, \vee, 1)$ generated by X. Especially, $(\mathbb{H}_{\setminus\emptyset}(X), \sqsubseteq)$ is the lattice $(\mathbb{H}_{\setminus\emptyset}(X), \cup, \cap, 2^X \setminus \{\emptyset\})$ of sets. The isomorphism $\varphi : \mathfrak{L}(X) \setminus \{0\} \to (2^{(2^X)} \setminus \{\{\emptyset\}\})/\equiv$ is given by Eq.* (1) *provided that $[\{\emptyset\}]$ is identified with $[\emptyset]$.*

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Representation of the Sign Dependent Expected Utility Functional as a Difference of Two Fuzzy Integrals

Endre Pap

Department of Mathematics and Informatics
University of Novi Sad
21000 Novi Sad, Serbia and Montenegro
pap@im.ns.ac.yu, pape@eunet.yu

1 Introduction

In modeling decision under uncertainty an important tool is the Choquet integral, see [6, 8–10, 12]. Then an universal set X is a space, its elements are state of nature and functions from X to $\mathbb{R}$ are prospects. The preference relation $\preceq$ is defined on the set of prospects and we say that a utility functional L represents a preference relation if and only if $L(f) \leq L(g)$ for all pairs of prospects f, g such that $f \preceq g$. Schmeidler [16] showed that preference can be represented by Choquet integral model, so called Choquet expected utility model (cumulative utility). Choquet expected utility model is not an appropriate tool when the gain and loss must be considered in the same time. In the field of decision theory the cumulative prospect theory (CPT), introduced by Tversky and Kahneman [15], see [3], combines cumulative utility and a generalization of expected utility, so called sign dependent expected utility, related to bipolar scale, see [14]. CPT holds if there exist two fuzzy measures, m^+ and m^-, which ensure that the utility functional L, model for preference representation, can be represented by the difference of two Choquet integrals, i.e.,

$$L(f) = (C)\int f^+ \, dm^+ - (C)\int f^- \, dm^-, \tag{1}$$

where $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$. Narukawa et al. proved in [11] that comonotone-additive and monotone functional can be represented as a difference of two Choquet integrals and gave the conditions for which it can be represented by one Choquet integral.

In the first part of the paper we consider the analogous situation for the Sugeno integral, based on [13]. An extension of the Sugeno integral in the spirit of the symmetric extension of Choquet integral proposed by M. Grabisch in [4] is useful as a framework for cumulative prospect theory in an ordinal context. In this paper we consider representation by two Sugeno integrals of the functional L defined on the class of functions $f : X \to [-1, 1]$ on a finite set X. In the case of infinitely countable set X we obtain as a consequence of results on general fuzzy rank and sign dependent functionals that the symmetric Sugeno integral is comonotone-$\otimes$-additive functional on the class of functions with finite support.

Starting from the needs of cumulative prospect theory and motivated by (1) in the second part of this paper we present some difference representations of asymmetric Choquet integral w.r.t a signed fuzzy measures.

2 Comonotone-$\otimes$-additive functional and its representation

The motivation for the paper [13] is based mainly on the axiomatic characterization of the preference relation $\preceq$ such that it is CPT, stated in [10], and our approach may be viewed as adequate base for an axiomatization for the preference representation in qualitative decision making. Let $f : X \to [-1,1]$ be a function on X with finite support. Consider the class of functions with finite support denoted by $\mathcal{K}_1(X)$:

$$\mathcal{K}_1(X) = \{f \mid f : X \to [-1,1],\ \text{card}(\text{supp}(f)) < \infty\},$$

where the support is given by $\text{supp}(f) = \{x \mid f(x) \neq 0\}$. $\mathcal{K}_1^+(X)$ and $\mathcal{K}_1^-(X)$ denote the class of non-negative and non-positive functions with finite support, respectively.

The *symmetric maximum* $\varovee : [-1,1]^2 \to [-1,1]$, originally introduced in [5], is defined by

$$a \varovee b = \begin{cases} -(|a| \vee |b|)\,, \ b \neq -a \text{ and } |a| \vee |b| = -a \text{ or } \ = -b, \\ 0\,, \ b = -a, \\ |a| \vee |b|\,, \text{ otherwise.} \end{cases}$$

The *symmetric minimum* $\varowedge : [-1,1]^2 \to [-1,1]$, introduced in [5], is defined by

$$a \varowedge b = \begin{cases} -(|a| \wedge |b|)\,, \ \text{sign}\, a \neq \text{sign}\, b, \\ |a| \wedge |b|\,, \text{ otherwise.} \end{cases}$$

We refer the reader to [5] for a detailed study of the properties of the introduced rules. Let $m : \mathcal{A} \to [0,1]$ be a fuzzy measure on the measurable space $(X, \mathcal{A})$. *The symmetric Sugeno integral* of $f \in \mathcal{K}_1(X)$ with respect to μ is defined by ([4]):

$$Ⓢ\int f\, dm = \Big((S)\int f^+\, dm\Big) \varovee \Big(-(S)\int f^-\, dm\Big),$$

where $f^+ = f \vee 0$ and $f^- = (-f) \vee 0 = -(f \wedge 0)$. In order to examine the $\varovee$-additivity of the symmetric Sugeno integral, it is useful to consider the concept of comonotone functions. Note that any function $f : X \to [-1,1]$ can be represented by symmetric maximum of two comonotone functions $f^+ \geq 0$ and $-f^- \leq 0$, i.e., $f = f^+ \varovee (-f^-)$. It is well known fact that the Sugeno integral of a non-negative function f is independent with respect to its comonotone maxitive representation, see [2]. This fact ensures that the symmetric Sugeno integral of function $f \in \mathcal{K}_1(X)$ is independent with respect to its comonotone $\varovee$-additive representation. Now we extend the notion of the symmetric Sugeno integral.

Definition 1. *A functional* $L : \mathcal{K}_1(X) \to [-1,1]$ *is a* fuzzy rank and sign dependent functional (f.r.s.d.) *on* $\mathcal{K}_1(X)$ *if there exist two fuzzy measures* m^+ *and* m^- *such that for all* $f \in \mathcal{K}_1(X)$

$$L(f) = \Big((S)\int f^+\, dm^+\Big) \varovee \Big(-(S)\int f^-\, dm^-\Big).$$

Note that in the case when $m^+ = m^-$ the fuzzy rank and sign dependent functional (f.r.s.d. functional for short) is exactly the symmetric Sugeno integral. If a f.r.s.d. functional L is the symmetric Sugeno integral then we have $L(-f) = -L(f)$.

Let $L : \mathcal{K}_1(X) \to [-1,1]$, be a functional on $\mathcal{K}_1(X)$: (i) L is *comonotone-$\varovee$-additive* iff $L(f \varovee g) = L(f) \varovee L(g)$ for all comonotone functions $f, g \in \mathcal{K}_1(X)$; (ii) L is *monotone* iff $f \leq g \quad \Rightarrow \quad L(f) \leq L(g)$ for all functions $f, g \in \mathcal{K}_1(X)$; (iii) L is positive *$\varowedge$-homogeneous* iff $L(a \varowedge f) = a \varowedge L(f)$ for all $f \in \mathcal{K}_1(X)$ and $a \in [0,1]$; (iv) L is weak *$\varowedge$-homogeneous* iff $L(a \varowedge \mathbf{1}_A) = a \varowedge L(\mathbf{1}_A)$ and $L(a \varowedge (-\mathbf{1}_A)) = a \varowedge L(-\mathbf{1}_A)$ for all $a \in [0,1]$ and $A \subseteq X$. Weak $\varowedge$-homogeneity does not imply positive $\varowedge$-homogeneity in general. In the case of finite set X and $\mathcal{K}_1(X)$ class of functions $f : X \to [-1,1]$ we have the next result.

Theorem 1. *Let X be a finite set. If $L : \mathcal{K}_1(X) \to [-1,1]$ is a comonotone-$\oslash$-additive, weak $\odot$-homogeneous and monotone functional on $\mathcal{K}_1(X)$, then L is a f.r.s.d functional, i.e., there exist two fuzzy measures m_L^+ and m_L^- such that*

$$L(f) = \Big((S)\int f^+\, dm_L^+\Big) \oslash \Big(-(S)\int f^-\, dm_L^-\Big).$$

Theorem 2. *Let X be an infinitely countable set. If $L : \mathcal{K}_1(X) \to [-1,1]$ is a f.r.s.d functional such that $L(f) \neq 0$, for all $f \in \mathcal{K}_1(X)$, $f \neq 0$, then it is a comonotone-$\oslash$-additive functional on the set $\mathcal{K}_1(X)$.*

A f.r.s.d. functional on $\mathcal{K}_1(X)$, where X is a finite set, is not always comonotone-$\oslash$-additive.

Corollary 1. *Let X be an infinitely countable set. The symmetric Sugeno integral is comonotone-$\oslash$-additive functional for functions $f \in \mathcal{K}_1(X)$ such that $Ⓢ\int f\, dm \neq 0$.*

3 Representation of the asymmetric Choquet integral with respect to signed fuzzy measure

Let $\mathcal{A}$ be a σ-algebra of subsets of X. We consider as extension of the notion of the fuzzy measure to set functions $m : \mathcal{A} \to [-\infty,\infty]$, as signed fuzzy measure, see [12]. The chain variation $\mid m \mid$ of real-valued set functions m, vanishing at the empty set, and the space BV were considered in [1, 12].

We shall give a representation of a signed fuzzy measure $m : \mathcal{A} \to [-\infty,\infty]$ which belongs to the space BV. We will correspond to it a signed measure μ defined on a σ-algebra $\mathcal{B}$ of subsets of a set Y. First, we will introduce an interpreter for measurable sets and a frame for representation [7], see [12].

Definition 2. *A mapping $H : \mathcal{A} \to \mathcal{B}$ is called an interpreter if H satisfies: (i) $H(\emptyset) = \emptyset$ and $H(X) = Y$; (ii) $H(E) \subset H(F)$, for all $E \subset F$. A triple $(Y,\mathcal{B},H)$ is called a frame of $(X,\mathcal{A})$, if H is an interpreter from $\mathcal{A}$ to $\mathcal{B}$.*

Definition 3. *Let m be a signed fuzzy measure defined on $\mathcal{A}$. A quadruple $(Y,\mathcal{B},\mu,H)$ is called a representation of m (or $(X,\mathcal{A},m)$) if H is an interpreter from $\mathcal{A}$ to $\mathcal{B}$, μ is a signed measure on $(Y,\mathcal{B})$, and $m = \mu \circ H$.*

Theorem 3. *Every signed fuzzy measure m, $m \in BV$, has its representation.*

We apply Theorem 3 to obtain a representation of the asymmetric Choquet integral of a measurable function f with respect to a signed fuzzy measure m.

Theorem 4. *If m is a signed fuzzy measure, $m \in BV$ and $f \in \overline{\mathcal{M}}$ (class of all measurable functions on X), then there exist two functions $I_f^1 : Y \to [0,\infty]$ and $I_f^2 : Y \to [0,\infty]$ such that the asymmetric Choquet integral has the following difference representation*

$$C_m(f) = \int I^1_{f^+}\, d\lambda - \int I^2_{f^-}\, d\lambda.$$

$C_m(f)$ does not depend of the representation of m by means of Theorem 3.

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Generalized Preference Structures Based on Interpolative Boolean Algebra

Dragan Radojević

Mihajlo Pupin Insitute
11000 Belgrade, Serbia & Montenegro
Dragan.Radojevic@imp-automatika.co.yu

Abstract. Classical (Aristotelian) two-valued realization of Boolean algebra is based on two-element Boolean algebra as its homomorphism. So, calculus and/or arithmetic for a two valued case is Boolean algebra of two elements. Interpolative Boolean algebra is MV realization of finite Boolean algebra. All axioms and all laws of Boolean algebra are preserved in a MV case. New approach is illustrated on the generalization of preference structures.

1 Introduction

Classical (Aristotelian) two-valued realization of Boolean algebra is based on two-element Boolean algebra. So, calculus and/or arithmetic for a two-valued case is two-valued Boolean algebra. Interpolative Boolean algebra [1] is a MV realization of finite Boolean algebra and/or it is consistent generalization of classical two-valued realization.

2 Interpolative Boolean algebra

Interpolative Boolean algebra has a finite number of elements which can have more than two values and in a general case all values from [0, 1]. Interpolative Boolean algebra has two levels: (a*) Symbolic or qualitative* – a matter of Boolean algebra and (b) *Semantic or valued* – a matter of interpolation.

2.1 Symbolic Level

A symbolic or qualitative level is value independent and, as a consequence, it is the same for all realizations on a valued level: classical (two-valued) and generalized MV-case. The main notions on the symbolic or qualitative level are: *a finite set of elements with corresponding Boolean operators – atomic Boolean algebra*. The finite set of elements of Boolean algebra is generated by a set of *primary elements – context*. No primary element can be realized as a Boolean function of the remaining elements from this set – context. An *order relation* on this level is based only on the operator of *inclusion*. A set of Boolean algebra is partially ordered on the basis of inclusion – a Boolean lattice. The *atomic elements* of Boolean algebra – Boolean lattice, are the simplest in the sense that any atomic element doesn't include in itself any other element except itself and a trivial zero constant. Meet (conjunction, intersection) of any two atomic elements is equal to a zero constant. Any element of Boolean algebra can be represented by join (disjunction, union) of relevant atoms – a disjunctive normal form. The *structure* of analyzed element of Boolean algebra is a characteristic function of the set of its relevant atoms. Calculus of structure of Boolean algebra elements is two-valued Boolean calculus based on the relation of inclusion. A consequence is the *principle of structural functionality*. The principle of structural functionality is value independent and, thus, it is a fundamental principle. The principle of *truth functionality* is isomorphism of the principle of structural functionality on a value level only for a classical (two valued) case.

2.2 Valued Level

On a valued level a result from the symbolic level is concretized in the sense of value. A partial order from the symbolic level, based on the relation of inclusion, is mapped into corresponding Boolean lattice on a valued level, based on the values and relation less or equal to. Values on a valued level correspond to the elements of Boolean algebra from symbolic level. (In the case of: null-ary relation - value of truth, unary relations - intensities of property for elements of analyzed universe, binary relations intensity of relation for elements of Cartesian product of universe, etc.). An element from a symbolic level on this level has obtained a value in a way which preserves all its characteristics. For example, to the order, which is determined by inclusion on a symbolic level, there corresponds the order on a valued level, determined by relation "less or equal to". The value of any element is equal to the value obtained by the superposition of values of relevant atomic elements. The value of atomic element is a function of the values of primary elements and a chosen operator of a *generalized product*. Atomic elements have non negative values, whose sum is equal to 1. All tautologies and contradictions from the symbolic level are tautologies and contradictions, respectively, on the valued level.

3 Generalized preference structures

Interpolative Boolean algebra as a MV algebra can be illustrated on the generalization of preference structures. A preference structure is the basic concept of preference modeling. Consider a set of alternatives *A* (objects, actions etc.) and suppose that a decision maker (DM) wants to judge them by pairwise comparison. Given two alternatives, the DM can act in one of the following three ways, [2]:

1. DM prefers one to the other - strict preference relations (>) or (<)
2. two alternatives are indefferent to DM- indifference relation (=)
3. DM is unable to compare the two alternatives – incomparability relation (<>)

For any $(a, b) \in A^2$, we classify:

$(a,b) \in (>) \;\Leftrightarrow\; a > b$ DM prefers a to b;
$(a,b) \in (=) \;\Leftrightarrow\; a = b$ a to b is indifferent to DM
$(a,b) \in (<>) \;\Leftrightarrow\; a <> b$ DM is unable to compare a and b.

A preference structure on A is a triplet $\{ (>),(=),(<>) \}$

The binary relation $(\geq) = (>) \vee (=)$ is called large preference relation of a given preference structure $\{ (>),(=),(<>) \}$.

The set of all possible binary relations generated by two primary relations $\Omega = \{(\leq),(\geq)\}$ (large preference relations) is a Boolean lattice given in the following figure:

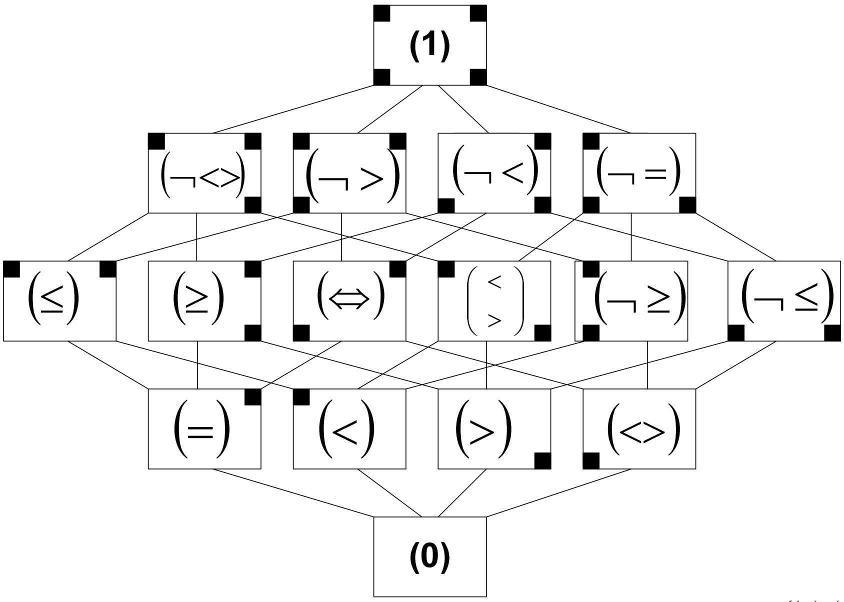

Fig. 1: Symbolic level: Boolean lattice generated by primary relations: $\Omega = \{(\leq), (\geq)\}$

Atomic Interpolative relations as functions of primary relations

$$\begin{aligned}
(=)(a,b) &= ((\leq) \wedge (\geq))(a,b), \\
(<)(a,b) &= ((\leq) \wedge (\neg \geq))(a,b), \\
(>)(a,b) &= ((\neg \leq) \wedge (\geq))(a,b), \\
(<>)(a,b) &= ((\neg \leq) \wedge (\neg \geq))(a,b), \qquad a,b \in A.
\end{aligned}$$

Interpolative relations based as two atomic relations

$$\begin{aligned}
(\leq)(a,b) &= (<)(a,b) + (=)(a,b), \\
(\geq)(a,b) &= (>)(a,b) + (=)(a,b),
\end{aligned}$$

$$\begin{aligned}
(\Leftrightarrow)(a,b) &= (=)(a,b) + (<>)(a,b) \\
\begin{pmatrix} < \\ > \end{pmatrix}(a,b) &= (<)(a,b) + (>)(a,b), \\
(\neg \geq)(a,b) &= (<)(a,b) + (<>)(a,b), \\
(\neg \leq)(a,b) &= (>)(a,b) + (<>)(a,b), \qquad a,b \in A.
\end{aligned}$$

Interpolative relations based on three atomic relations

$$\begin{aligned}
(\neg <>)(a,b) &= (=)(a,b) + (<)(a,b) + (>)(a,b), \\
(\neg >)(a,b) &= (=)(a,b) + (<)(a,b) + (<>)(a,b), \\
(\neg <)(a,b) &= (=)(a,b) + (>)(a,b) + (<>)(a,b), \\
(\neg =)(a,b) &= (<)(a,b) + (>)(a,b) + (<>)(a,b), \qquad a,b \in A.
\end{aligned}$$

Universal Interpolative relation as function of atomic relations

$$(1)(a,b) = (=)(a,b) + (<)(a,b) + (>)(a,b) + (<>)(a,b), \quad a,b \in A.$$

Values (intensity) of atomic Interpolative relations as functions of intensity of primary relations

$$(=)(a,b)=(\leq)(a,b)\otimes(\geq)(a,b),$$
$$(<)(a,b)=(\leq)(a,b)-(\leq)(a,b)\otimes(\geq)(a,b),$$
$$(>)(a,b)=(\geq)(a,b)-(\leq)(a,b)\otimes(\geq)(a,b),$$
$$(<>)(a,b)=1-(\leq)(a,b)-(\geq)(a,b)+(\leq)(a,b)\otimes(\geq)(a,b),$$
$$a,b\in A,$$

where, $\otimes$ is an operator for generalized product [1].

For different generalized product operators we have obtained the following results for Interpolative atomic relations:

Values (intensity) of atomic relations for $\otimes := \min$

$$(=)(a,b)=\min((\leq)(a,b),(\geq)(a,b)),$$
$$(<)(a,b)=(\leq)(a,b)-\min((\leq)(a,b),(\geq)(a,b)),$$
$$(>)(a,b)=(\geq)(a,b)-\min((\leq)(a,b),(\geq)(a,b)),$$
$$(<>)(a,b)=1-(\leq)(a,b)-(\geq)(a,b)+\min((\leq)(a,b),(\geq)(a,b)),$$
$$a,b\in A.$$

Values (intensity) of atomic relations for $\otimes := *$

$$(=)(a,b)=(\leq)(a,b)*(\geq)(a,b),$$
$$(<)(a,b)=(\leq)(a,b)-(\leq)(a,b)*(\geq)(a,b),$$
$$(>)(a,b)=(\geq)(a,b)-(\leq)(a,b)*(\geq)(a,b),$$
$$(<>)(a,b)=1-(\leq)(a,b)-(\geq)(a,b)+(\leq)(a,b)*(\geq)(a,b),$$
$$a,b\in A.$$

Values (intensity) of atomic relations for $a\otimes b := \max(a+b-1,0)$

$$(=)(a,b)=\max((\leq)(a,b)+(\geq)(a,b)-1,\ \ 0),$$
$$(<)(a,b)=(\leq)(a,b)-\max((\leq)(a,b)+(\geq)(a,b)-1,\ \ 0),$$
$$(>)(a,b)=(\geq)(a,b)-\max((\leq)(a,b)+(\geq)(a,b)-1,\ \ 0),$$
$$(<>)(a,b)=1-(\leq)(a,b)-(\geq)(a,b)+\max((\leq)(a,b)+(\geq)(a,b)-1,\ \ 0)$$
$$a,b\in A.$$

All results (a. b. and c.) correspond to known results for fuzzy preference structures [3] but crucially new is the fact that these results are direct generalizations of classical result, contrary to the [2].

The values of all other relations for all $a,b\in A$ generated by two primary relations $\Omega=\{(\leq),(\geq)\}$ as the function of intensities of primary relations can be generalized in the following way too:

$$(\Leftrightarrow)(a,b)=1-(\leq)(a,b)-(\geq)(a,b)+2(\leq)(a,b)\otimes(\geq)(a,b),$$
$$\begin{pmatrix}<\\>\end{pmatrix}(a,b)=(\leq)(a,b)+(\geq)(a,b)-2(\leq)(a,b)\otimes(\geq)(a,b),$$
$$(\neg\geq)(a,b)=1-(\geq)(a,b),$$
$$(\neg\leq)(a,b)=1-(\leq)(a,b),\qquad a,b\in A.$$

$$(\neg <>)(a,b) = (\leq)(a,b) + (\geq)(a,b) - (\leq)(a,b) \otimes (\geq)(a,b),$$
$$(\neg >)(a,b) = 1 - (\geq)(a,b) + (\leq)(a,b) \otimes (\geq)(a,b),$$
$$(\neg <)(a,b) = 1 - (\leq)(a,b) + (\leq)(a,b) \otimes (\geq)(a,b),$$
$$(\neg =)(a,b) = 1 - (\leq)(a,b) \otimes (\geq)(a,b), \qquad a,b \in A.$$

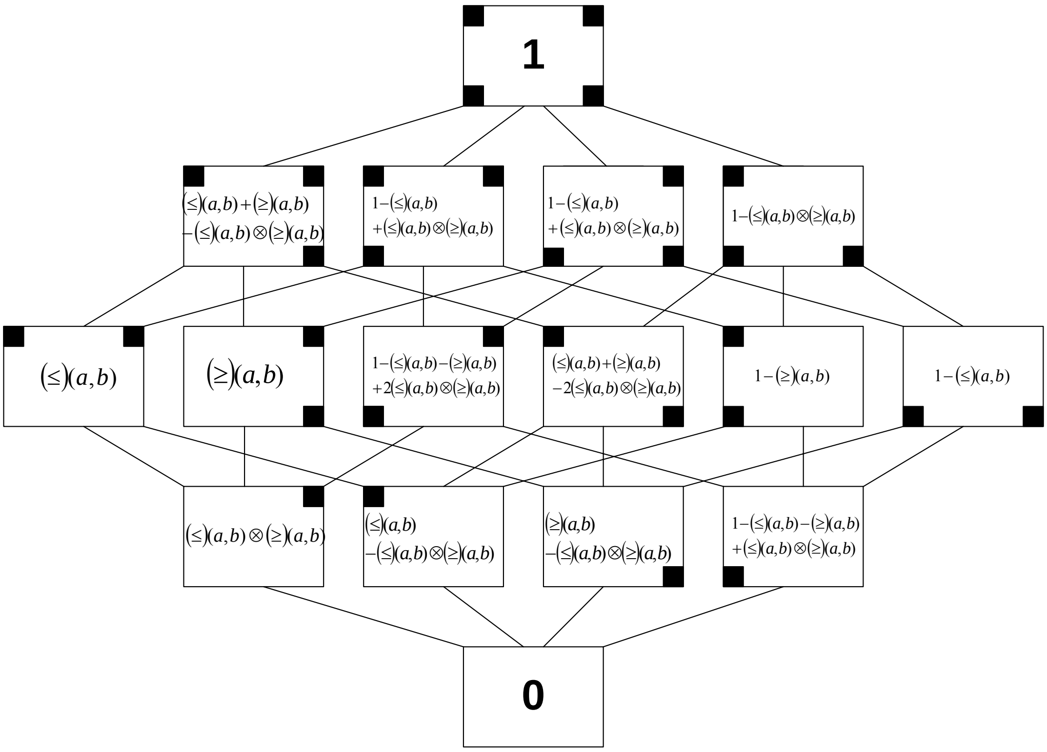

Fig. 2: Valued level: Boolean lattice of relational functions based on *e*($\curlyvee$) and *e*($\llbracket$)

4 Conclusions

Interpolative Boolean algebra is a MV realization of finite Boolean algebra. All axioms and all laws of Boolean algebra are preserved in a MV case. Generalized preference structures as well as generalization of all binary relations generated by relations “less or equal to” and “more or equal to” are obtained from a classical result straightaway.

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On Consensus Functions in the Bipolar Case

Susanne Saminger[1], Didier Dubois[2], and Radko Mesiar[3,4]

[1] Department of Knowledge-Based Mathematical Systems
Johannes Kepler University
4040 Linz, Austria
susanne.saminger@jku.at
[2] IRIT
31062 Toulouse Cedex, France
dubois@irit.fr
[3] Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering
Slovak University of Technology
81 368 Bratislava, Slovakia
mesiar@math.sk
[4] Institute of the Theory of Information and Automation
Czech Academy of Sciences, Prague, Czech Republic

1 Motivation

In decision theory the evaluation of sets of criteria by some individual can be modelled by a normalized, monotone set function , i.e., some function $m\colon \mathcal{A} \to [0,1]$ with $\mathcal{A} \subseteq 2^X$ for some universe X which fulfills $m(\emptyset) = 0$, $m(X) = 1$ and for all $A, B \in \mathcal{A} : A \subseteq B \Rightarrow m(A) \leq m(B)$. Moreover, so called *decomposable* measures have been introduced in [4, 17] for the representation of the importance of a group of criteria by generalizing the additive structure of probability measures. Such a decomposable measure m additionally fulfils

$$A \cap B = \emptyset \quad \Rightarrow \quad m(A \cup B) = S(m(A), m(B)) \tag{1}$$

for all $A, B \in \mathcal{A}$ and some binary function S on $[0,1]$. Note that for some finite X and due to the property of decomposability it is enough to know the measure of every singleton $s \in X$ in order to compute the measure of any $A \in \mathcal{A}$. Having in mind that X might be finite and $\mathcal{A} = 2^X$ it is natural, even if not compulsory, to assume that S is associative, commutative and non-decreasing in each argument. Moreover, due to the normalization of m and its decomposability, 0 is a neutral element of S such that S is t-conorm and m a so-called S-measure.

When representing the opinion of a group of n individuals, whose evaluations are expressed by some S_i-measures m_i, a *consensus* function is needed which maps all these measures to another S-measure m representing the opinion of the group. We assume that the aggregation of the group opinion just depends on the individuals' opinions, such that the group's opinion can be computed by

$$m(A) = \mathbf{A}(m_1(A), \ldots, m_n(A))$$

for all $A \in \mathcal{A}$ with $\mathbf{A}$ some aggregation operator. Note that the choice of an aggregation operator is reasonable since its monotonicity and boundary conditions guarantee the preservation of the monotonicity and normalization of the measures involved. Clearly, the admissibility of aggregation operators depends on the decomposability of the measures being aggregated as well as of the aggregated measure itself. Note that the question of appropriate consensus functions has already been investigated for continuous Archimedean t-conorms as well as the maximum in, e.g., [3, 5, 6]. The solutions are related to unary functions fulfilling a generalized Cauchy equation w.r.t. the t-conorms involved (for more details on such functions, see, e.g., [1, 6]).

Additionally to these aspects, investigations of measures on bipolar scales for decision making have become rather popular during the last years (see, e.g., [2, 8–13] or for earlier investigations also, e.g., [16]). Most often the bipolar scale is assumed to be $[-1,1]$ whereas the explicit range of the scale is of minor importance. Most relevant is that the level of neutrality lies within this interval, usually 0 on $[-1,1]$, and as such separates the scale into two parts — the positive and the negative one.

Note that different approaches for the treatment of evaluations on bipolar scales can be distinguished, depending on whether the (unipolar) positive and negative scales are kept separatly or make up a single bipolar scale:

The simplest such setting is when the universe X can be partitioned into a positive, negative and neutral part, i.e., $X = X^+ \cup X_0 \cup X^-$ and each $A \subseteq X$ is viewed as $A^+ \cup A_0 \cup A^-$. The evaluation of a set A is done by a measure of positiveness m^+ on A^+ and a measure of negativeness m^- on A^-. The set functions $m^+, m^- \colon X \to [0,1]$ fulfil monotonicity conditions w.r.t. set inclusion and the usual limit conditions. A property of neutrality invariance expresses that X_0 does not play any role in the preference representation, so that the evaluation of A is expressed by the pair $(m^+(A^+), m^-(A^-))$ of numbers on unipolar scales revealing its positive and negative information.

Dubois and Fargier [2] take a more qualitative view on this approach, where the chosen measure fulfills $m^+ = m^-$ and is a possibility measure, and they consider the partial order relation obtained from the separate comparison of the positive and negative parts. Moreover, such a relation has to fulfill additional monotonicity and limit conditions as well as neutrality invariance and unanimity.

In the second approach, normalized bi-capacities have been introduced (see [7, 8]) as functions $v \colon \mathcal{Q}(X) \to [-1,1]$ fulfilling

(i) $v(\emptyset,\emptyset) = 0$,
(ii) $A \subseteq B$ implies that $v(A,C) \leq v(B,C)$ and $v(C,A) \geq v(C,B)$ for all $C \in X \setminus B$,
(iii) $v(X,\emptyset) = -1$, $v(\emptyset,X) = 1$

with $\mathcal{Q}(X) = \{(A,B) \in 2^X \times 2^X \mid A \cap B = \emptyset\}$ the set of all pairs of disjoint subsets of some finite universe X. Note that bi-capacities can be interpreted as evaluations of ternary alternatives on a bipolar scale. Further note that (A^+, A^-) as introduced before is an element of $\mathcal{Q}(X)$ such that $v(A^+, A^-)$ can be interpreted as a combination of two functions $v^+(A^+)$ and $v^-(A^-)$ of the previous approaches leading to a value on a bipolar scale. For example, the cumulative prospect theory of Tversky and Kahneman [16] considers the difference of two measures m^+, m^- for such a combination, i.e., $v(A^+, A^-) = m^+(A^+) - m^-(A^-)$. Lexicographic refinements of the proposals of Dubois and Fargier prove to be of that form [2].

Finally, bipolar capacities [12, 13] act on $\mathcal{Q}(X)$ but introduce a measure of positiveness and a measure of negativeness, i.e., bipolar capacities are functions $c \colon \mathcal{Q}(X) \to [0,1]^2$ fulfilling analogous properties as bi-capacities but revealing positive and negative information as a pair of numbers instead of a number on a bipolar scale. All three approaches have a distinction between positive and negative sets in common and reflect that neutrality on the input side should correspond to the neutrality level on the output side.

Supposing now that several individuals express their evaluation of events w.r.t. some bipolar scale the determination of a group's opinion leads naturally again to the question of consensus, of consensus in the bipolar case. Moreover, we will also focus on decomposability of bipolar measures, particularly on decomposable bi-capacities as introduced in [15] and will therefore restrict our considerations in the sequel mainly on bi-capacities. We will now briefly recall the consensus of bipolar measures in general and the concept of decomposability in the bipolar case in order to introduce the necessary basics for a discussion and presentation of results for the consensus of decomposable measures in the bipolar scale as it is intended for the presentation at the seminar.

2 Consensus of bipolar measures

As in the unipolar case we might assume that the group's opinion on some event $A \in \mathcal{A}$ just depends on the evaluations by all its members (independence of irrelevant alternatives). Further the monotonicity of the bi-capacities involved as well as the normalization conditions and the neutral element should be preserved. Special and appropriate aggregation operators on $[-1,1]$, namely bipolar aggregation operators, have already been introduced in [14].

Definition 1. *An arbitrary mapping* $\mathbf{B}\colon \bigcup_{n\in\mathbb{N}}[-1,1]^n \to [-1,1]$ *is called* bipolar aggregation operator *if it fulfills the following properties for arbitrary* $n \in \mathbb{B}$ *([14])*

(i) $\mathbf{B}(x_1,\ldots,x_n) \leq \mathbf{B}(y_1,\ldots,y_n)$ *whenever* $x_i \leq y_i$ *for all* $i \in \{1,\ldots,n\}$,
(ii) $\mathbf{B}(x) = x$ *for all* $x \in [-1,1]$,
(iii) $\mathbf{B}(d,\ldots,d) = d$ *for all* $d \in \{-1,0,1\}$.

Bipolar aggregation operators are general aggregation operators on $[-1,1]$ and possess the additional property that not only the boundaries of the interval are idempotent elements but also the middle element of the bipolar scale, namely 0. As such they guarantee that the function $v\colon \mathcal{Q}(X) \to [-1,1]$ defined by

$$v(A,B) = \mathbf{B}(v_1(A,B),\ldots,v_n(A,B))$$

is again a bi-capacity for arbitrary bi-capacities $v_i\colon \mathcal{Q}(X) \to [-1,1]$, with $i \in \{1,\ldots,n\}$, $n \in \mathbb{B}$.

As a consequence any idempotent bipolar aggregation operator as, e.g., minimum, maximum, arithmetic mean, or weighted means, is an appropriate candidate for aggregating bi-capacities.

3 Decomposability in the bipolar case

Turning back to unipolar measures $m\colon \mathcal{A} \to [0,1]$ note that decomposability w.r.t. some t-conormis equivalent to the fulfillment of the valuation property, i.e.

$$S(m(A\cap B), m(A\cup B)) = S(m(A), m(B)) \tag{2}$$

for all $A,B \in \mathcal{A}$, a property which takes all lattice operations on 2^X into account. Based on these consideration a concept for decomposability has been introduced for bi-capacities in [15]. Note that the operation w.r.t. which the bi-capacity is decomposable has to preserve the level of neutrality namely 0 and the monotonicity conditions. Further associativity has been assumed for the operation such that uninorms U on some interval I, i.e., symmetric, associative, non-decreasing operations on $I \supseteq [-1,1]$ with neutral element 0 are the appropriate candidates for describing the decomposition in case of bi-capacities.

Definition 2. *Consider some interval* $I \supseteq [-1,1]$ *and some uninorm* $U : I^2 \to I$ *with neutral element 0. A bi-capacity* $v : \mathcal{Q}(X) \to [-1,1]$ *is called* decomposable *if for all* $(A,B),(C,D) \in \mathcal{Q}(X)$ *the following equation is fulfilled*

$$U(v(A,B), v(C,D)) = U(v(A\cup C, B\cap D), v(A\cap C, B\cup D)). \tag{3}$$

Note that Eq. (3) exactly expresses the valuation property for the lattice $Q(X)$ w.r.t. the uninorm U. As a consequence of the definition, a U-decomposable bi-capacity $v : Q(X) \to [-1,1]$ can be constructed by fixing the values of $v(\{i\},\emptyset)$ and $v(\emptyset,\{j\})$ for all $i, j \in X$ and as such the complexity of determining v can again be reduced. Additionally conditions on U and v are implied in order for keeping the normalization conditions of v.

As already indicated before, in the presentation, we will also present how the consensus with U-decomposable bi-capacities can be modelled without losing the decomposability. We will further discuss how appropriate consensus functions can be constructed depending on the uninorm involved and will present some examples for particular uninorms.

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Fuzzy Interaction Values

Wolfgang Sander

Computational Mathematics, Technical University of Braunschweig
38106 Braunschweig, Germany
w.sander@tu-bs.de

We present a framework for fuzzy interaction values.
Let $X_m = \{1,2,...,m\}$ be a set of players where the players belong to coalitions with gradual degrees, so that fuzzy coalitions can be described by fuzzy subsets of X_m.
Moreover, for $m,s \in \mathbb{N}, s \leq m$, let V_m^s be the vector space of all functions $f : [0,1]^m \to \mathbb{R}$ for which $D^s f$ is continuous on the diagonal
$\{t\xi_{X_m} : t \in [0,1]\}$ and satisfies $f(0) = 0$ (here ξ_A is the characteristic function of $A, A \subset X_m$, so that $t\xi_{X_m} \equiv t(1,...,1) \equiv \underline{t}_m$).
Together with the scalar product

$$(f,g)_{m,s} = \int_0^1 < D^s f(t\xi_{X_m}) \, , D^s g(t\xi_{X_m}) > dt =$$
$$= \sum_{i_1=1}^{m} \dots \sum_{i_s=1}^{m} \int_0^1 D_{i_1}...D_{i_s} f(t\xi_{X_m}) \, D_{i_1}...D_{i_s} g(t\xi_{X_m}) \, dt \, ,$$

V_m^s becomes a pre-Hilbert space.
If $L^s(\mathbb{R}^m, \mathbb{R})$ is the set of all s-linear, continuous mappings from $\mathbb{R}^m$ into $\mathbb{R}$ and if e_i (or e_i^m, if it is needed) denotes the i-th standard basis vector of $\mathbb{R}^m$ then we call any linear and continuous function

$$\varphi_m^s : V_m^s \to L^s(\mathbb{R}^m, \mathbb{R}) \quad , \quad m,s \in \mathbb{N} \, , s \leq m$$

fuzzy interaction value. The real numbers

$$(\varphi_m^s f) \; (e_{i_1}, ..., e_{i_s})$$

are called fuzzy interaction indices of $S = \{i_1, ..., i_s\}$(note that repetitions of indices are allowed) and will be interpreted as the s-dimensional power (for example multigain or multiloss) of S .
In analogy to results from the theory of semi values it can be expected that fuzzy interaction values behave like higher dimensional differential operators of the type

$$\varphi_m^s f = \int_0^1 D^s f(t\xi_{X_m}) \, g(t) dt \; , \qquad f \in V_m^s$$

where g is a nonnegative $L_\infty(0,1)$ function with $\int_0^1 g(t)dt = 1$. Using that $f \in V_m^s$ and $D^s f : \mathbb{R}^m \to L^s(\mathbb{R}^m, \mathbb{R})$ we distinguish in

$$(\varphi_m^s f) \, (e_{i_1}, ..., e_{i_s}) = \int_0^1 D^s f(\underline{t}_m) \, (e_{i_1}, ..., e_{i_s}) \, g(t) dt = \int_0^1 D_{i_1}...D_{i_s} f(\underline{t}_m) g(t) dt$$

between the " classical case " where all $i_1, ..., i_s$ are different (and form a crisp set $S = \{i_1, ..., i_s\}$ with $|S| = s$) and the " multiset case " where the number of repetitions of at least one $i_\sigma, 1 \leq \sigma \leq s$ is greater than 1.

From now on we here restrict to the classical case and present characterizations of the

$$\underline{\text{fuzzy Shapley interaction value}}\, Sh-\varphi_m^s f=\int_0^1 D^s f(\underline{t}_m)\,dt$$

$$\underline{\text{fuzzy chain interaction value}}\, C-\varphi_m^s f=\int_0^1 D^s f(\underline{t}_m)\,st^{s-1}\,dt \quad \text{and the}$$

$$\underline{\text{fuzzy Banzhaf interaction value}}\, B-\varphi_m^s f=D^s f(\underline{c}_m)=\int_0^1 D^s f(\underline{c}_m)dt,$$

where $c\in[0,1]$ is a constant and $f\in V_M^s$.
In each case we need 3 axioms, which we are going to describe.
To formulate these axioms we need a generalized " dual mapping " : To each $s\in\mathbb{N}$ and $\beta\in L(\mathbb{R}^m,\mathbb{R}^k)$ we define $\beta^{\star s}: L^s(\mathbb{R}^k,\mathbb{R})\to L^s(\mathbb{R}^m,\mathbb{R})$ by

$$(\beta^{\star s}T)(v_1,...,v_s):=T(\beta(v_1),...,\beta(v_s))$$

for all $T\in L^s(\mathbb{R}^k,\mathbb{R})$ and $v_1,...,v_s\in\mathbb{R}^m$.
After this more technical remark, we want to " justify " our axioms. Thus these axioms must be in some respect a mirror of the structure in V_m^s. If we assume that φ_m^s is linear and continuous then by the Stone-Weierstrass theorem $f\in\varphi_m^s$ can be uniformly approximated by polynomials in m variables in such a manner that also the derivatives of f up to order s can be approximated by the corresponding derivatives of the polynomials. Thus we may assume w.l.o.g. that $f\in\varphi_m^s$ has the form $f(x_1,...,x_m)=x_1^{k_1}\cdots x_m^{k_m}$. We rewrite f as $f(x_1,...,x_m)=\underbrace{x_1\cdots x_1}_{k_1}\cdots\underbrace{x_m\cdots x_m}_{k_m}=$
$f_k(x_1,...,x_1,...,x_m,...,x_m)=(f_k\circ B_P)((x_1,...,x_m)$ where $k=k_1+\cdots+k_m$, where f_k is a symmetric function (in all its variables) and where B_P is a (k,m)-matrix which belongs to a natural partition P of " k players into m subsets $A_j, 1\le j\le m$ of players with $|A_j|=k_j, 1\le j\le m$ ". The matrix $B_P=(b_{ij})$ is given by

$$b_{ij}=\begin{cases}1 & i\in A_j\\ 0 & i\notin A_j\end{cases}\quad 1\le i\le k, 1\le j\le m.$$

Now, the <u>symmetry axiom</u> says that the fuzzy interaction index of the crisp set $S=\{i_1,...,i_s\}\subset X_m$ is independent upon the order of the players within S, ore more exactly : To each permutation $\pi: X_m\to X_m$ we associate the linear map $\alpha_\pi:\mathbb{R}^m\to\mathbb{R}^m$, represented by the matrix $A_\pi=(\delta_{\pi^{-1}(i),j)})$,
$1\le i,j\le m$. The symmetry axiom requires that

$$\varphi_m^s(f\circ\alpha_\pi)=\alpha_\pi^{\star s}(\varphi_m^s\circ f) \tag{1}$$

for each symmetric $f\in V_m^s$, $s\le m$ (which means - using the above remark - that $f(x_1,...,x_m)=x_1\cdots x_m$).
Concerning the <u>partition axiom</u>, we associate to each partition $P=\{A_1,...,A_m\}$ of subsets of players of a set X_k of k players a linear mapping β_P given by the above matrix B_P. Then the partition axiom states that

$$\varphi_m^s(f\circ\beta_P)=\beta_P^{\star s}(\varphi_k^s\circ f) \tag{2}$$

for all $f\in V_k^s$, $s\le m\le k$. This means that for all $j_1,...,j_s\in\{1,...,m\}$

$$\varphi_m^s(f\circ\beta_P)(e_{j_1}^m,...,e_{j_m}^m)=\sum_{i_1\in A_{j_1}}\cdots\sum_{i_s\in A_{j_s}}(\varphi_k^s\circ f)(e_{i_1}^k,...,e_{i_s}^k)$$

and gives the natural connection of s-dimensional powers in games of different sizes (if $f \in V_k^s$ then $f \circ \beta_P \in V_m^s$).
The third axiom is an efficiency axiom for all three fuzzy interaction indices. We here treat only the case of the fuzzy Shapley interaction indices. Let $S = \{i_1, ... i_s\}$ be a fixed crisp set of players in X_m. We consider the coalition $S \setminus \{i_s\}$ and the opposite coalition $X_m \setminus (S \setminus \{i_s\})$. Then the efficiency axiom requires

$$\sum_{j_k \in X_m \setminus (S \setminus \{i_s\})} (\varphi_m^s f)(e_{i_1}, ..., e_{i_{s-1}}, j_k) = f(\underline{1}_m), \tag{3}$$

that is, the sum of the shares of players from $X_m \setminus (S \setminus \{i_s\})$ which they are willing to invest for participation in S is the value of the grand coalition.
In the two other cases we have also very simple expressions for the efficiency axiom, and we can prove the following result.

Theorem. Let $S = \{i_1, ..., i_s\}$ be a crisp set with $|S| = s$, let $m \in \mathbb{N}, s \leq m$, and let (φ_m^s) be a sequence of linear and continuous fuzzy interaction values. Then $Sh - \varphi_m^s(i_1, ..., i_s)$ (and analogously, $C - \varphi_m^s(i_1, ..., i_s)$ and $B - \varphi_m^s(i_1, ..., i_s)$) is the only sequence satisfying the axioms of symmetry, partition and efficiency.

Let us add some remarks.

(1) The 3 results differ only in the different form of the efficiency axiom, but the proofs are nearly the same. Moreover the proofs are not long.

(2) If we add one more (complicated) axiom (which corresponds to the multiset case) then we can even give characterizations of the three different fuzzy interaction values (and not only of the fuzzy interaction indices).

(3) In the special case of the multilinear extension $f_v \in V_m^s$ of a game
$v : 2^{X_m} \to \mathbb{R}, v(\emptyset) = 0$, which is given by the two expressions

$$f_v(x) = f_v(x_1, ..., x_n) = \sum_{T \subset X_m} a_v(T) \prod_{i \in T} x_i = \sum_{T \subset X_m} v(T) \prod_{i \in T} x_i \prod_{i \notin T} (1 - x_i) \tag{4}$$

(here $a_v(T) = \sum_{L \subset T} (-1)^{|T|-|L|} v(L)$, $T \subset X_m$ is the Möbius transform of v),
we get - if $S = \{i_1, ..., i_s\}$ is a crisp set -

$$D_{i_1} ... D_{i_s} f_v(x) = \sum_{T \supset S} a_v(T) \prod_{i \in T \setminus S} x_i = \sum_{T \subset X_m \setminus S} \delta_S v(T \cup S) \prod_{j \in T} x_j \prod_{j \notin T \cup S} (1 - x_j)$$

and thus

$$(\varphi_m^s f_v)\ (e_{i_1}, ..., e_{i_s}) = \sum_{T \supset S} a_v(T) \underbrace{\int_0^1 x^{t-s} g(x) dx}_{\beta_t^s} = \tag{5}$$

$$= \sum_{T \subset X_m \setminus S} \delta_S v(T \cup S) \underbrace{\int_0^1 x^t (1-x)^{m-(t+s)} g(x) dx}_{p_t^s(m)}, \tag{6}$$

where $\delta_S v$ is the usual $|S|$-th derivative of v, that is
$\delta_S v(T \cup S) = \sum_{L \subset S} (-1)^{|S|-|L|} v(L \cup T)$. Thus we have two possibilities to characterize the usual interaction indices by using the "β_t^s-version" or the "$p_t^s(m)$-version". The "β_t^s-version" seems to have advantages, at first, β_t^s is not dependent upon m (in comparison with $p_t^s(m)$), and secondly, using the usual basis representation

$$v = \sum_{T \subset X_m} a_v(T) v_T \tag{7}$$

(where $v_T(A) = 1$ if $T \subset A$ and $v_T(A) = 0$ otherwise) (5) goes over into

$$(\varphi_m^s f_v)\ (e_{i_1}, \ldots, e_{i_s}) = \sum_{T \supset S} a_v(T) (\varphi_m^s f_{v_T})\ (e_{i_1}, \ldots, e_{i_s}). \tag{8}$$

Thus axioms for fuzzy interaction indices can be used as axioms for interaction indices. Note that because of the first representation of f_v in (4), $f_{v_T} = \prod_{i \in T} x_i$ is symmetric and thus the partition axiom gives no additional information and must be replaced by another axiom. In this way we can give new characterizations for the Shapley interaction index, the chain interaction index, and the (generalized) Banzhaf interaction index (including a characterization of the Möbius transform and the co-Möbius transform by putting c = 0 and c = 1, respectively in the fuzzy Banzhaf interaction value).

(4) Using the considerations in (3) it is possible to compare the existing different characterizations for interaction indices.

(5) It seems that the above results can be generalized to fuzzy Aumann-Shapley values on more general spaces (in the sense of the results, given in the book "Triangular norm-based measures and Games with Fuzzy Coalitions" of Butnariu and Klement).

Uninorm-based Approximate Reasoning Models

Márta Takács

Institute of Intelligent Engineering Systems, Budapest Tech
Budapest, Hungary
takacs.marta@nik.bmf.hu

The practical motivations for the introduction of uninorms were the applications from multicriteria decision making, where the aggregation is one of the key issues. Some alternatives are evaluated from several points of view. Each evaluation is a number from the unit interval. If the level of the satisfaction is $e \in]0,1[$, than if all criteria are satisfied to at least *e*-extent then we would like to assign a high aggregated value to this alternative. The opposite of that is if all evaluations are below *e* then we would like to assign a low aggregated value to this alternative. But if there are evaluations below and above *e*, an aggregated value ought to be assigned somewhere in between. Such situations can be modeled by uninorms [1] and distance-based operators [12].

The structure and the representation of uninorms and the mathematical background of uninorm-based fuzzy applications where studied extensively in many sources [2], [3]. Applying the uninorm operators and distance-based operators with the changeable parameter *e* in fuzzy approximate reasoning systems is born in mind that the underlying notions of soft-computing systems are flexibility and the human mind. The choice of the fuzzy environment must support the efficiency of the system, it must comply to the real world. This is more important than trying to fit the real world into the inflexible models. [4], [5], [6].

Furthermore, the applications of the tree-structure, the hierarchical fuzzy control systems in decision-making in a given moment enables us to choose the most efficient system parameters, for example and environment factors and by this achieve the desired state as soon as possible [7].

Parameterized approximate reasoning with distance-based uninorms

Generally, the fundamental of the decision making in fuzzy based real systems is the approximate reasoning, which is a rule-based system. Knowledge representation in a rule-based system is done by means of IF…THEN rules. Furthermore, approximate reasoning systems allow fuzzy inputs, fuzzy antecedents and fuzzy consequents. The computational rule of inference plays a curricular role in fuzzy control, but also in approximate reasoning [8]. This theory provides a powerful framework for reasoning in the face of imprecise and uncertain information between the input and the output space (for a fuzzy input and fuzzy output, by means of fuzzy relations) [9], [10].

The strict modus ponens in those systems, (where the rule premise is fuzzy set *A* and the actual system input is *A'* are membership functions on the universe *X*, the rule consequence is *B* on the universe *Y*) is replaced with the expectation: let be $B' \supset B$, where *B'* is a cut of *B*, that is the Generalized Modus Ponens (GMP) Usually the general rule consequence for one rule from the *i*-th rule base system is obtained by

$$B'(y) = \sup_{x \in X}\left(OPDis1\left(A'(x), OPDis2\left(A(x), B(y)\right)\right)\right), \; x \in X, \; y \in Y \quad \mathbf{(1)}$$

The connections *OPDis1* and *OPDis2* are generally defined, and they can be some type of fuzzy disjunctive operators [11]. The membership function of the consequence in the *i*-th rule B_i' is defined by

$$B_i'(y) = \sup_{x\in X}\left(OPDis\left(A'(x), OPDis\left(A_i(x), B_i(y)\right)\right)\right) \quad (2)$$

where *OPDis* is a fuzzy disjunctive operator. Using the operator properties, from the above expression follows

$$B_i'(y) = OPDis\left(\sup_{x\in X} OPDis\left(A'(x), A_i(x)\right), B_i(y)\right), \quad (3)$$

Generally speaking, the consequence (rule output) is given with a fuzzy set *B'*(*y*), which is derived from rule consequence *B*(*y*), as a cut of the *B*(*y*). This cut,

$$DOF_i = \sup_{x\in X} OPDis\left(A'(x), A_i(x)\right) \quad (4)$$

is the generalized degree of firing level of the rule, considering actual rule base input *A'*(*x*), and usually depends on the covering over *A*(*x*) and *A'*(*x*). Rule base output B'_{out} is an aggregation of all rule consequences $B_i'(y)$ from the rule base. As aggregation operator a conjunctive fuzzy operator is usually used.

$$B'_{out}(y) = OPCon(B_n'(y), OPCon(B_{n-1}'(y), OPCon(\ldots, OPConS(B_2'(y), B_1'(y))))). \quad (5)$$

If in the applications a crisp FLC output y_{out} is needed, it is constructed as a crisp value calculated with a defuzzification method from rule base output.

It can be conclude, that in decision making approximate reasoning the (*OPDis*, *OPCon*) pair of operators is used.

The operators *OPDis* and *OPCon* can be chosen from the group of distance based operators, which contains uninorms too [12], [18]. Considering the structure of distance based operators, namely that they are constructed by the *min* and *max*, it was worth trying to move away from the strictly applied max (disjunctive) and min (conjunctive) operator pair in approximate reasoning. Therefore, in the simulation systems and applications different operators from the group of distance based operators were applied as disjunctive and conjunctive. Moreover, the distance based operators are parameterized by the parameter *e*, therefore the program, which performs the task of decision making in the simulation system, has global, optional, variables (*OPDis*, *OPCon*, *e*), where *Opdis* is the operator applied by GMP, and the *OPCon* is the aggregation operator for the calculation of the B'_{out}. The neutral element of the *OPDis* operator is parameter *e*, and the neutral element of the *OPCon* operator is parameter 1-*e*. Details about the simulation results can be found in [13], [14]. Hence and because by the simulation the triple (*OPDis*, *OPCon*, *e*) can be chosen by even running of the simulation system, it enables the parameters to be set at every running of the system in order to achieve greater efficiency.

In reality the other elements of the system (gains, product elements) are also system dependent and changeable and it can be expected that the operators used in decision making can be tuned to these elements for greater efficiency. It could be implemented in a fuzzy rule system which is of such type:

IF *the system elements(gains,…)* ARE … ,

THEN *the chosen triple of operators and its parameters* IS (*OPDis, OPCon, e*) .

The system presented above, from the input to the output will be the lower level of the hierarchical system, while on the upper level decisions will be made about the choice of operators in decision making system depending on the temporary state of other system elements, (gains, etc.).

Further possibilities: similarity measures based and residuum based approximate reasoning with distance-based operators

In several decision making systems, for example in system control, one would intuitively expect: to make the powerful coincidence between fuzzy sets stronger, and the weak coincidence even weaker. The distance-based operators group satisfy these properties, but the covering over $A(x)$ and $A'(x)$ are not really reflected by the *sup* of the membership function of the $min_e^{max}(A_i(x),A'(x))$, see (4), therefore a Degree of Coincidence (*Doc*) for those fuzzy sets has been initiated. This is actually the proportion of area under membership function of the distance-based intersection of those fuzzy sets, and the area under membership function of their union (using *max* as the fuzzy union).

$$Doc_i = \int_X \min_e^{\max}\left(A_i(x), A'(x)\right)dx \Big/ \int_X \max\left(A_i(x), A'(x)\right)dx \quad \textbf{(6)}$$

This definition has two advantages: it consider the width of coincidence of A_i and A', and not only the "height", the *sup*, and the rule output is weighted with a measure of coincidence of A_i and A' in each rule.

Based on definition of similarity measures from [14] and [15], we can give a generalization of this reason. The Jackard measure for fuzzy sets:

$$Doc_{R5}(i) = \int_X \max_e^{\min}\left(\mu_{A_i}(x), \mu_{A'}(x)\right)dx \Big/ \int_X \max_{1-e}^{\max}\left(\mu_{A_i}(x), \mu_{A'}(x)\right)dx \quad \textbf{(7)}$$

The modified cardinality measure for fuzzy sets:

$$Doc_{R6}(i) = \frac{\int_X \max_e^{\min}\left(\mu_{A_i}(x), \mu_{A'}(x)\right)dx}{\int_X \max_{1-e}^{\max}\left(\mu_{A_1}(x), \max_{1-e}^{\max}\left(\mu_{A2}(x), \max_{1-e}^{\max}\left(\mu_{A3}(x), \ldots\right)\right)\right)dx} \quad \textbf{(8)}$$

The rule output can be the cut of the rule consequence, see (3), in this case

$$B_i'(y) = \min\left(Doc_{similarity}(i), B_i(y)\right), Doc_{similarity}(i) \in \left\{Doc_{R5}, Doc_{R6}\right\} \quad \textbf{(9)}$$

Furthermore, in [17] for the conjunctive left-continuous idempotent uninorm $\max_{0.5}^{\min}$ with the unary operator $g(x) = 1 - x$, its residual implicator $Imp_{\max_{0.5}^{\min}}$, and the residuum-based approximate reasoning with this distance based operator is given.

Currently these methods are used in medical, urological diagnostical systems, moreover, there are environmental studies taking place with the application of the given uninorm-based approximate reasoning methods.

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Learning Fuzzy Logic Aggregation for Multicriterial Querying With User Preferences

Peter Vojtáš[1,2,3] and Marta Vomlelová[1]

[1] Faculty of Mathematics and Physics
Charles University
Praha, Czech Republic
{Peter.Vojtas|Marta.Vomlelova}@mff.cuni.cz
[2] Institute of Computer Science
Czech Academy of Science
Praha, Czech Republic
[3] Faculty of Science
P. J. Šafárik University
Košice, Slovak Republic

In this work we discuss different formal models dealing with the same task on the same data.

A user U is looking for a resource $id \in \mathcal{R}$ best fitting his/her preferences. For a sample set $\mathcal{R}_0 \subseteq \mathcal{R}$ we have an ordinal classification $C_U : \mathcal{R}_0 \to [0,1]$ of user preferences. This problem is important for web applications [2, 1]. With the problem of finding best (top k) answers we dealt in [4, 7, 6]. Here, the preference value structure can be an arbitrary ordinal scale (linear, partially ordered, ...).

To learn user preferences and extend them to the whole $\mathcal{R}$ we use the information on resource attribute values, typically stored in a database

$$\mathcal{D}(id) = \langle a_1^{id}, \ldots, a_n^{id} \rangle \in D_1 \times \ldots \times D_n$$

Now the task is to learn a function $A_U : D_1 \times \ldots \times D_n \to [0,1]$ such that $A_U(D(id)) = C_U(id)$.

In a fuzzy setting we can find fuzzy sets

$$f_U^j : D_j \to [0,1]$$

describing user preference on values from j-th attribute domain D_j and a fuzzy aggregation function

$$@_U : [0,1]^n \to [0,1]$$

such that

$$@_U(f_U^1(a_1^{id}), \ldots, f_U^n(a_n^{id})) = C_U(id)$$

This has been studied for generalized annotated programs in [8]. In a bayesian setting we assume the data attribute j is a random variable over the domain D_j. Each resource represents a sample from an unknown joint distribution over $D_1 \times \ldots \times D_n$. We can estimate this distribution by learning a Bayesian network.

We can try to find dependences between C_U (a joint distribution) and distributions of attributes.

In [7] we tackled this problem extending results of [3] on Bayesian logic programs to many valued case.

In this work we follow two goals. First, we compare learning of fuzzy aggregation function [8] and induction of Bayesian network [5].

Second, we compare direct translation from GAP to BLP [7] with the composition of a translation from GAP to classical logic programs (LP) and the translation from LP to BLP introduced by [3].

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Dominance of Continuous Triangular Norms and its Transitivity

Peter Sarkoci*

Department of Mathematics, IAEIM
Slovak University of Technology, Bratislava, Slovakia
E-mail: peter.sarkoci@stuba.sk

The notion of dominance was originally introduced within the theory of probabilistic metric spaces [5, 6]; in particular dominance becomes important when constructing cartesian products of probabilistic metric spaces. We say that a t-norm T_1 dominates a t-norm T_2 ($T_1 \gg T_2$ in symbols) if for all $x, y, u, v \in [0, 1]$ we have

$$T_1(T_2(x, y), T_2(u, v)) \geq T_2(T_1(x, u), T_1(y, v)).$$

Thanks to associativity and commutativity any t-norm dominates itself; therefore dominance of t-norms is a *reflexive* relation. Moreover from the commutativity together with a fact that that all t-norms have common neutral element it follows that the dominance is a refinement of a standard point-wise ordering of t-norms – from $T_1 \gg T_2$ follows $T_1 \geq T_2$. Thus dominance of t-norms is an *antisymmetric* relation. The old open problem was whether dominance of t-norms is a transitive relation (Problem 12.11.3 in [5, 6]). If it were true dominance would be a partial order.

In our talk we will show that the dominance is *not* transitive *even* on continuous t-norms; we will provide an infinitude of counterexamples. The simplest one, perhaps, is this one:

$$\begin{aligned} (\langle 0, 1/2, T_{\mathbf{L}} \rangle) &\gg (\langle 0, 1/2, T_{\mathbf{L}} \rangle, \langle 1/2, 1, T_{\mathbf{L}} \rangle) \\ (\langle 0, 1/2, T_{\mathbf{L}} \rangle, \langle 1/2, 1, T_{\mathbf{L}} \rangle) &\gg T_{\mathbf{L}} \\ (\langle 0, 1/2, T_{\mathbf{L}} \rangle) &\not\gg T_{\mathbf{L}} \end{aligned}$$

In our proofs we refer to results from [1–4].

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Fuzzy Logic Laboratorium Linz-Hagenberg
Dept. of Knowledge-Based Mathematical Systems
Johannes Kepler Universität
A-4040 Linz, Austria
Tel.: +43 732 2468 9194
Fax: +43 732 2468 1351
E-Mail: info@flll.jku.at
WWW: http://www.flll.jku.at/

Software Competence Center Hagenberg
Hauptstrasse 99
A-4232 Hagenberg, Austria
Tel.: +43 7236 3343 800
Fax: +43 7236 3343 888
E-Mail: sekr@scch.at
WWW: http://www.scch.at/

FUZZY.LOGIC.LAB.LINZ

scch

software competence center hagenberg

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