From Single Agent to Many Agents. Agent Logics of Dynamic Belief and Knowledge

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joint work with
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Overview

Part 1: Single-agent framework for actions and beliefs.
: We study combinations of *PDL* and the well-known logics of belief and knowledge extended with extra axioms of interaction of the action and informational modalities and select an appropriate *decidable* and *complete* logic which represents beliefs and actions of a single agent the most adequately.

Part 2: From single agent to many agents.
: We show how to increase the language expressive power and combine a single agent logics form Part 1 into a real multi-agent framework preserving decidability and completeness.

Outline

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Standard Axioms to Represent Beliefs and Knowledge

(D)	$\Box p \rightarrow \neg\Box\neg p$
(T)	$\Box p \rightarrow p$
(B)	$p \rightarrow \Box\neg\Box\neg p$
(4)	$\Box p \rightarrow \Box\Box p$
(5)	$\neg\Box p \rightarrow \Box\neg\Box p$

Interaction Axioms

(*NL*) $$[a]\Box p \rightarrow \Box[a]p$$

(*PR*) $$\Box[a]p \rightarrow [a]\Box p$$

(*CR*) $$\neg\Box\neg[a]p \rightarrow [a]\neg\Box\neg p$$

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PDL Language

- $\mathrm{AtAc} = \{a, b, \dots\}$ is a set of atomic actions.
- $\mathrm{Var} = \{p, q, \dots\}$ is a set of propositional variables.
- Formula connectives: $\bot$, $\to$, $\Box$.
- Action connectives: $;$, $\cup$, $*$.
- Mixed operators: $?$, $[\cdot]$.
- $\mathbf{For}$ and $\mathbf{Ac}$ are the smallest sets such that:
  - $\mathrm{AtAc} \subseteq \mathbf{Ac}$ and $\mathrm{Var} \cup \{\bot\} \subseteq \mathbf{For}$
  - if $\phi, \psi \in \mathbf{For}$, $\alpha, \beta \in \mathbf{Ac}$
    then $\alpha^*, \alpha \cup \beta, \alpha;\beta, \phi? \in \mathbf{Ac}$, and $\Box\phi, \phi \to \psi, [\alpha]\phi \in \mathbf{For}$

PDL Semantics

Model M is a tuple $\langle S, Q, \models \rangle$, where all Q are defined on all the actions and $\models$ is a truth relation on M such that [1]:

- $Q(\alpha \cup \beta) \stackrel{\text{def}}{=} Q(\alpha) \cup Q(\beta)$,
- $Q(\alpha;\beta) \stackrel{\text{def}}{=} Q(\alpha) \circ Q(\beta)$,
- $Q(\alpha^*) \stackrel{\text{def}}{=} Q(\alpha)^* =$
  $= \{(x, y) \in S^2 \mid \exists n \geq 0 \exists x_0 = x, x_1, \ldots, x_{n-1}, x_n = y \ (x_i, x_{i+1}) \in Q(\alpha)\}$,
- $Q(\phi?) \stackrel{\text{def}}{=} \{(x, x) \in S^2 \mid x \models \phi\}$,
- $M, x \not\models \bot$,
- $M, x \models \phi \to \psi \stackrel{\text{def}}{\iff} (M, x \models \phi$ implies $M, x \models \psi)$,
- $M, x \models [\alpha]\phi \stackrel{\text{def}}{\iff} (x, y) \in Q(\alpha)$ implies $M, y \models \phi$ for all $y \in S$.

[1] Q^* is the transitive and reflexive closure of Q.

Fusions of Modal Logics

$L_1 \otimes L_2$ is a logic where all modal operators of L_1 and L_2 are treated separately and its Boolean part is the only common part with both L_1 and L_2.

...

Logics Considered

For any $Ax \subseteq \{NL, PR, CR\}$

$$\textbf{(test-free)}\mathit{PDL} \otimes \mathit{K45} \oplus Ax,$$
$$\textbf{(test-free)}\mathit{PDL} \otimes \mathit{KD45} \oplus Ax,$$
$$\textbf{(test-free)}\mathit{PDL} \otimes \mathit{S5} \oplus Ax,$$

with either
weak substitution rule (substitutions of formulae for propositional variables are allowed only) or
full substitution rule (substitutions of formulae for propositional variables and of arbitrary actions for atomic actions are both allowed).

Admissibility of the Full Substitution Rule

Theorem

$$\textit{PDL} \otimes L = (\textit{PDL} \otimes L)_w$$
$$\textit{test-free PDL} \otimes L = (\textit{test-free PDL} \otimes L)_w$$

Theorem

Let $\varnothing \neq Ax \subseteq \{NL, PR, CR\}$ and L be contained in the logic of the two-element cluster (for example, K45, KD45 or S5). Then

$$\textit{PDL} \otimes L \oplus Ax \neq (\textit{PDL} \otimes L \oplus Ax)_w$$

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Let $Ax \subseteq \{PR, CR\}$ and L be K45, KD45 or S5. Then

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Extensions of *KB*

For any $L \supseteq \mathit{KB}$

$$(\mathit{PDL} \otimes L \oplus \{\mathit{NL}\})_w = (\mathit{PDL} \otimes L \oplus \{\mathit{CR}\})_w$$
$$(\textit{test-free PDL} \otimes L \oplus \{\mathit{NL}\})_w = (\textit{test-free PDL} \otimes L \oplus \{\mathit{CR}\})_w$$

and, consequently,

$$\mathit{PDL} \otimes L \oplus \{\mathit{NL}\} = \mathit{PDL} \otimes L \oplus \{\mathit{CR}\}$$
$$\textit{test-free PDL} \otimes L \oplus \{\mathit{NL}\} = \textit{test-free PDL} \otimes L \oplus \{\mathit{CR}\}$$

For any $L \subseteq \mathit{S5}$

$$(\mathit{PDL} \otimes L \oplus \{\mathit{PR}\})_w \not\supseteq \not\subseteq (\mathit{PDL} \otimes L \oplus \{\mathit{CR}\})_w$$
$$\textit{test-free PDL} \otimes L \oplus \{\mathit{PR}\} \not\supseteq \not\subseteq \textit{test-free PDL} \otimes L \oplus \{\mathit{CR}\}$$

but for any $L \supseteq T$

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Collapse of Belief Operator

Theorem

Let $\varnothing \neq Ax \subseteq \{NL, PR, CR\}$. For every unimodal logic L, $PDL \otimes L \oplus Ax \vdash p \to \Box p$.

Theorem

Let $L \supseteq T$ and $\varnothing \neq Ax \subseteq \{NL, PR, CR\}$. If the logic $PDL \otimes L \oplus Ax$ is consistent then it is equal to $PDL \otimes K \oplus \{p \leftrightarrow \Box p\}$ and, consequently, is deductively equivalent to PDL.

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Completeness and the Effective Finite Model Property

Let L be *K45*, *KD45* or *S5*, and $\varnothing \neq Ax \subseteq \{\mathit{NL}, \mathit{PR}, \mathit{CR}\}$.
Then the following logics are complete and have the effective finite model property with the upper bound $\mu(n)$ for the sizes of models.

$\mu(n) = 2^n \cdot 2^{2^n}$	$\mu(n) = 2^n$
$(\mathit{PDL} \otimes L \oplus \{\mathit{PR}\})_w$	$\mathit{PDL} \otimes \mathit{S5} \oplus Ax$
$(\mathit{PDL} \otimes L \oplus \{\mathit{CR}\})_w$	
$(\mathit{PDL} \otimes L \oplus \{\mathit{PR}, \mathit{CR}\})_w$	
test-free $\mathit{PDL} \otimes L \oplus \{\mathit{PR}\}$	
test-free $\mathit{PDL} \otimes L \oplus \{\mathit{CR}\}$	
test-free $\mathit{PDL} \otimes L \oplus \{\mathit{PR}, \mathit{CR}\}$	

Test operators

- Classical test:

  Axiomatisation $[\phi?]\psi \leftrightarrow (\phi \rightarrow \psi)$

  Semantics $Q(\phi?) = \{(s, s) \in S^2 \mid s \models \phi\}$

  Example

  [(pass_exam?; celebrate) ∪ (¬pass_exam?; go_to_pub)]drunk

- Informational test:

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  Example [know_subject??]self-confident

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- Informational test:

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Example $[\text{know\_subject}⁇]\text{self-confident}$

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Properties of the Informational Test

- $\Box p \leftrightarrow [\top ⁈]p \in (\mathit{PDL} \otimes \mathit{K})^{⁈}$.
- $[p⁈]\Box q \to \Box[p⁈]q$, $\Box[p⁈]q \to [p⁈]\Box q$ and $\Diamond[p⁈]q \to [p⁈]\Diamond q$ belong to $(\mathit{PDL} \otimes \mathit{K45})^{⁈}$.
- Let L be *K45*, *KD45*, or *S5*. Then any extension of $(\mathit{PDL} \otimes L)^{⁈}$ by the axioms *PR* and/or *CR* with the weak substitution rule
  - admits the rule of full substitution,
  - has the effective finite model property with the upper bound $2^n \cdot 2^{2^n}$ for the model size,
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Embedding of *PDL* into $(PDL \otimes S5)^{⁇}$

$$\sigma p = \Box p \qquad \sigma\bot = \bot$$

$$\sigma a = a \qquad \sigma(\psi?) = (\sigma\psi)⁇$$

$$\sigma(\alpha \cup \beta) = \sigma\alpha \cup \sigma\beta \qquad \sigma(\alpha;\beta) = \sigma\alpha;\top⁇;\sigma\beta$$

$$\sigma(\alpha^*) = (\sigma\alpha;\top⁇)^*$$

$$\sigma(\phi \to \psi) = \Box(\sigma\phi \to \sigma\psi) \qquad \sigma([\alpha]\psi) = \Box[\sigma\alpha]\sigma\psi$$

Theorem

$$\phi \in PDL \iff \sigma\phi \in (PDL \otimes S5)^{⁇}$$

Embedding of *PDL* into $(PDL \otimes S5)^{??}$

$$
\begin{aligned}
\sigma \boldsymbol{p} &= \Box \boldsymbol{p} & \sigma\bot &= \bot \\
\sigma \boldsymbol{a} &= \boldsymbol{a} & \sigma(\psi?) &= (\sigma\psi)?? \\
\sigma(\alpha \cup \beta) &= \sigma\alpha \cup \sigma\beta & \sigma(\alpha;\beta) &= \sigma\alpha;\top??;\sigma\beta \\
\sigma(\alpha^*) &= (\sigma\alpha;\top??)^* & & \\
\sigma(\phi \to \psi) &= \Box(\sigma\phi \to \sigma\psi) & \sigma([\alpha]\psi) &= \Box[\sigma\alpha]\sigma\psi
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Theorem

$$\phi \in PDL \iff \sigma\phi \in (PDL \otimes S5)^{??}$$

Summary

- A class of logics relevant to agent theory is considered.
- A behaviour of the logics with respect to weak and full substitution rule is studied.
- A semantics and axiomatisation for a new informational test operator is proposed.
- The effective finite model property, completeness and decidability is proved for a number of the logics with either classical or informational test operator.

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Why Abstract Actions?

- It is natural to distinguish abstract and concrete actions in many real applications. For instance, ‘process’ and ‘process with user permissions’.

Example

Abstract action: eat

Concrete actions: $\text{eat}_{\text{Michael}}$ and $\text{eat}_{\text{Jerry}}$

I.e. ‘Michael eats’ and ‘Jerry eats’ are particular instances of ‘to eat’.

- It is easy to extend the language of the logic. For example, operators of ‘pipeline’ | and ‘grouping’ + can be introduced on the set of agents. Let α be an abstract action.

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  Let α be an abstract action.

$$\alpha_{i+j} = \alpha_i \cup \alpha_j \qquad \alpha_{i|j} = \begin{cases} \beta_i;\gamma_j, & \alpha = \beta;\gamma \\ \alpha_i, & \text{otherwise} \end{cases}$$

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$$\alpha_{i+j} = \alpha_i \cup \alpha_j \qquad \alpha_{i|j} = \begin{cases} \beta_i;\gamma_j, & \alpha = \beta;\gamma \\ \alpha_i, & \text{otherwise} \end{cases}$$

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Language of *BDL*

Agents i,j

Abstract actions $\alpha,\beta \stackrel{\text{def}}{=} a \mid \phi?? \mid \alpha^* \mid \alpha\cup\beta \mid \alpha;\beta$

Concrete actions $\gamma,\delta \stackrel{\text{def}}{=} \alpha_i \mid \gamma^* \mid \gamma\cup\delta \mid \gamma;\delta$

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Examples

Let be two agents p — programmer and d — program designer:

$$\mathbf{B}_p[\text{develop\_model}_d]\text{model\_is\_consistent}\wedge$$
$$[\text{develop\_model}_d;\text{implement\_model}_p]\neg\mathbf{B}_p\text{model\_is\_consistent}$$

Let John do the following sequence α of actions to make Mary happy:

$$\alpha = (\neg\text{Mary\_is\_happy})?;(\langle\text{kiss\_Mary}^*_{\text{John}}\rangle\text{Mary\_is\_happy})?;\text{kiss\_Mary}$$

It is possible for John to make Mary happy:

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Semantics of *BDL*

Standard Kripke style semantics:
Model $M = \langle S, Q, \{R_i\}_{i \in \mathrm{Ag}}, \models \rangle$

- S is set of states,
- $Q(\alpha)$ and R_i are binary relations on S for any concrete action α and agent i,
- R_i is a transitive and Euclidean.
- $\models$ is a truth relation,
- semantics for ?:

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Properties of Test Operators
(B. van Linder, W. van der Hoek, J.-J.Ch. Meyer)

- An abstract action α is *informative* with respect to a formula ϕ in a logic L, if the formula $[\alpha_i](\mathbf{B}_i\phi \vee \mathbf{B}_i\neg\phi)$ belongs to L.
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- An abstract action α *preserves beliefs* in logic L, if the formula $\mathbf{B}_i\phi \to [\alpha_i]\mathbf{B}_i\phi$ belongs to L for any formula ϕ.

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The action $\phi? \cup \neg\phi?$ is informative and truthful with respect to ϕ and preserves beliefs.

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Expressiveness of the Language

Let $I = \{i_0, \ldots, i_m\}$ be a finite set of agents.

'Everyone in I believes that...' operator $\mathbf{E}_I$:

$$\mathbf{E}_I p \leftrightarrow [(\top ?)_{i_0} \cup \cdots \cup (\top ?)_{i_m}]p$$

Common belief operator $\mathbf{C}_I$ (relative to I):

$$\mathbf{C}_I p \leftrightarrow [((\top ?)_{i_0} \cup \cdots \cup (\top ?)_{i_m})^*]\mathbf{E}_I p$$

BDL is *more expressive* than the fusion of infinite copies (for each agent) of the fusion of *PDL* and *S5*

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Substitution rule

- Informal restrictions on the substitutions are:

If a formula says about an agent then, after substitution of action, it must still say about the same agent. (Similarly for actions.)

- Problem: Substitutions in extra interaction axiom

$$[a_i]\mathbf{B}_i p \leftrightarrow \mathbf{B}_i[a_i]p$$

must be limited. E.g. the instance

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Two Forms of Substitution

Propositional style substitution for agent variables, propositional variables, *abstract* action variables:

$$([a_i]\mathbf{B}_i p \to \mathbf{B}_i [a_i] p)\{(b;c)/a\} = [(b;c)_i]\mathbf{B}_i p \to \mathbf{B}_i [(b;c)_i] p$$

Substitution for concrete actions:

$$([(a_i)^*]p \to [a_i][(a_i)^*]p)\{(b_j;c_k)/a_j\} = [(b_i;c_k)^*]p \to [b_i;c_k][(b_i;c_k)^*]p$$

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Axiomatisation of *BDL*

1. Axioms of classical propositional logic
2. *PDL*-like axioms for test-free actions:
   1. $[a_i](p \rightarrow q) \rightarrow ([a_i]p \rightarrow [a_i]q)$
   2. $[a_i \cup b_j]p \leftrightarrow [a_i]p \wedge [b_j]p$
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3. *K45* axioms for the belief operators:
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Properties of *BDL* and Its Extensions

Theorem (Completeness)

BDL is complete.

Theorem (The effective finite model property)

If ϕ is satisfiable in some BDL-model then ϕ is satisfiable in a finite model with no more than $2^n \cdot (2^{2^n})^m$ states, where

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All extensions of BDL by the axioms

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Summary

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- A new informational test operator is proposed.
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