Max-min and min-max approximation problems for normal matrices revisited

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joint work with

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$Ax = b$, $A \in \mathbb{C}^{n \times n}$ is nonsingular, $b \in \mathbb{C}^n$,

$x_0 = 0$ and $\|b\| = 1$ for simplicity.

GMRES computes $x_k \in \mathcal{K}_k(A, b)$ such that $r_k \equiv b - Ax_k$ satisfies

$$\|r_k\| = \min_{p \in \pi_k} \|p(A)b\| \quad \text{(GMRES)}$$

$$\leq \max_{\|b\| = 1} \min_{p \in \pi_k} \|p(A)b\| \quad \text{(worst-case GMRES)}$$

$$\leq \min_{p \in \pi_k} \|p(A)\| \quad \text{(ideal GMRES)}$$

where $\pi_k = \text{degree } \leq k$ polynomials with $p(0) = 1$. 
Two bounds on the GMRES residual norm

$$\max_{\|b\| = 1} \min_{p \in \pi_k} \|p(A)b\| \leq \min_{p \in \pi_k} \|p(A)\|$$

- They are equal if $A$ is normal.
  
  [Greenbaum, Gurvits '94; Joubert '94].

- The inequality can be strict if $A$ is non-normal.

  [Toh '97; Faber, Joubert, Knill, Manteuffel '96].
How to prove the equality for normal matrices?

If \( A \) is normal, then

\[
\max_{\|b\| = 1} \min_{p \in \pi_k} \| p(A)b \| = \min_{p \in \pi_k} \| p(A) \|.
\]

- [Joubert '94] Proof using analytic methods of optimization theory, for real or complex data, only in the GMRES context.

- [Greenbaum, Gurvits '94]: Proof based mostly on matrix theory, only for real data but in a more general form.

- Is there a straightforward proof that uses, e.g., known classical results of approximation theory?
Outline

1. Normal matrices and classical approximation problems
2. Best polynomial approximation for $f$ on $\Gamma$
3. Proof
4. Connection to results by Greenbaum and Gurvits
A is normal iff $A = Q\Lambda Q^*$, $Q^*Q = I$.

$\Gamma \equiv \{\lambda_1, \ldots, \lambda_n\}$ is the set of eigenvalues of $A$.

For any function $g$ defined on $\Gamma$ denote

$$\|g\|_\Gamma \equiv \max_{z \in \Gamma} |g(z)|. $$

$p \in \pi_k$ means

$$p(z) = 1 - \sum_{i=1}^{k} \alpha_i z^i.$$ 

Then

$$\min_{p \in \pi_k} \|p(A)\| = \min_{p \in \pi_k} \|QP(\Lambda)Q^*\| = \min_{p \in \pi_k} \max_{\lambda_i} |p(\lambda_i)|$$

$$= \min_{\alpha_1, \ldots, \alpha_k} \|1 - \sum_{i=1}^{k} \alpha_i z^i\|_\Gamma.$$
Instead of 1 we consider a general function $f$ defined on $\Gamma$. Instead of $\{z^i\}_{i=1}^k$ we consider general basis functions $\varphi_i$.

We ask whether

$$\max_{\|b\|=1} \min_{p \in \mathcal{P}_k} \|f(A)b - p(A)b\| = \min_{p \in \mathcal{P}_k} \|f(A) - p(A)\|$$

where $A$ is normal and $p$ is of the form

$$p(z) = \sum_{i=1}^k \alpha_i \varphi_i(z) \in \mathcal{P}_k.$$

A comment on $\mathbb{R}$ versus $\mathbb{C}$ → coefficients $\alpha_i$.

As in the previous

$$\min_{p \in \mathcal{P}_k} \|f(A) - p(A)\| = \min_{p \in \mathcal{P}_k} \|f(z) - p(z)\|_{\Gamma}.$$
A polynomial of best approximation for $f$ on $\Gamma$

Definition and notation

$p_\ast \in \mathcal{P}_k$ is a **polynomial of best approximation** for $f$ on $\Gamma$ when

$$\|f - p_\ast\|_\Gamma = \min_{p \in \mathcal{P}_k} \|f - p\|_\Gamma.$$

For $p \in \mathcal{P}_k$, define

$$\Gamma(p) \equiv \{ z \in \Gamma : |f(z) - p(z)| = \|f - p\|_\Gamma \}.$$
Characterization of best approximation for \( f \) on \( \Gamma \)

[Chebyshev, Berstein, de la Vallée Poussin, Haar, Remez, Zuhovickiĭ, Kolmogorov]
[Rivlin, Shapiro '61], [Lorentz '86]

**Characterization theorem (complex case)**

\( p_\ast \in P_k \) is a **polynomial of best approximation** for \( f \) on \( \Gamma \) if and only if

there exist \( \ell \) points \( \mu_i \in \Gamma(p_\ast) \) where \( 1 \leq \ell \leq 2k + 1 \), and \( \ell \) real numbers \( \omega_1, \ldots, \omega_\ell > 0 \) with \( \omega_1 + \cdots + \omega_\ell = 1 \), such that

\[
\sum_{j=1}^{\ell} \omega_j \frac{p(\mu_j)}{p_\ast(\mu_j)} \left[ f(\mu_j) - p_\ast(\mu_j) \right] = 0, \quad \forall \ p \in P_k.
\]

Denote

\[
\delta \equiv \| f - p_\ast \|_\Gamma = | f(\mu_j) - p_\ast(\mu_j) |, \quad j = 1, \ldots, \ell.
\]
Proof I

It suffices to prove that

$$\max \min \frac{||f(A)b - p(A)b||}{||b||=1} p \in \mathcal{P}_k \geq \min \frac{||f(A) - p(A)||}{p \in \mathcal{P}_k} = \min \frac{||f(z) - p(z)||_\Gamma}{p \in \mathcal{P}_k}.$$

From the previous,

$$0 = \sum_{j=1}^\ell \omega_j p(\mu_j) [f(\mu_j) - p_*(\mu_j)] \quad \forall \ p \in \mathcal{P}_k.$$

Let $\lambda_j$ be numbered such that $\lambda_j = \mu_j$, $j = 1, \ldots, \ell$. Define

$$\xi \equiv [\sqrt{\omega_1}, \ldots, \sqrt{\omega_\ell}, 0, \ldots, 0]^T \quad \text{and} \quad w = Q^T \xi.$$

Then

$$\sum_{j=1}^\ell \omega_j p(\mu_j) [f(\mu_j) - p_*(\mu_j)] = \xi^H p(\Lambda)^H [f(\Lambda) - p_*(\Lambda)] \xi = w^H p(A)^H [f(A) - p_*(A)] w.$$
In other words,

\[ f(A)w - p_*(A)w \perp p(A)w, \quad \forall p \in \mathcal{P}_k, \]

or, equivalently,

\[ \| f(A)w - p_*(A)w \| = \min_{p \in \mathcal{P}_k} \| f(A)w - p(A)w \|. \]

Moreover

\[ \| f(A)w - p_*(A)w \|^2 = \| \left[ f(\Lambda) - p_*(\Lambda) \right] \xi \|^2 \]

\[ = \sum_{j=1}^{\ell} \xi_j^2 |f(\mu_j) - p_*(\mu_j)|^2 \]

\[ = \sum_{j=1}^{\ell} \omega_j \delta^2 = \delta^2 \]

\[ = \| f(A) - p_*(A) \|^2. \]
In summary, for $p_* \in \mathcal{P}_k$ we have constructed $w \in \mathbb{C}^n$ such that

$$\min_{p \in \mathcal{P}_k} \| f(A) - p(A) \| = \| f(A) - p_*(A) \|$$

$$= \| f(A)w - p_*(A)w \|^2$$

$$= \min_{p \in \mathcal{P}_k} \| f(A)w - p(A)w \|$$

$$\leq \max \min_{\|b\|=1} \| f(A)b - p(A)b \| .$$

The proof for complex $A$ is finished.
A note on the real case

- Assume that $A$, $f(A)$ and $\varphi_i(A)$ are real. We look for a polynomial of a best approximation with real coefficients.

- **Technical problem:** $A$ can have complex eigenvalues but we look for a real vector $b$ that maximizes

  \[
  \min_{p \in P_k} \| f(A)b - p(A)b \|.
  \]

- $\Gamma$ is a set of points that appear in complex conjugate pairs.

- This symmetry with respect to the real axes has been used to find a real $b$ and to prove the equality [Liesen, T. 2013].
Theorem [Greenbaum, Gurvits ’94]

Let $A_0, A_1, \ldots, A_k$ be normal matrices that commute. Then

$$\max_{\|v\|=1} \min_{\alpha_1, \ldots, \alpha_k} \|A_0v - \sum_{i=1}^{k} \alpha_i A_i v\| = \min_{\alpha_1, \ldots, \alpha_k} \|A_0 - \sum_{i=1}^{k} \alpha_i A_i\|.$$ 

Theorem [Theorem 2.5.5, Horn, Johnson ’90]

Commuting normal matrices can be simultaneously unitarily diagonalized, i.e., there exists a unitary $U$ so that

$$U^H A_i U = \Lambda_i, \quad i = 0, 1, \ldots, k.$$
Using the theorem by Horn and Johnson we can equivalently rewrite the problem

\[
\min_{\alpha_1, \ldots, \alpha_k} \| A_0 - \sum_{i=1}^{k} \alpha_i A_i \|
\]

in our notation

\[
\min_{\alpha_1, \ldots, \alpha_k} \| f(A) - \sum_{i=1}^{k} \alpha_i \varphi_i(A) \|
\]

where \( A \) is any diagonal matrix with distinct eigenvalues and \( f \) and \( \varphi_i \) are any functions satisfying

\[
f(A) = \Lambda_0, \quad \varphi_i(A) = \Lambda_i, \quad i = 1, \ldots, k.
\]
Inspired by the convergence analysis of GMRES, **we formulated** two general approximation problems involving normal matrices.

We used a **direct link** between

- approximation problems involving normal matrices,
- classical approximation problems

and **proved** that

$$\max_{\|b\|=1} \min_{p \in \mathcal{P}_k} \| f(A)b - p(A)b \| = \min_{p \in \mathcal{P}_k} \| f(A) - p(A) \|. $$

**Our results**

- represent a **generalization** of results by [Joubert '94],
- offer **another point of view** to [Greenbaum, Gurvits '94].
Related papers


Thank you for your attention!