Graphon branching processes and fractional isomorphism

arXiv:2408.02528

Jan Hladký, Anna Margarethe Limbach

Institute of Computer Science, Czech Academy of Sciences

Eng Keat Hng

Institute for Basic Science in Daejeon, South Korea

Slides+this video: https://www.cs.cas.cz/~hladky/papers

The talk in 3 minutes

Two models: (a) Erdős–Rényi random graphs and inhomogeneous versions thereof, (b) uniform spanning tree.

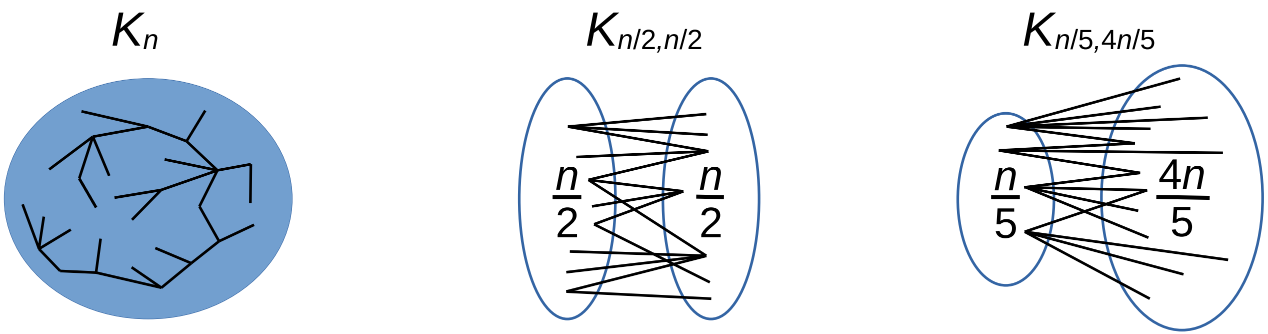
Using the language of dense graph limits, we characterize when two dense graph give a similar distribution of these random subgraphs.

The talk in 2 minutes

Dens
subg
unifo

Using
characteriz … e dense graph give a
similar … ribution of these random subgraphs.

Example: Uniform spanning tree (UST)



	UST(K_n)	≈	UST($K_{n/2,n/2}$)	≠	UST($K_{n/5,4n/5}$)
#leaves:	$0.367n$		$0.367n$		$0.626n$

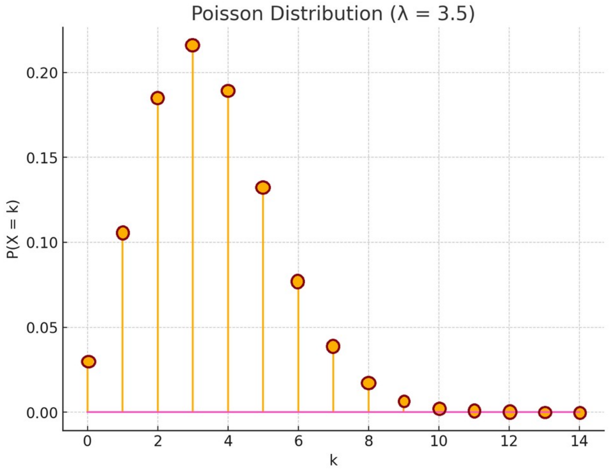
Program

- Probability recap (Poisson distribution, Galton-Watson branching processes)
- Erdős–Rényi random graphs and component structure
- Bollobás-Janson-Riordan inhomogeneous random graphs and component structure
- Main Theorem I
- Graph and graphon fractional isomorphism
- Uniform spanning tree
- Main Theorem II

Poisson distribution

Definition: $\lambda \geq 0$. Poisson(λ) is $\mathbb{N}_0$-valued distribution

$$\mathbb{P}[X = k] = \frac{\exp(-\lambda)\lambda^k}{k!}$$

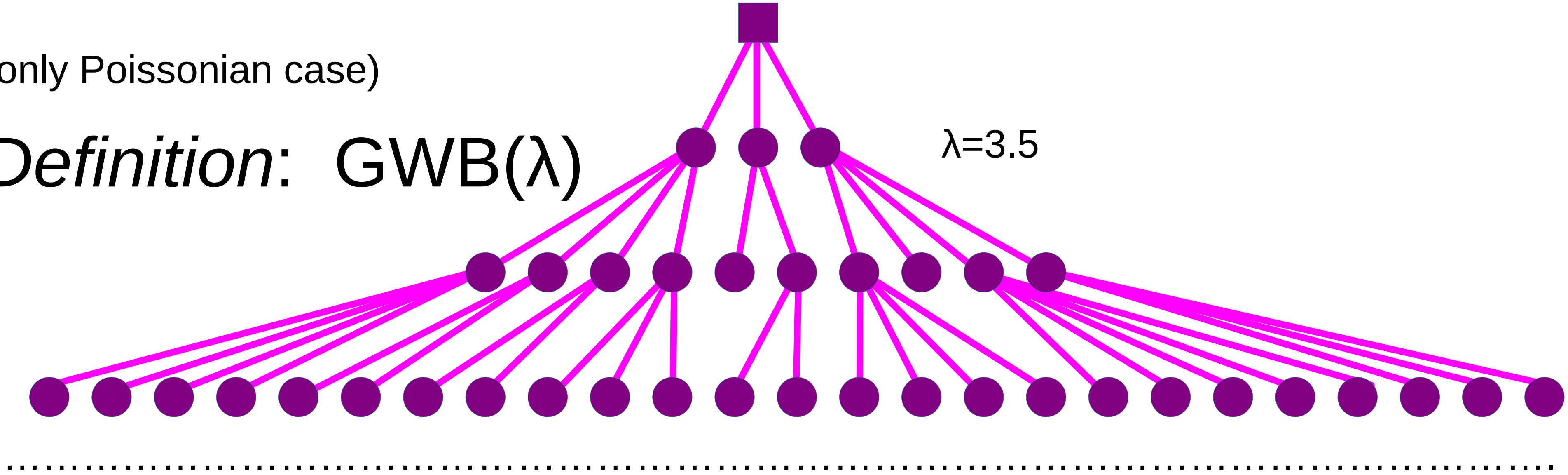


- *Key property:* Poisson(λ) is limit of Binomial(n,λ/n)

Galton-Watson branching process

(only Poissonian case)

Definition: GWB(λ)



- *Key property:* For λ≤1, GWB(λ) survives with prob=0

  For λ>1, GWB(λ) survives with prob s(λ)∈(0,1)

Erdős–Rényi random graphs

G(n,p) n fixed vertices {1,2,…,n}
each pair forms an edge with prob p

p = 0.3, 0.7, …: Easy; G(n,p) is connected aas, ...

p = 1/√n, log n/n, **const/*n***: Harder

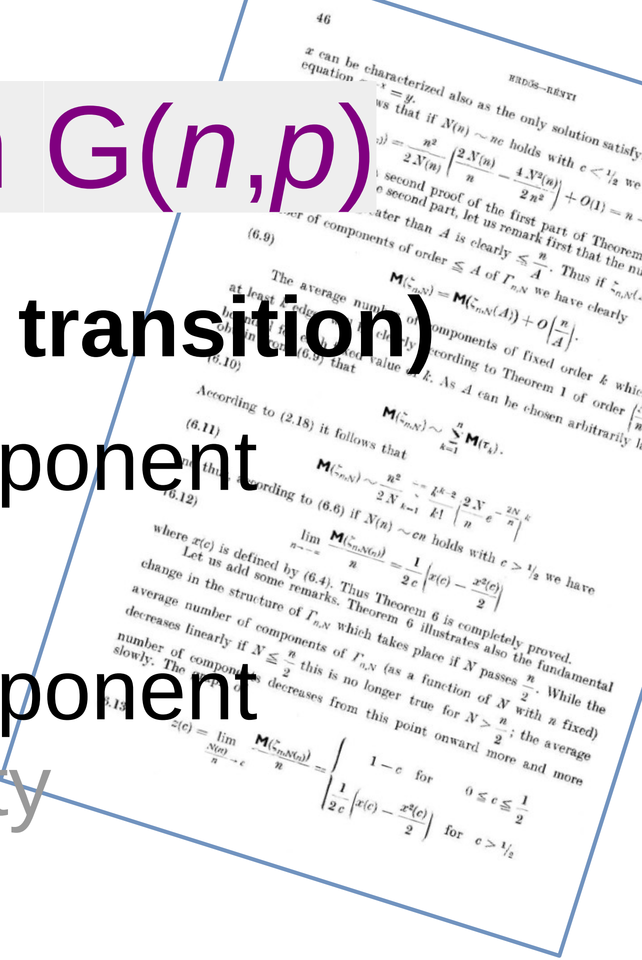
Erdős–Rényi 1959: Phase transition in G(n,p)

Component structure in G(n,p)

Erdős–Rényi 1959 ("the" phase transition)

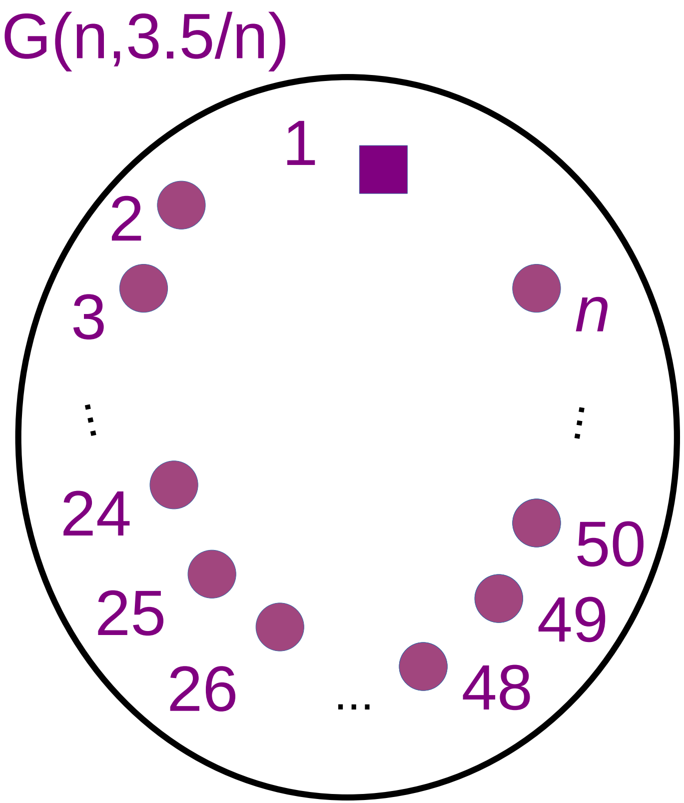
For c<1, G(n,c/n) has largest component of order Θ(log n). subcriticality

For c>1, G(n,c/n) has largest component of order (s(c)±o(1))n. supecriticality



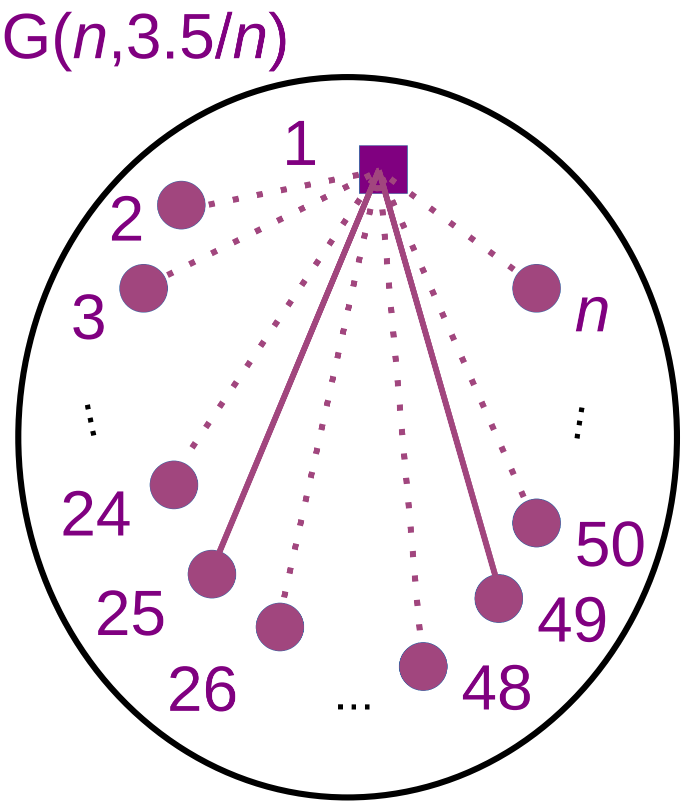
Early proofs enumerative. Karp 1990: Lets use GW branching processes.

Component structure in G(n,3.5/n)

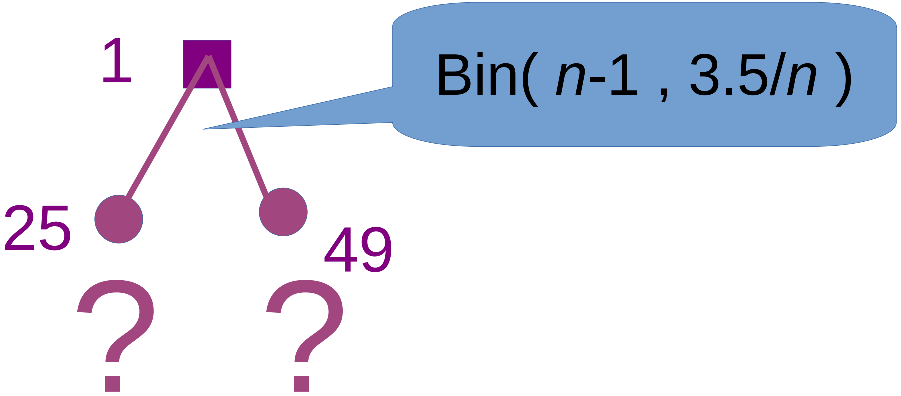


What is the neighborhood structure of vertex 1?

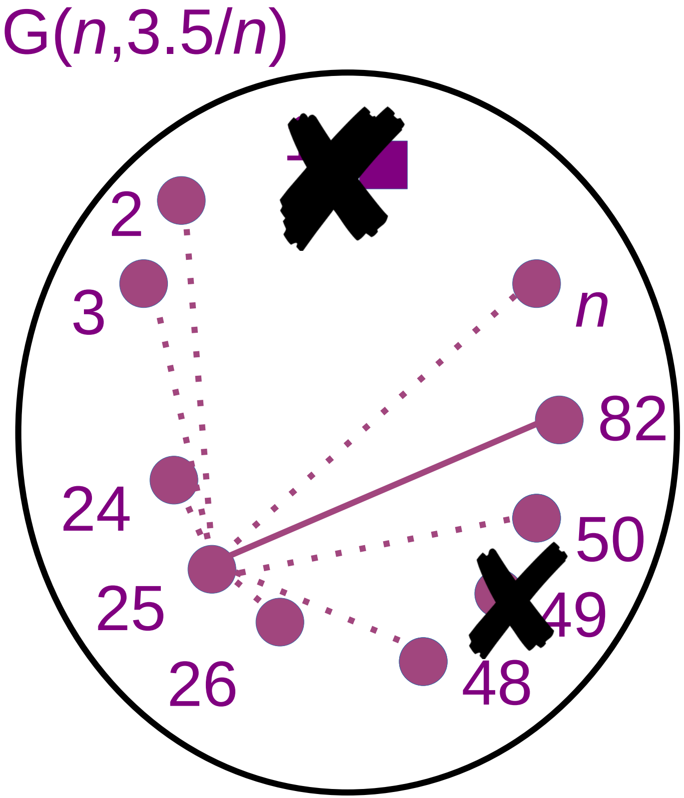
Component structure in G(n,3.5/n)



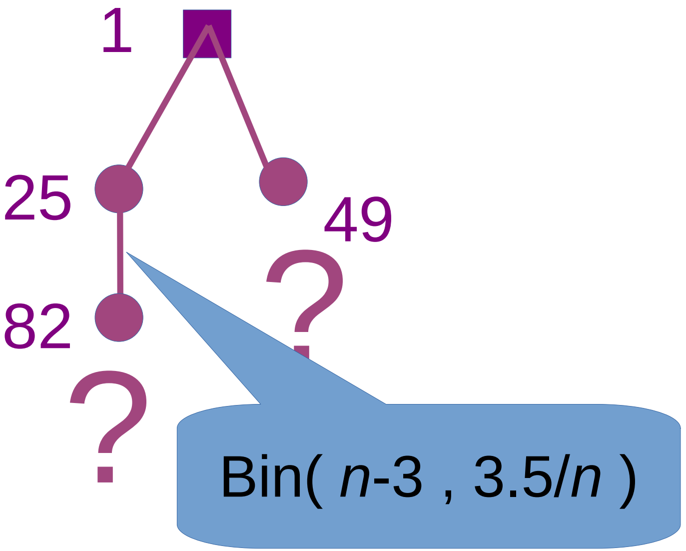
What is the neighborhood structure of vertex 1?



Component structure in G(n,3.5/n)

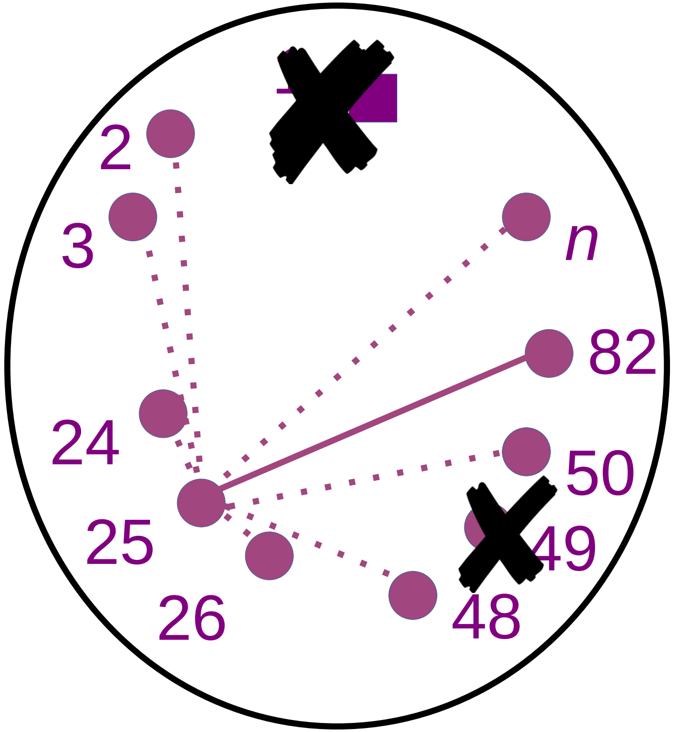


What is the neighborhood structure of vertex 1?

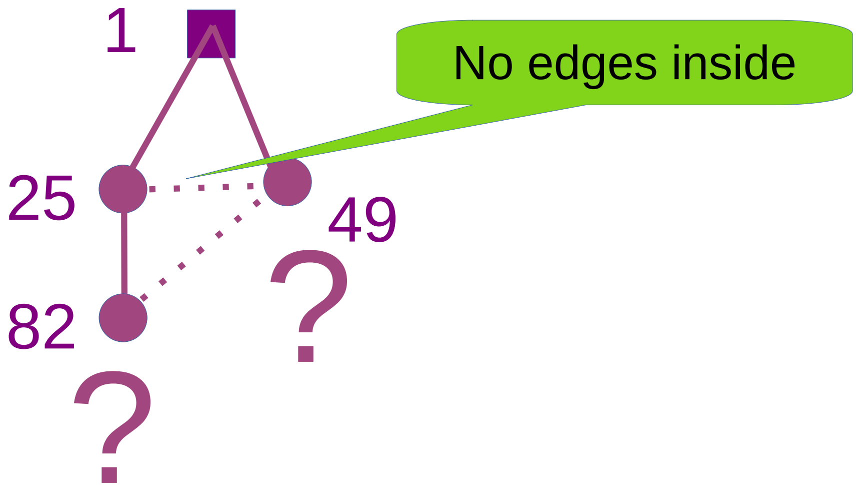


Component structure in G(n,3.5/n)

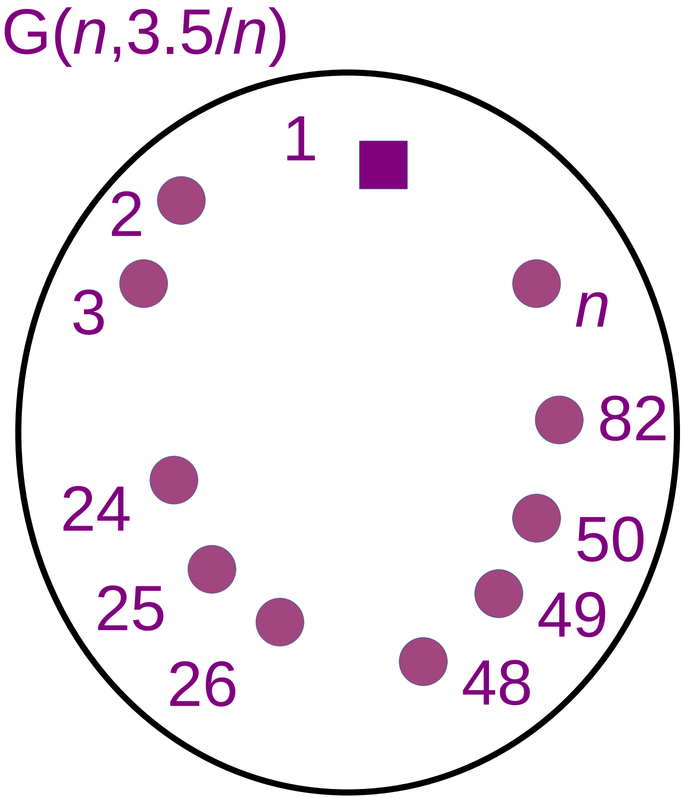
G(n,3.5/n)



What is the neighborhood structure of vertex 1?



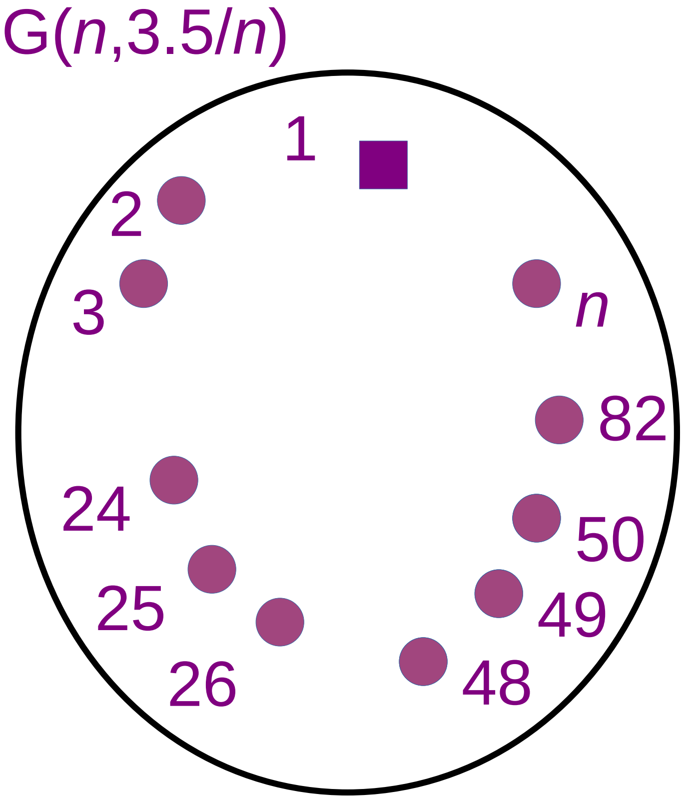
Component structure in G(n,3.5/n)



What is the neighborhood structure of vertex 1?

What is the size of the giant component?

Component structure in G(n,3.5/n)

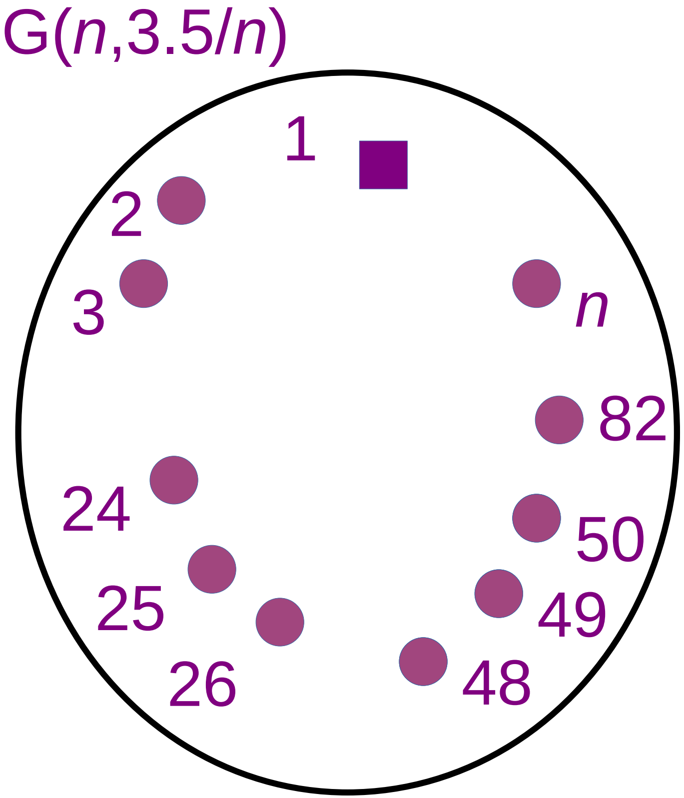


What is the neighborhood structure of vertex 1?

What is the size of the giant component?

= Number of vertices in a giant component

Component structure in G(n,3.5/n)



What is the neighborhood structure of vertex 1?

What is the size of the giant component?

= Number of vertices in a giant component

= n × survival probability of GW(3.5)

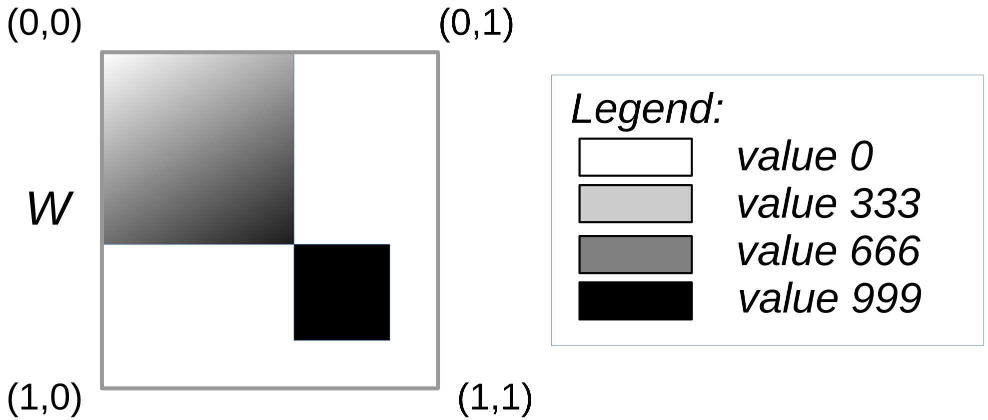
= (s(c)±o(1))n

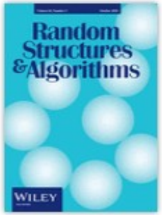
Inhomogeneous random graphs

Bollobás-Janson-Riordan 2005

subsequent work by B., Borgs., Chayes, J., R.

- Main idea G(n,3.5/n) ⇝ G(n,W/n), where W:$[0,1]^2$➜ [0,999] is a “symmetric kernel”.





Random Structures & Algorithms: Volume 31, Issue 1

Pages: 1-122

August 2007

‹ Previous Issue | Next Issue ›

Export Citation(s)

Ubiquitous percolation

Pages: 1-2 | First Published: 17 April 2007

PDF | Request permissions

The phase transition in inhomogeneous random graphs

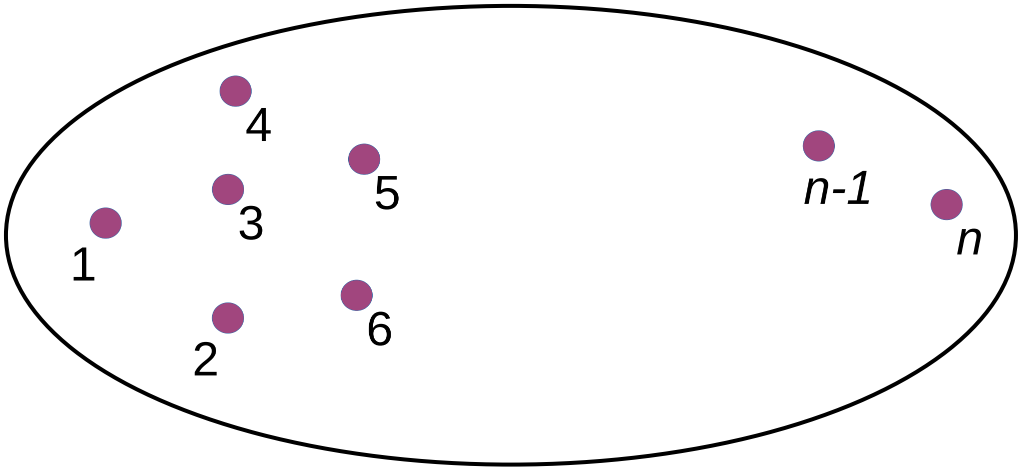
Béla Bollobás, Svante Janson, Oliver Riordan

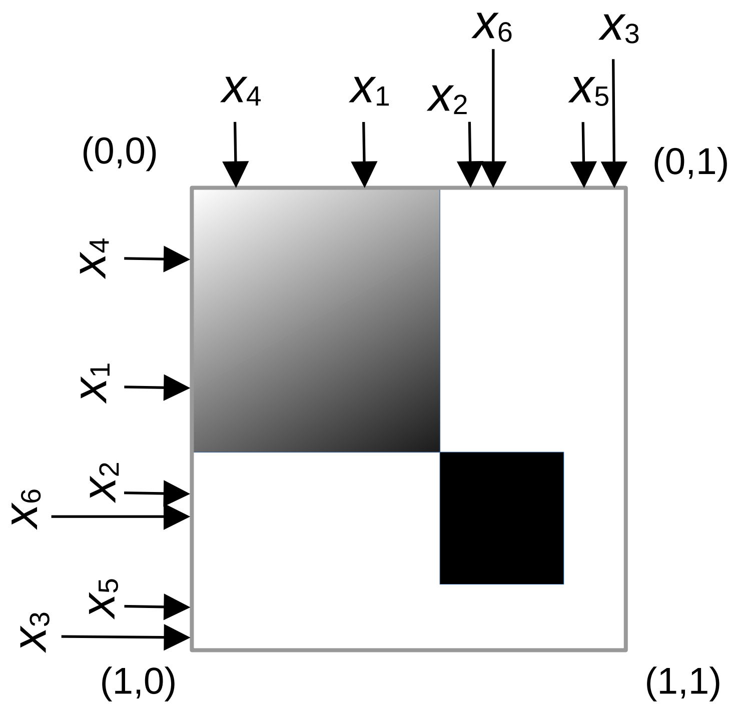
Pages: 3-122 | First Published: 01 June 2007

Abstract | PDF | References | Request permissions

Generating G(n,W/n)

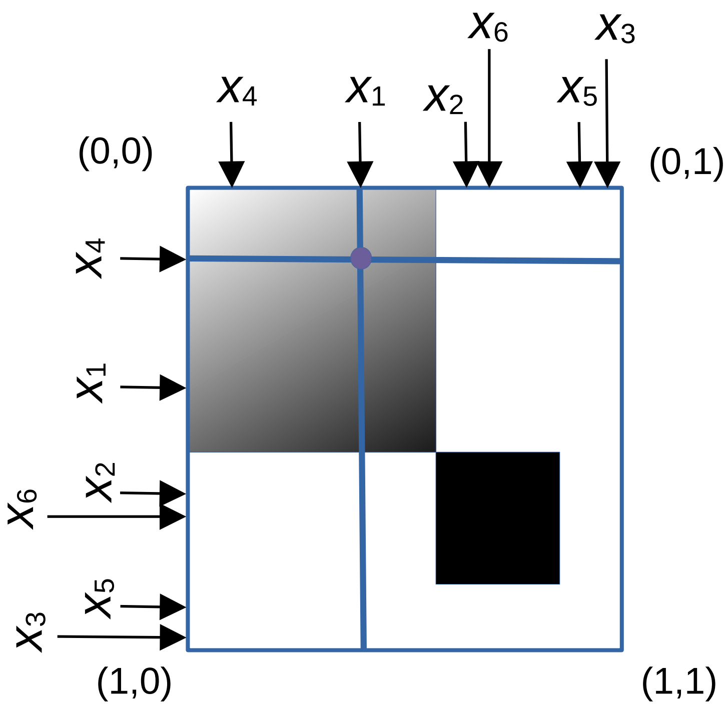
- Vertex set {1,2,…,n}
- Generate x_1, x_2, …, $x_n \in [0,1]$ at random





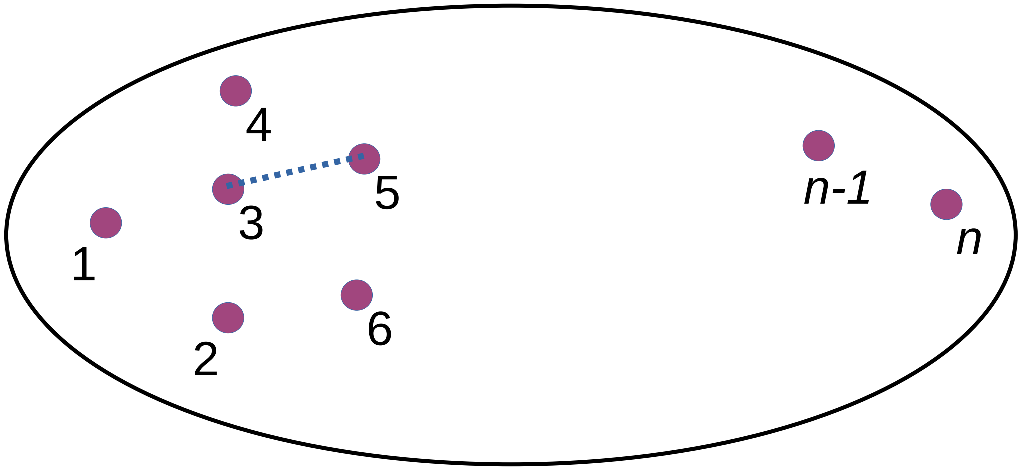
Generating G(n,W/n)

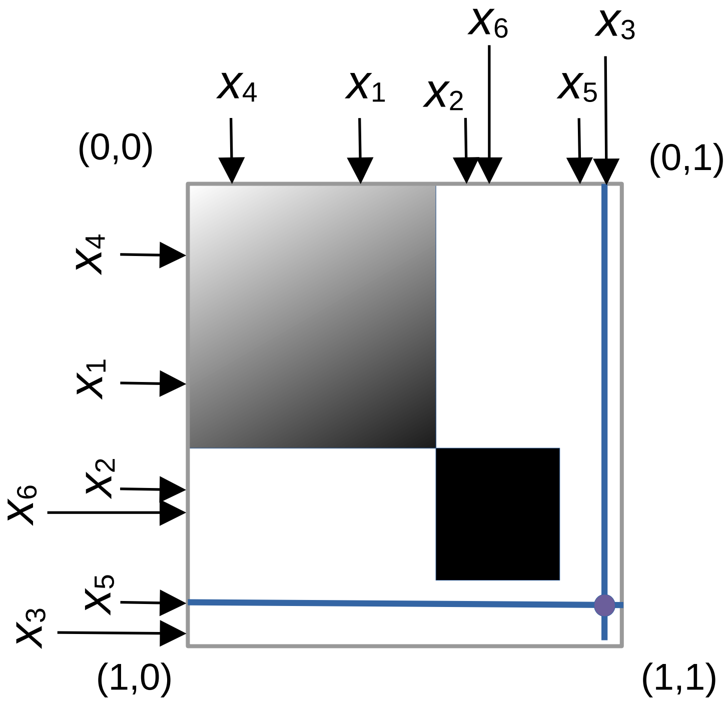
- Vertex set {1,2,…,n}
- Generate $x_1, x_2, \ldots, x_n \in [0,1]$ at random
- Make { i,j } an edge with probability $W(x_i,x_j)/n$



Generating G(n,W/n)

- Vertex set {1,2,…,n}
- Generate x_1, x_2, …, $x_n \in [0,1]$ at random
- Make { i,j } an edge with probability $W(x_i,x_j)/n$





Component structure in G(n,W/n)

Bollobás-Janson-Riordan 2005

Neighborhood structure of vertex 1?

- In G(n,3.5/n) … **GWB(3.5)**
- In G(n,W / n) … **GWB(W)**

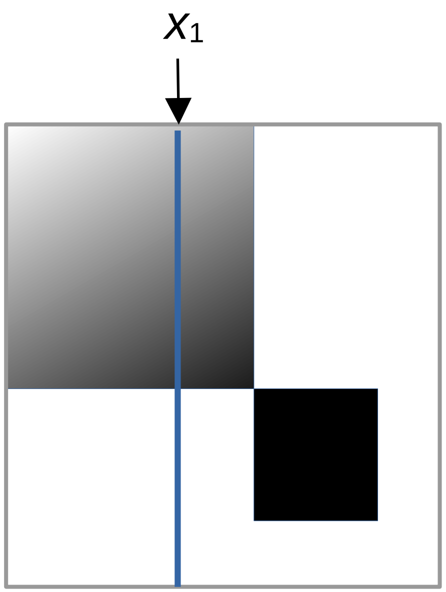
Component structure in G(n,W/n)

Bollobás-Janson-Riordan 2005

Neighborhood structure of vertex 1?

GWB(W) ■ **type=0.42** **(at random in [0,1])**

?



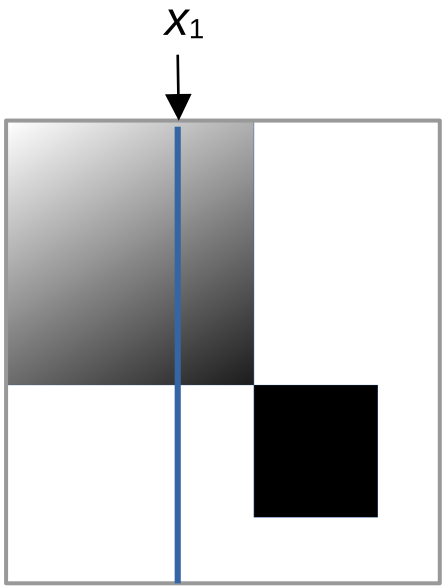
Component structure in G(n,W/n)

Bollobás-Janson-Riordan 2005

Neighborhood structure of vertex 1?

GWB(W) ■ **type=0.42 (at random in [0,1])**

?



$$deg(x_1)=\int W(x_1,y)\,dy$$

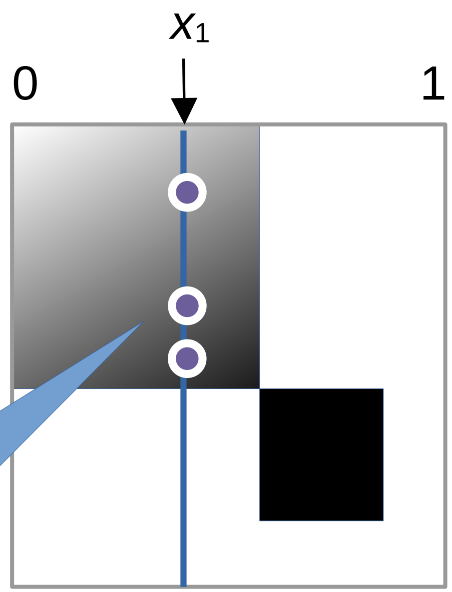
Component structure in G(n,W/n)

Bollobás-Janson-Riordan 2005

Neighborhood structure of vertex 1?

GWB(W) ■ **type=0.42 (at random in [0,1])**

?



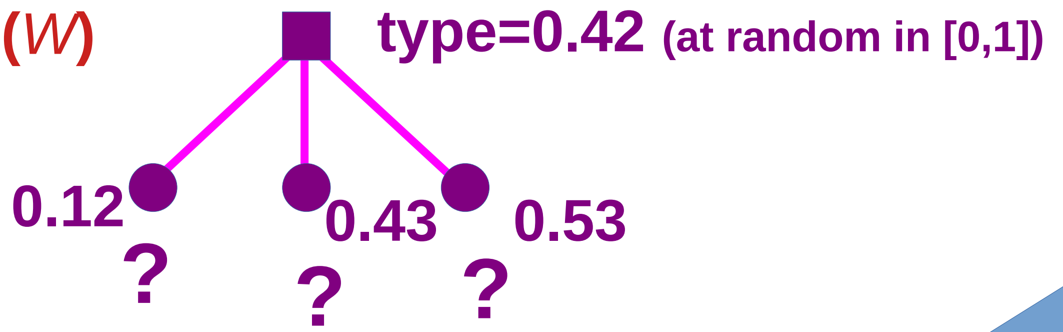
Poisson point process
with intensity $W(x_1,\circ)$

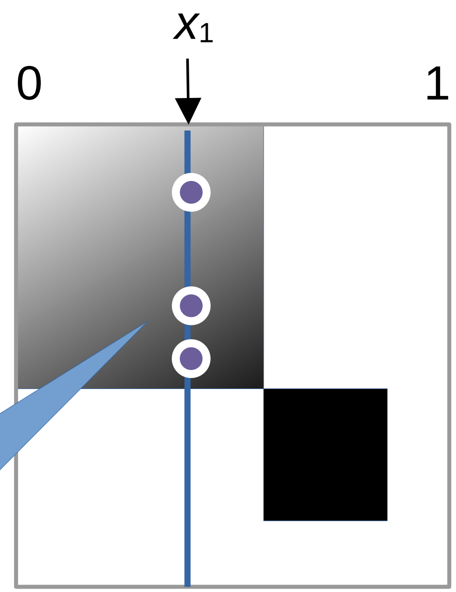
Component structure in G(n,W/n)

Bollobás-Janson-Riordan 2005

Neighborhood structure of vertex 1?

GWB(W)





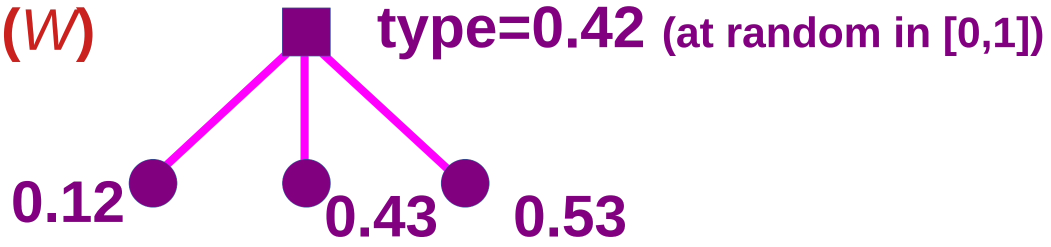
Poisson point process
with intensity $W(x_1,\circ)$

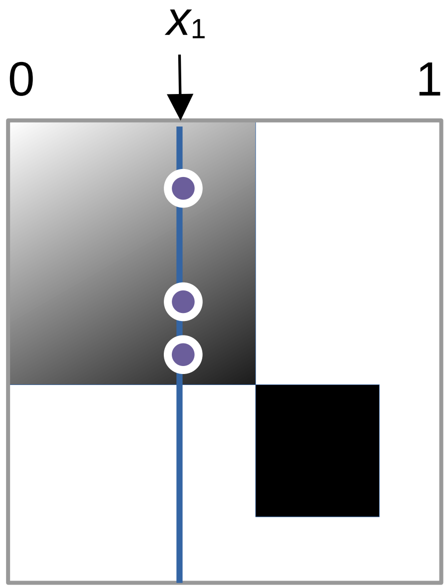
Component structure in G(n,W/n)

Bollobás-Janson-Riordan 2005

Neighborhood structure of vertex 1?

GWB(W)



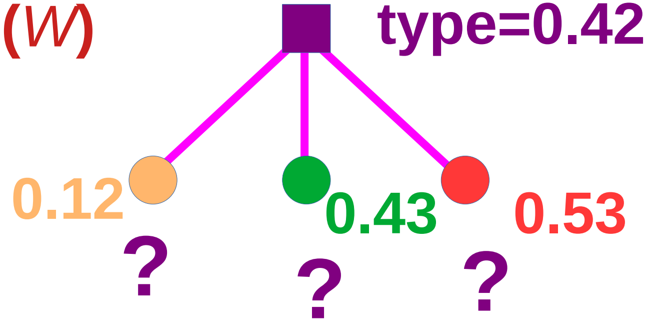


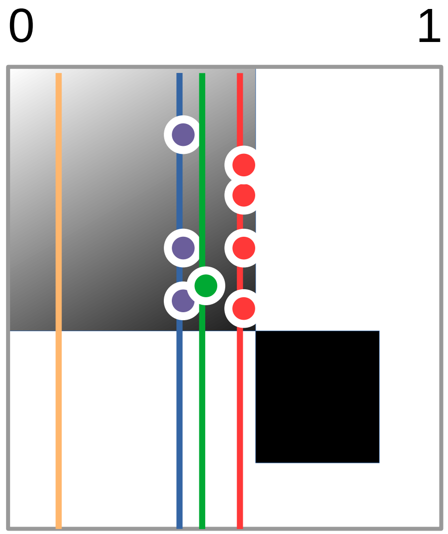
Component structure in $G(n,W/n)$

Bollobás-Janson-Riordan 2005

Neighborhood structure of vertex 1?

GWB(W) **type=0.42 (at random in [0,1])**



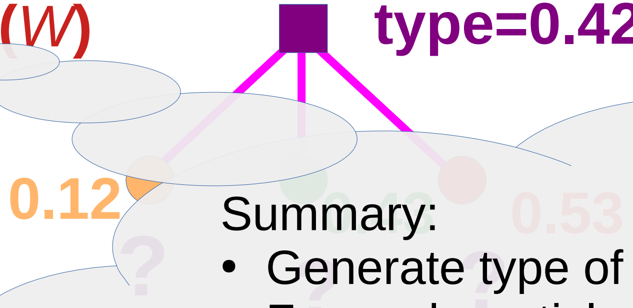


Component structure in G(n,W/n)

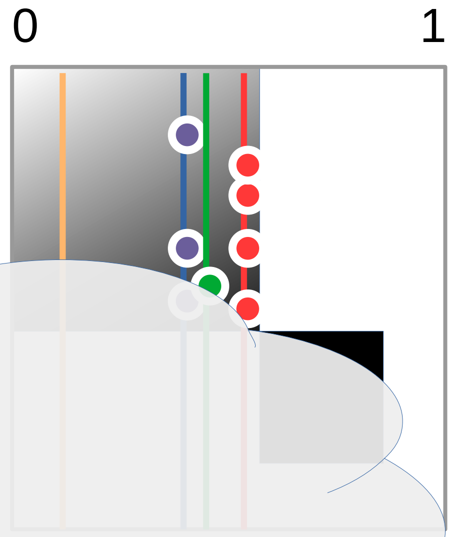
Bollobás-Janson-Riordan 2005

Neighborhood structure of vertex 1?

GWB(W)



type=0.42 **(at random in [0,1])**



Summary:

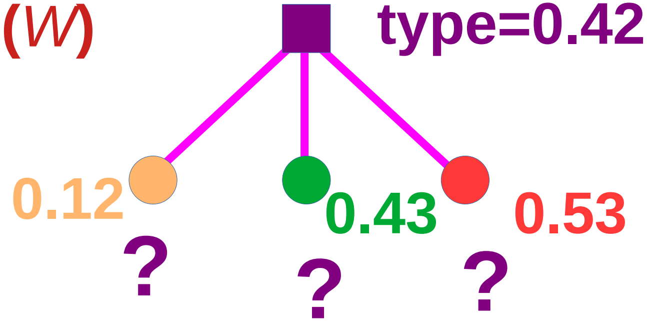
- Generate type of root in [0,1] at random
- For each particle of given type $x\in[0,1]$,
  - Generate its children according to Poisson point process with intensity $W(x,\circ)$. (Distribution of the number is Poisson(deg(x)).

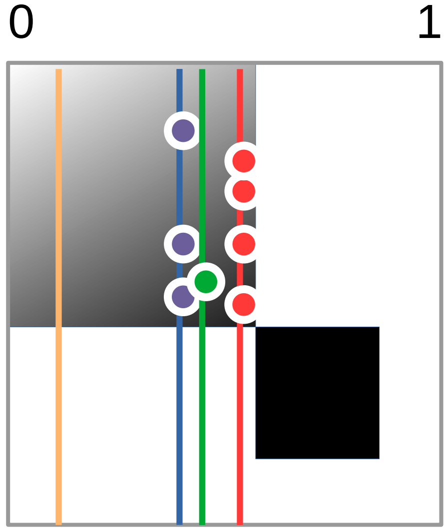
Component structure in G(*n*,*W*/*n*)

Bollobás-Janson-Riordan 2005

Neighborhood structure of vertex 1?

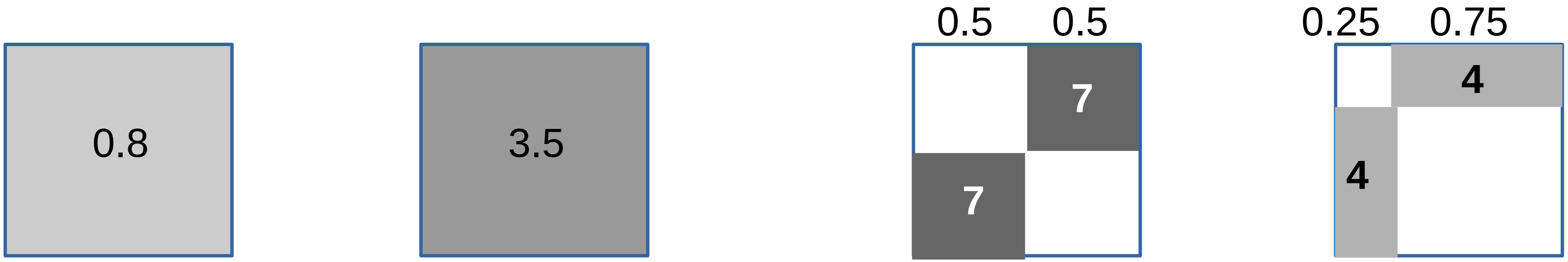
GWB(*W*) **type=0.42 (at random in [0,1])**





Importance: Let s(*W*) be the probability of survival of GWB(W).
Then the largest component in G(*n*, *W*/*n*) is of size (s(W)+o(1))n.

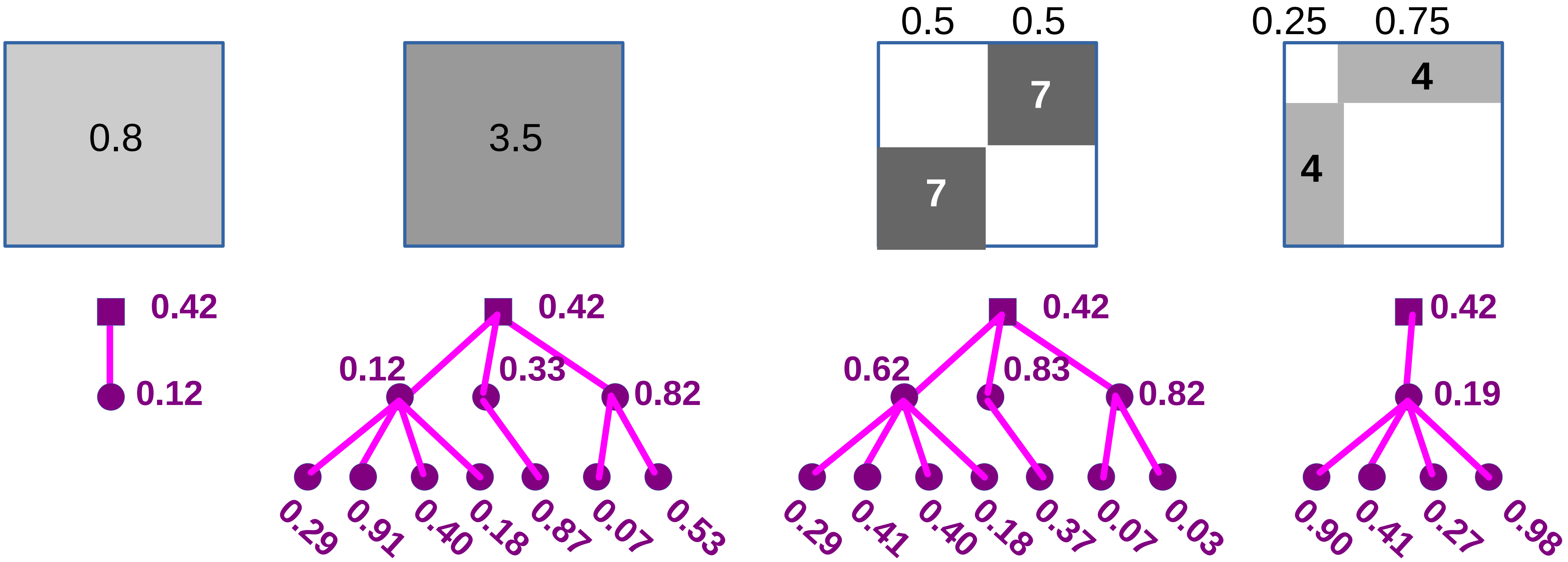
Examples



Definition:

- Generate type of root in [0,1] at random
- For each particle of given type $x \in [0,1]$:
  Generate its children according to Poisson point process with intensity $W(x,\circ)$.

Examples

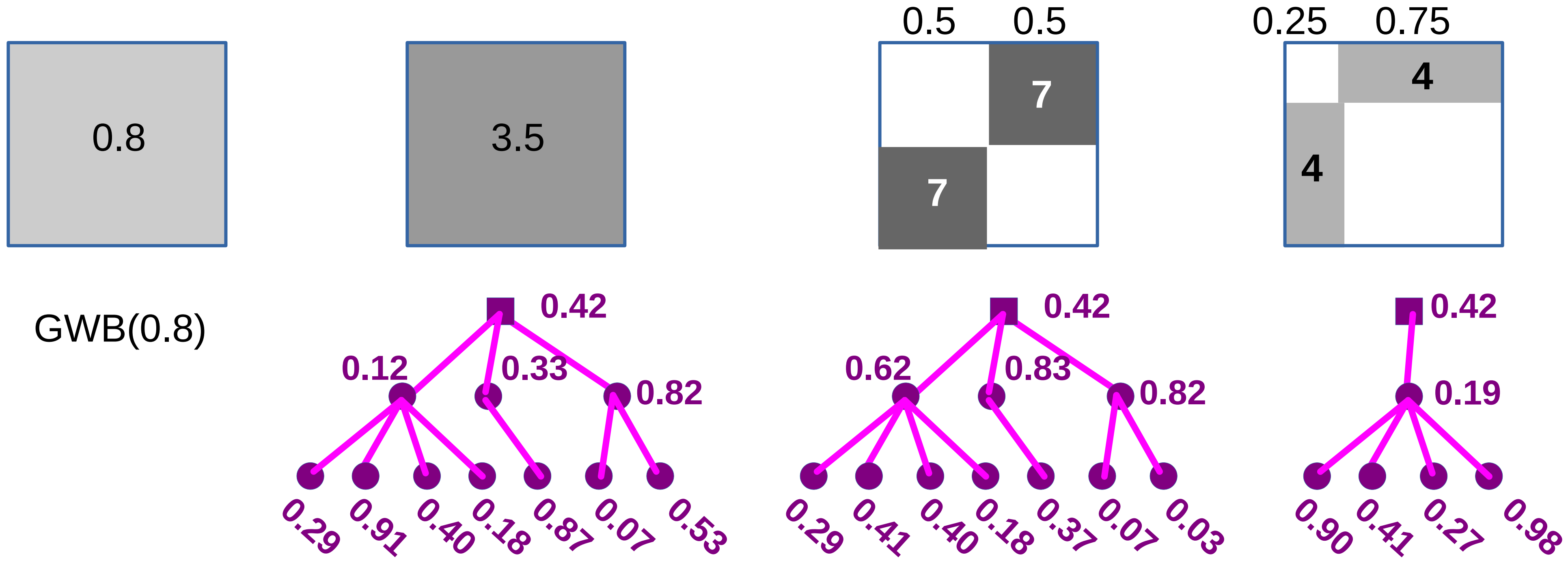


Definition:

- Generate type of root in [0,1] at random
- For each particle of given type $x \in [0,1]$:

  Generate its children according to Poisson point process with intensity $W(x,\circ)$.

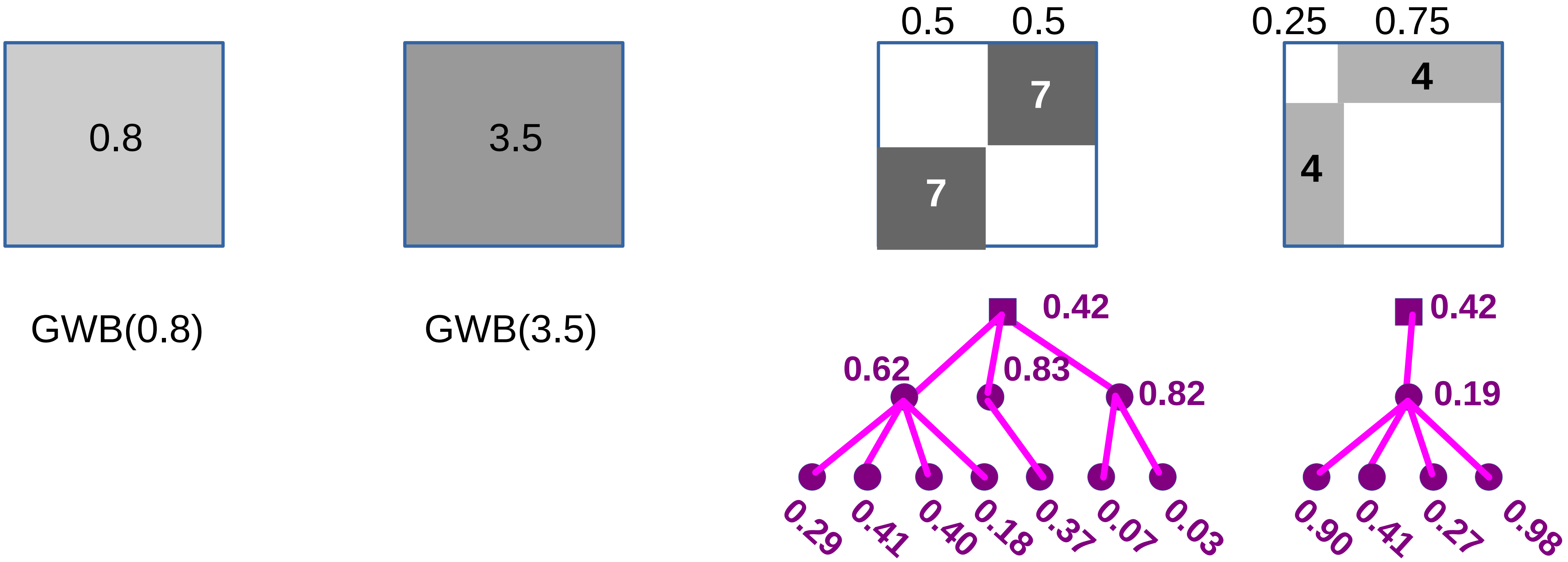
Examples



Definition:

- Generate type of root in [0,1] at random
- For each particle of given type $x \in [0,1]$:
  Generate its children according to Poisson point process with intensity $W(x,\circ)$.

Examples



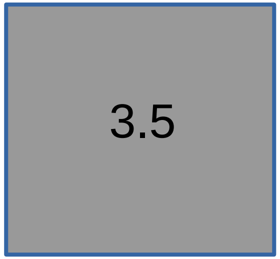
Definition:

- Generate type of root in [0,1] at random
- For each particle of given type $x \in [0,1]$:

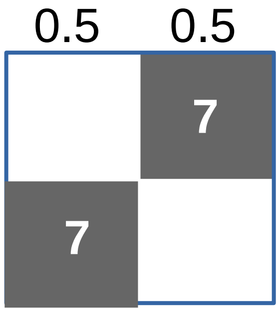
  Generate its children according to Poisson point process with intensity $W(x,\circ)$.

Examples

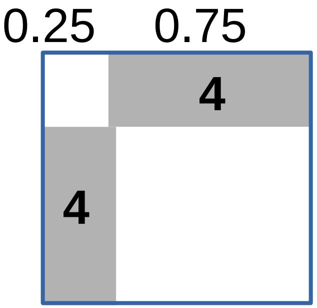
GWB(0.8)

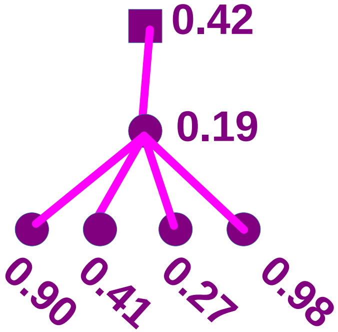


GWB(3.5)



GWB(3.5)



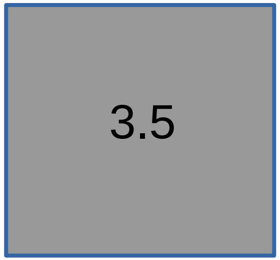


Definition:

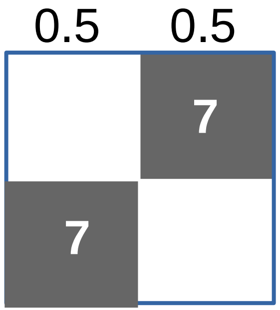
- Generate type of root in [0,1] at random
- For each particle of given type $x \in [0,1]$:
  Generate its children according to Poisson point process with intensity $W(x,\circ)$.

Examples

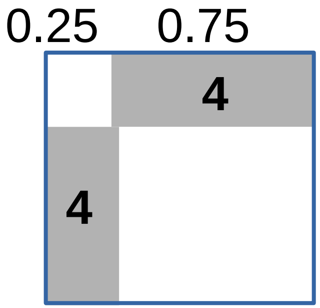
GWB(0.8)



GWB(3.5)



GWB(3.5)



0.25*GWB(e=3,o=1)
+0.75*GWB(o=3,e=1)

Definition:

- Generate type of root in [0,1] at random
- For each particle of given type $x \in [0,1]$:
  Generate its children according to Poisson point process with intensity $W(x,\circ)$.

Main theorem I

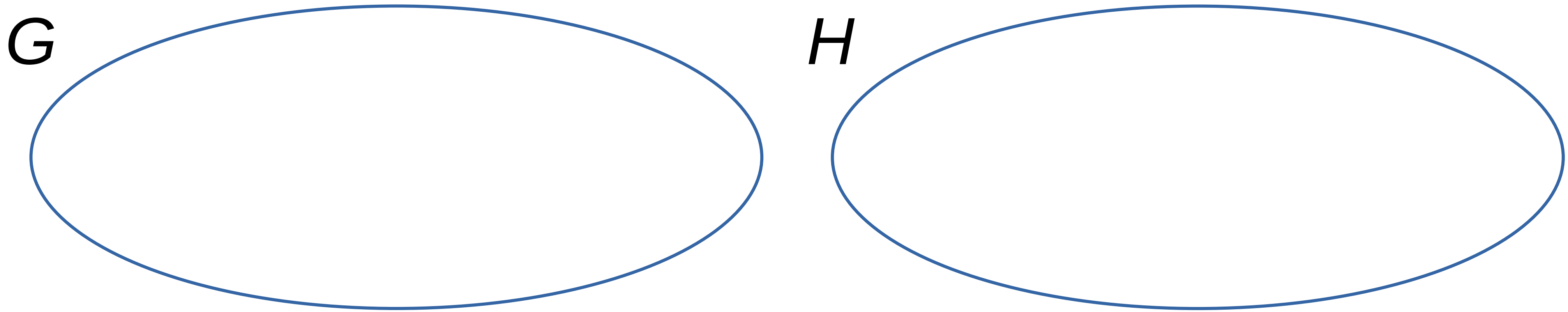
H., Hng, Limbach 2024
For two kernels U, W, the following are equivalent
- GWB(U) and GWB(W) have the same distribution,
- U and W are fractionally isomorphic.

Fractional isomorphism of **graphs**

Idea: Relaxation of isomorphism. If two graphs are isomorphic then they have the same degree sequence. Iterate.

Tinhofer 1986, Ramana-Scheinerman-Ullman 1994

Color refinement algorithm:

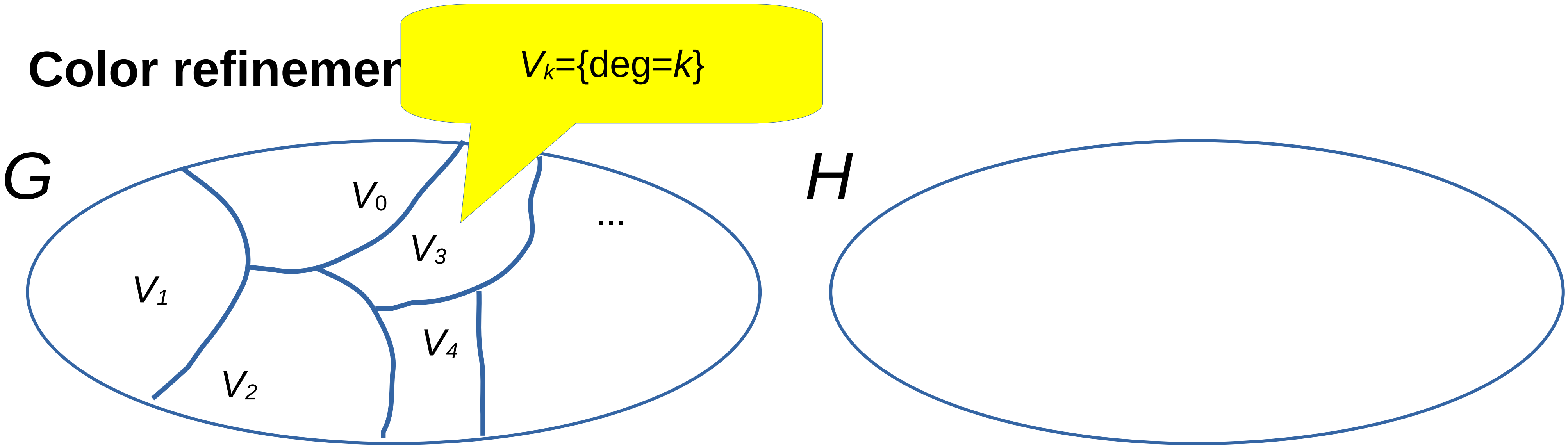


Fractional isomorphism of **graphs**

Idea: Relaxation of isomorphism. If two graphs are isomorphic then they have the same degree sequence. Iterate.

Tinhofer 1986, Ramana-Scheinerman-Ullman 1994

Color refinemen



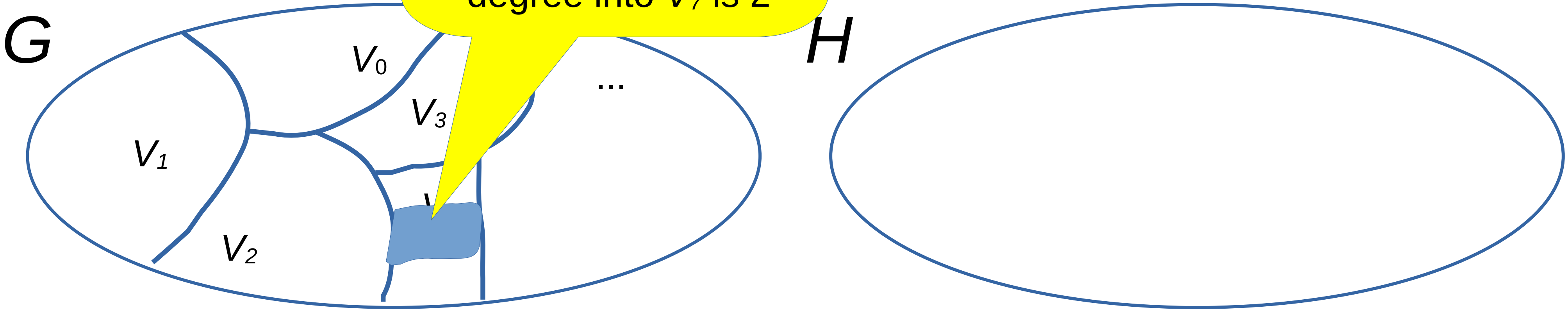
Fractional isomorphism of **graphs**

Idea: Relaxation of isomorphism. If two graphs are isomorphic then they have the same degree sequence. Iterate.

Tinhofer 1986, R ... **nan-Ullman 1994**

$V_{4;(V2:1,V4:1,V7:2)}$:
- degree is 4
- degree into V_2 is 1
- degree into V_4 is 1
- degree into V_7 is 2

Color refinemen



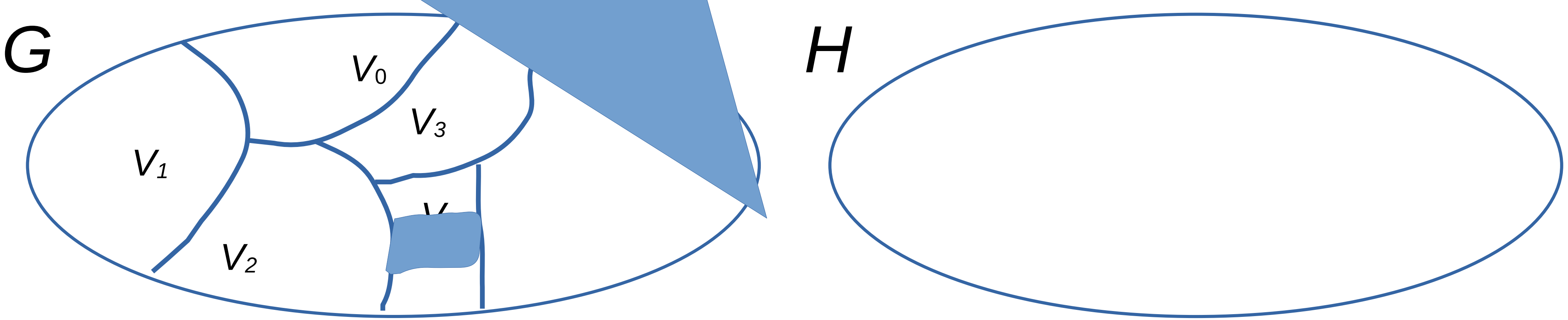
Fractional isomorphism of **graphs**

Idea: Relaxation of isomorphism. If two graphs are isomorphic then they h

Tinhofe

Color re

Definition:
***G* and *H* are fractionally isomorphic if all the sets of all possible “iterated degrees” are of the same cardinalities.**



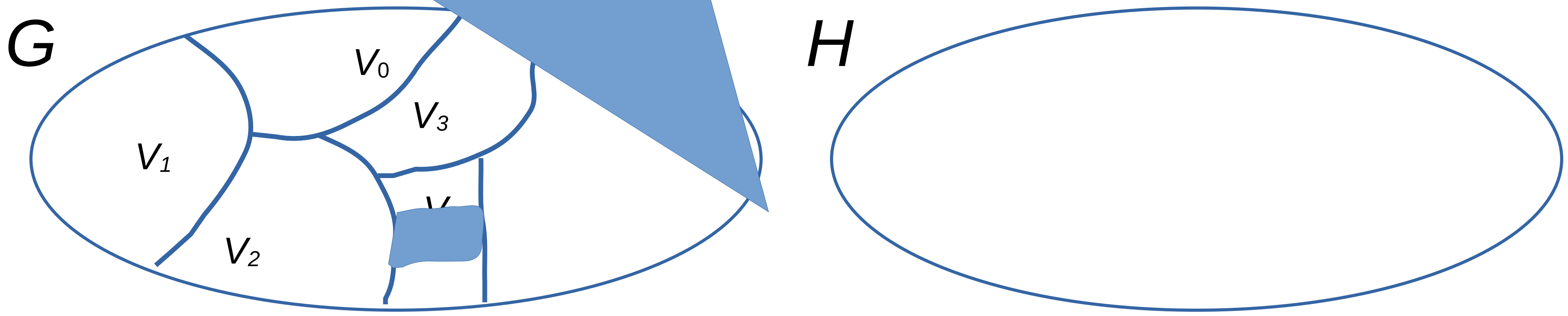
Fractional isomorphism of **graphs**

Idea: Relaxation of isomorphism. If two graphs are isomorphic then they h

Tinhofe

Color re

Definition (equivalent):
***G* and *H* are fractionally isomorphic if there exists a bijection between *V*(*G*) and *V*(*H*) preserving iterated degrees.**



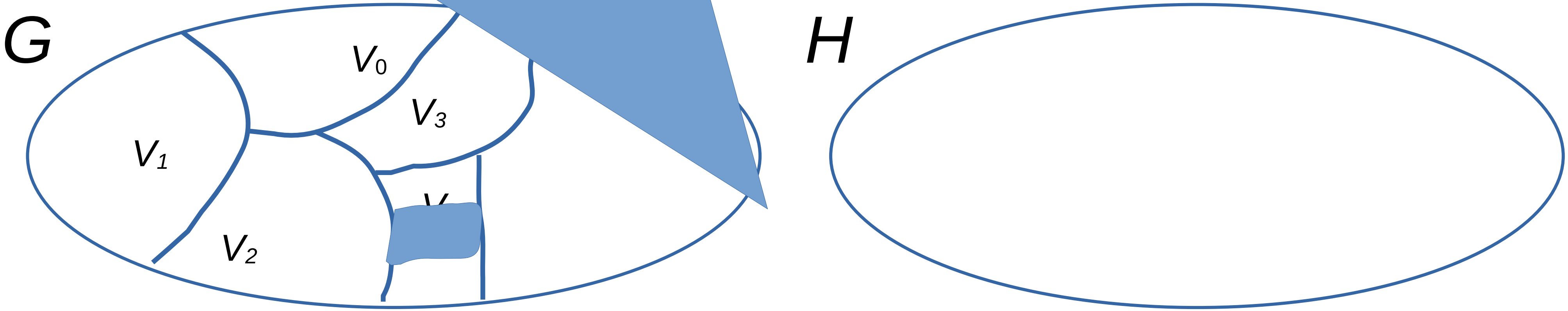
Fractional isomorphism of **graphs**

Idea: Relaxation of isomorphism. If two graphs are isomorphic then they h

Tinhofe

Color re

Definition (equivalent):
G and H are fractionally isomorphic if there exists a coupling between $V(G)$ and $V(H)$ preserving iterated degrees.



Fractional isomorphism

For graphs:

G and *H* are fractionally isomorphic, if:

- The color refinement algorithm gives the same output
- The number of copies of each tree is the same
- $A_H = P^{-1} A_G P$ for a bistochastic matrix P

Fractional isomorphism

For graphs:

G and H are fractionally isomorphic, if:

- The color refinement algorithm gives the same output
- The number of copies of each tree is the same
- $A_H = P^{-1} A_G P$ for a bistochastic matrix P

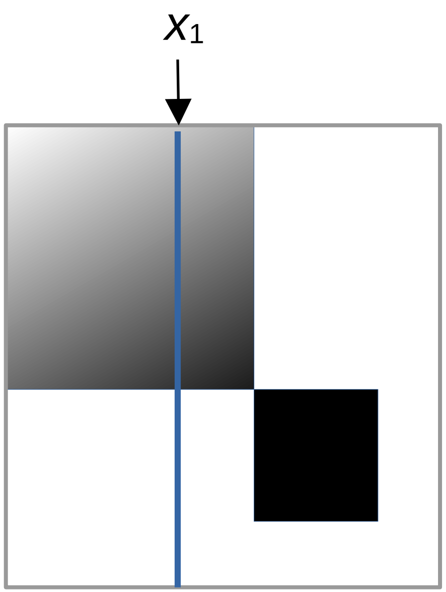
For symmetric kernels:

Grebik, Rocha 2022

“Fractional isomorphism of graphons”

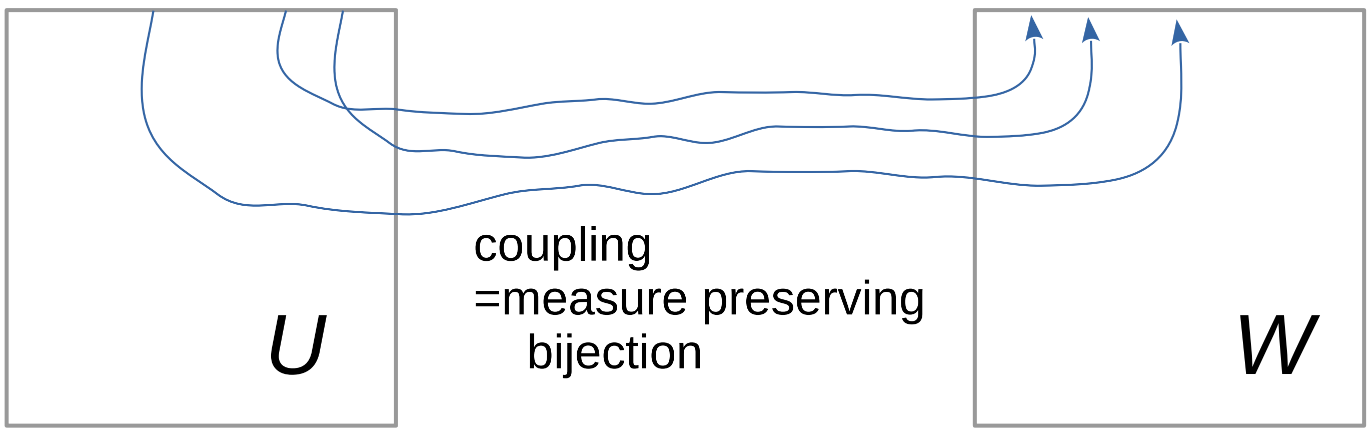
Counterparts to the above

Key: Degree has counterpart

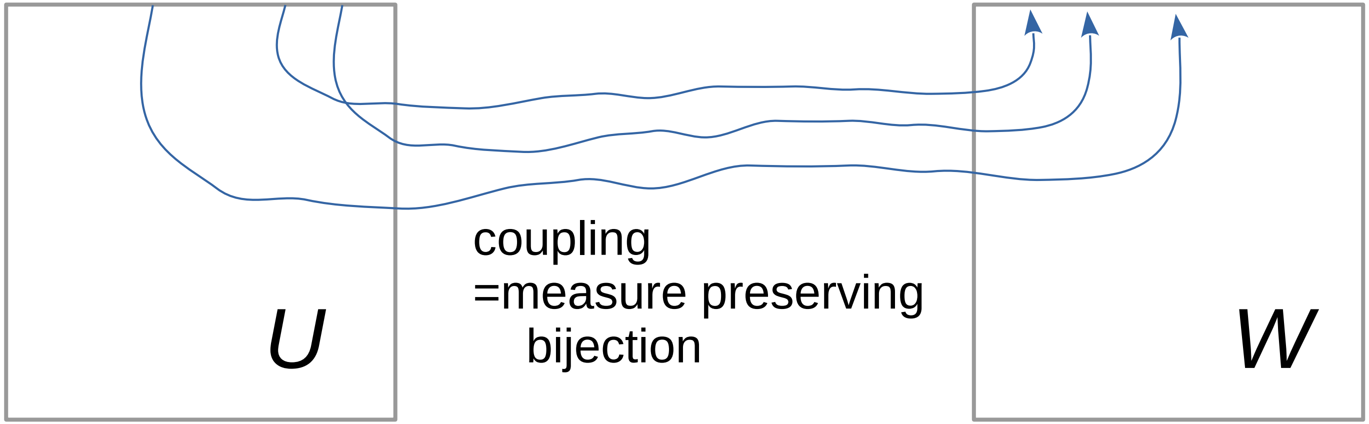


$$deg(x_1) = \int W(x_1, y)\, dy$$

Fractional isomorphism of ~~symm~~ kernels



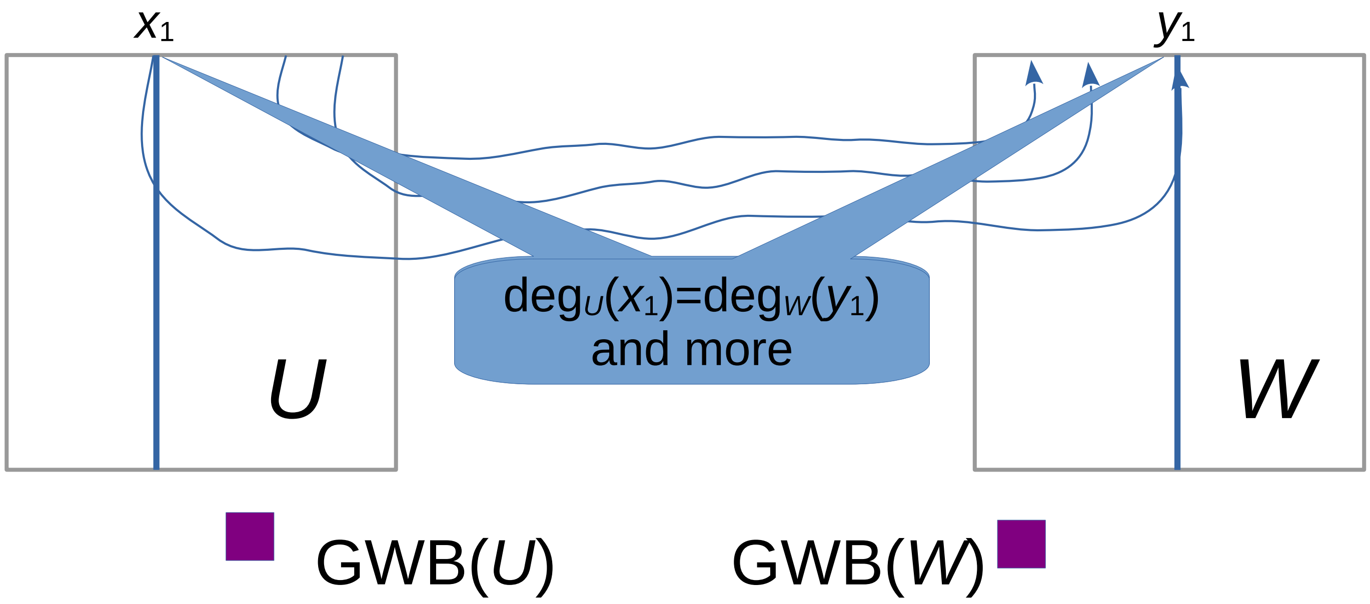
Fractional isomorphism of ~~symm~~ kernels



GWB(U) GWB(W)

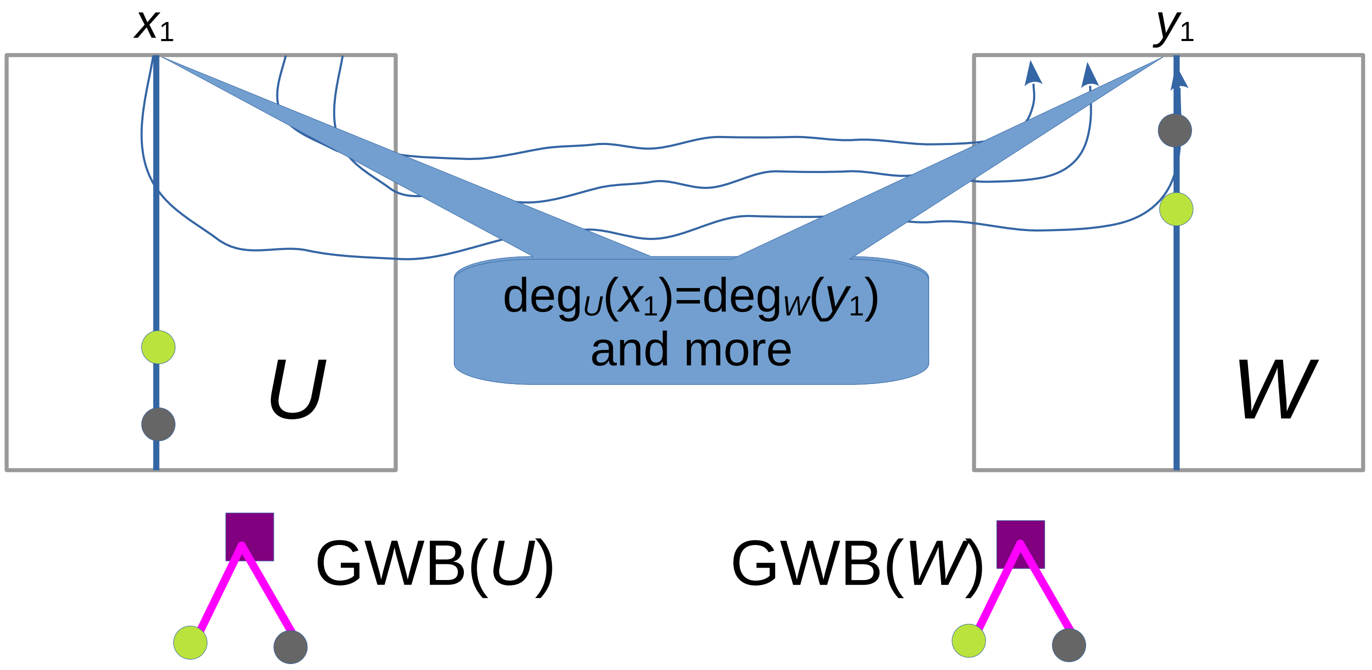
Main Theorem I: GWB(U) and GWB(W) have the same distribution
U and W are fractionally isomorphic

Fractional isomorphism of ~~symm~~ kernels



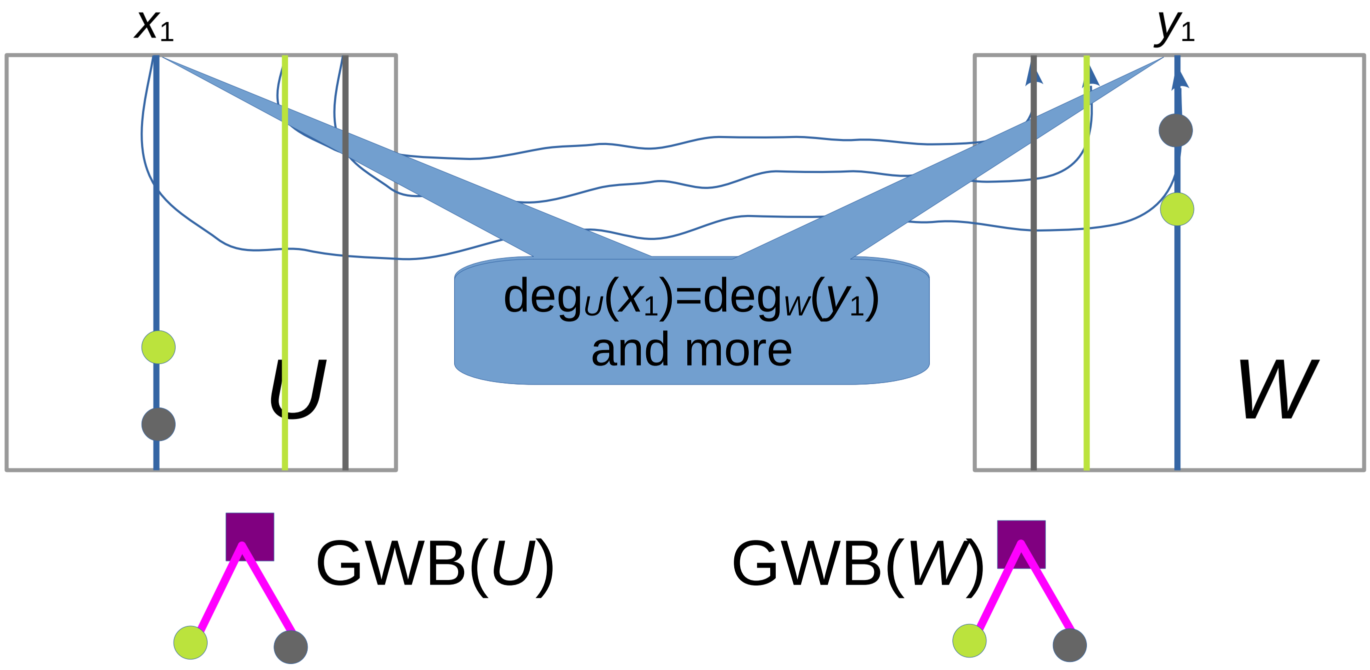
Main Theorem I: GWB(U) and GWB(W) have the same distribution
$\Updownarrow$ U and W are fractionally isomorphic

Fractional isomorphism of ~~symm~~ kernels



Main Theorem I: GWB(U) and GWB(W) have the same distribution ⇑ U and W are fractionally isomorphic

Fractional isomorphism of ~~symm~~ kernels



Main Theorem I: GWB(U) and GWB(W) have the same distribution

$\Uparrow$ U and W are fractionally isomorphic

Uniform spanning tree

Definition: G connected graph. $\mathcal{T}(G)$={all spanning trees of G}

UST(G)...uniform measure on $\mathcal{T}(G)$

Probability/statistical physics:

- electrical networks, Wilson's algorithms, Aldous-Broder algorithm
- 2D lattices, Schramm-Loewner evolution, scaling limit, SLE(8)

Uniform spanning tree

Definition: G connected graph. T(G)={all spanning trees of G}

UST(G)…uniform measure on T(G)

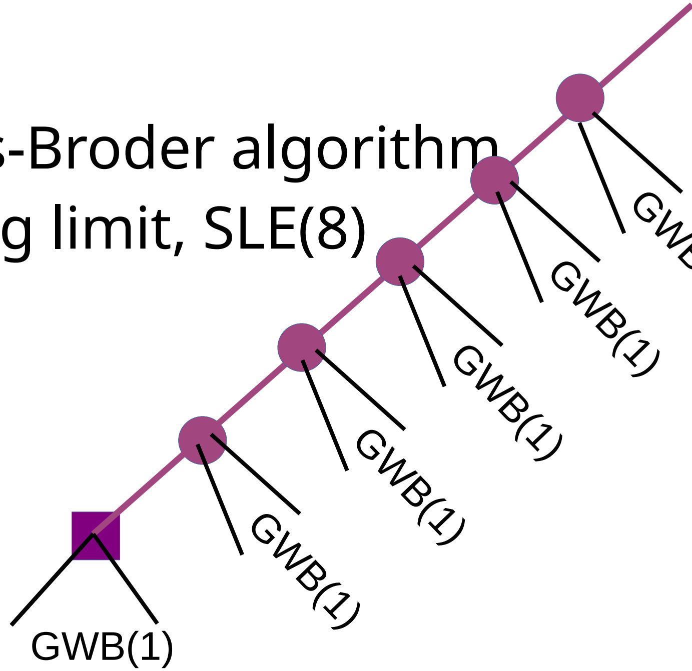
Probability/statistical physics:

- electrical networks, Wilson's algorithms, Aldous-Broder algorithm
- 2D lattices, Schramm-Loewner evolution, scaling limit, SLE(8)

Here: dense graphs

Theorem (Kolchin 1977 / Grimmet 1980):

UST(K_n) around vertex 1 converges to GWB(1) conditioned on survival, n→∞.



Uniform spanning tree

Theorem (Kolchin 1977 / Grimmet 1980) Local structure of K_n

Borgs, Chayes, Lovász, Sós, Szegedy, Vesztergombi 2004-2012,
Description of dense graphs using "graphons"

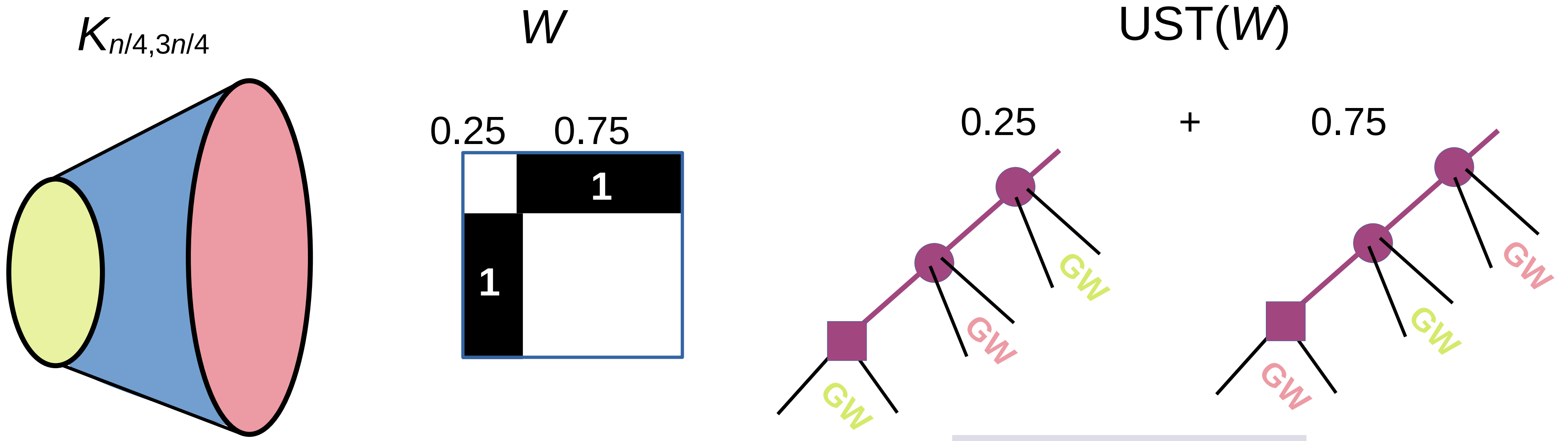
Theorem (Hladký, Nachmias, Tran, 2018)
Suppose that $(G_n)_n$ is a sequence of connected graphs that converges to a graphon W. Suppose that W has positive minimum degree.
Then UST(G_n) around a randomly chosen vertex converges to a branching process UST(W).

Example

Hladký-Nachmias-Tran, 2018, simplified

$(G_n)_n$ converges to W.

Then UST(G_n) around a randomly chosen vertex converges to UST(W).



GW=GWB(e=3,o=1/3)
GW=GWB(e=1/3,e=3)

Uniform spanning tree

Main Theorem II (simplified)

The following are equivalent for two connected graphons *U* and *W*:

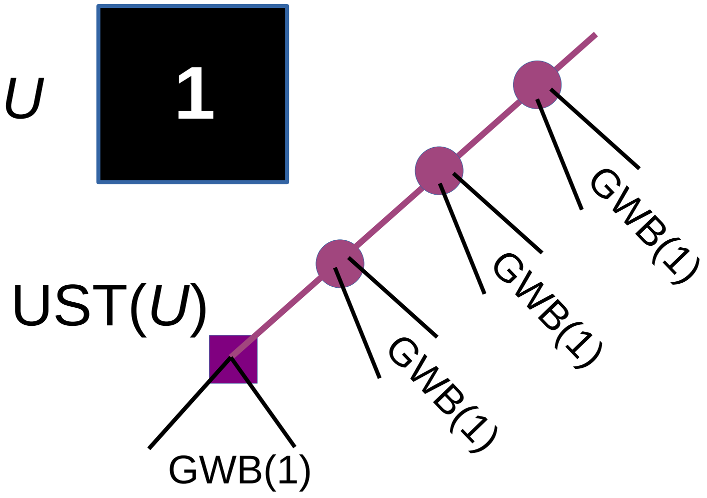
- UST(*U*) and UST(*W*) have the same distribution
- *U* and *W* are fractionally isomorphic ………………………….

Uniform spanning tree

Main Theorem II (simplified)

The following are equivalent for two connected graphons *U* and *W*:

- UST(*U*) and UST(*W*) have the same distribution
- *U* and *W* are fractionally isomorphic ....

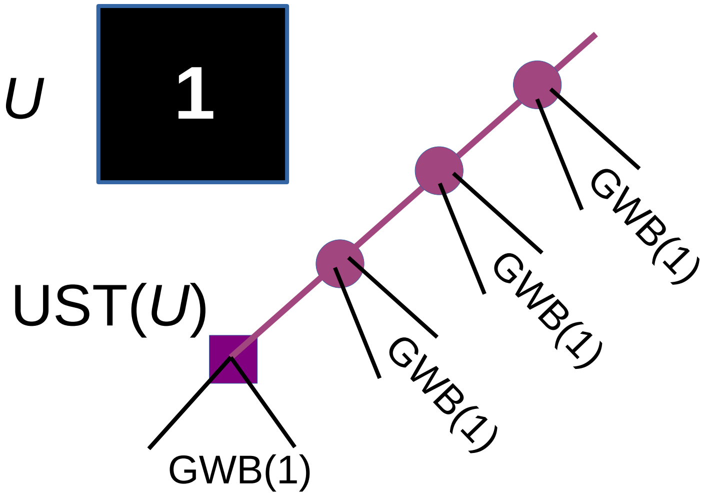


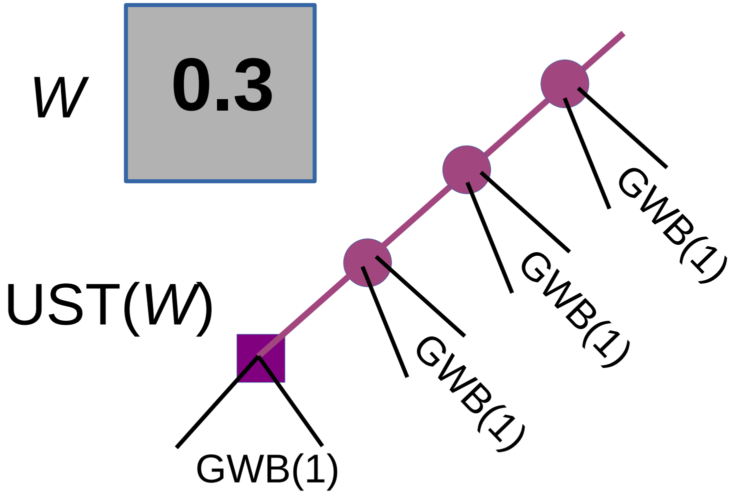
Uniform spanning tree

Main Theorem II (simplified)

The following are equivalent for two connected graphons *U* and *W*:

- UST(*U*) and UST(*W*) have the same distribution
- *U* and *W* are fractionally isomorphic ....





Uniform spanning tree

Main Theorem II (simplified)

The following are equivalent for two connected graphons U and W:

- UST(U) and UST(W) have the same distribution
- U and W are fractionally isomorphic up to multiple

("affinely fractionally isomorphic")