Institute of Computer Science
Academy of Sciences of the Czech Republic

A Manual of Results on Interval Linear Problems

Dedicated to my grandchildren Matthias, William and Beata, and also to those who might yet come

Jiří Rohn
http://uivtx.cs.cas.cz/~rohn

Technical report No. V-1164

10.08.2004 / 11.02.2005 / 05.07.2012

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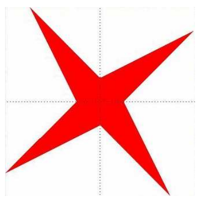
Abstract:

This is a torso of an unfinished book project which the author started in August 2004, abandoned in February 2005, left unchanged since and finally made public in July 2012. It contains almost everything the author knew about the matter by February 2005 (253 theorems, about half of them proved, other proofs referenced). The text was left unchanged in its original February 2005 form; errata, comments and a list of new results will be possibly published in the wake of it. [1]



Keywords:
Interval linear problems, auxiliary results, basic tools, interval matrices, systems of interval linear equations (square case), systems of interval linear equations and inequalities (rectangular case), interval linear programming, and many others.

[1]Above: logo of interval computations and related areas (depiction of the solution set of the system $[2,4]x_1 + [-2,1]x_2 = [-2,2]$, $[-1,2]x_1 + [2,4]x_2 = [-2,2]$ (Barth and Nuding [3])).

Contents

Chapter 1

Introduction, notations, and auxiliary results

1.1 General introduction

1.2 Notations

We shall use the following notations. The ith row of a matrix A is denoted by $A_{i\cdot}$, the jth column by $A_{\cdot j}$. For two matrices A, B of the same size, inequalities like $A \le B$ or $A < B$ are understood componentwise. A is called nonnegative if $0 \le A$ and symmetric if $A^T = A$; A^T is the transpose of A. $A \circ B$ denotes the Hadamard (entrywise) product of $A, B \in \mathbb{R}^{m\times n}$, i.e., $(A \circ B)_{ij} = A_{ij}B_{ij}$ for each i, j. The absolute value of a matrix $A = (a_{ij})$ is defined by $|A| = (|a_{ij}|)$. We shall use the following easy-to-prove properties valid whenever the respective operations and inequalities are defined:

(i) $A \le B$ and $0 \le C$ imply $AC \le BC$,
(ii) $A \le |A|$,
(iii) $|A| \le B$ if and only if $-B \le A \le B$,
(iv) $|A + B| \le |A| + |B|$,
(v) if $A \circ B \ge 0$, then $|A + B| = |A| + |B|$,
(vi) if $|A - B| < |B|$, then $A \circ B > 0$,
(vii) $||A| - |B|| \le |A - B|$,
(viii) $|AB| \le |A||B|$.

The same notations and results also apply to vectors which are always considered one-column matrices. Hence, for $a = (a_i)$ and $b = (b_i)$, $a^Tb = \sum_i a_ib_i$ is the scalar product whereas ab^T is the matrix (a_ib_j). Maximum (or minimum) of two vectors a, b is understood componentwise, i.e., $(\max\{a, b\})_i = \max\{a_i, b_i\}$ for each i. In particular, for vectors a^+, a^- defined by $a^+ = \max\{a, 0\}$, $a^- = \max\{-a, 0\}$ we have $a = a^+ - a^-$, $|a| = a^+ + a^-$, $a^+ \ge 0$, $a^- \ge 0$ and $(a^+)^Ta^- = 0$. I denotes the unit matrix, e_j is the jth column of I, $e = (1, \dots, 1)^T$ is the vector of all ones and $E = ee^T \in \mathbb{R}^{m\times n}$ is the matrix of all ones (in these cases we do not designate explicitly the dimension which can always be inferred from the context). In our descriptions to follow, important role is played by the set Y_m of all ± 1 vectors in $\mathbb{R}^m$, i.e.,

$$Y_m = \{y \in \mathbb{R}^m;\ |y| = e\}.$$

Obviously, the cardinality of Y_m is 2^m. For each $x \in \mathbb{R}^m$ we define its sign vector $\operatorname{sgn} x$ by

$$(\operatorname{sgn} x)_i = \begin{cases} 1 & \text{if } x_i \ge 0, \\ -1 & \text{if } x_i < 0 \end{cases} \qquad (i = 1, \dots, m),$$

so that $\operatorname{sgn} x \in Y_m$. For a given vector $y \in \mathbb{R}^m$ we denote

$$T_y = \begin{pmatrix} y_1 & 0 & \dots & 0 \\ 0 & y_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y_m \end{pmatrix}. \tag{1.1}$$

With a few exceptions we shall use the notation T_y for vectors $y \in Y_m$ only, in which case we have $T_{-y} = -T_y$, $T_y^{-1} = T_y$ and $|T_y| = I$. For each $x \in \mathbb{R}^m$ we can write

$|x| = T_z x$, where $z = \operatorname{sgn} x$; we shall often use this trick to remove the absolute value of a vector. Notice that $T_z x = (z_i x_i)_{i=1}^m = z \circ x$. $\lambda_{\min}(A)$, $\lambda_{\max}(A)$ denote the minimal and maximal eigenvalue of a symmetric matrix A, respectively. As is well known, $\lambda_{\min}(A) = \min_{\|x\|_2=1} x^T A x$ and $\lambda_{\max}(A) = \max_{\|x\|_2=1} x^T A x$ hold. $\sigma_{\min}(A)$, $\sigma_{\max}(A)$ denote the minimal and maximal singular value of A, and $\varrho(A)$ is the spectral radius of A. *All notations are summed up in the next two subsections.*

1.2.1 Linear algebraic notations

A	matrix
$A_{i\cdot}$	the ith row of A
$A_{\cdot j}$	the jth column of A
A^{-1}	inverse matrix
A^+	the Moore-Penrose inverse of A
A^T	transpose of A
$\|A\|_{\infty,1}$	$= \max_{\|x\|_\infty=1} \|Ax\|_1$
$A \le B$	$A_{ij} \le B_{ij}$ for each i, j
$A < B$	$A_{ij} < B_{ij}$ for each i, j
$A \ge B$	$\Leftrightarrow B \le A$
$A > B$	$\Leftrightarrow B < A$
$A \circ B$	$= (a_{ij}b_{ij})$ for $A = (a_{ij})$, $B = (b_{ij})$ (Hadamard product)
a	column vector
$a^T b$	$= \sum_i a_i b_i$ (scalar product)
ab^T	outer product ($(ab^T)_{ij} = a_i b_j$ for each i, j)
a^+	$= \max\{a, 0\}$
a^-	$= \max\{-a, 0\}$
$\mathbb{C}$	the set of complex numbers
$\mathbb{C}^n$	complex vector space
$\mathrm{Conv}\, X$	the convex hull of X
$\det A$	determinant of A
$\mathrm{diag}\,(A)$	$= (A_{11}, \ldots, A_{nn})^T$ for $A \in \mathbb{R}^{n\times n}$ (diagonal of A)
E	$= ee^T \in \mathbb{R}^{m\times n}$ (the matrix of all ones)
e	$= (1, 1, \ldots, 1)^T$
e_j	the jth column of the unit matrix I
$\square$	end of proof ("halmos")
I	unit (or identity) matrix
$\lambda_i(A)$	ith eigenvalue of a symmetric A ($\lambda_1(A) \ge \ldots \ge \lambda_n(A)$)
$\lambda_{\max}(A)$	$= \lambda_1(A)$ (maximum eigenvalue of a symmetric A)
$\lambda_{\min}(A)$	$= \lambda_n(A)$ (minimum eigenvalue of a symmetric A)
$\max\{A, B\}$	componentwise maximum of matrices (vectors)
$\min\{A, B\}$	componentwise minimum of matrices (vectors)
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^{m\times n}$	the set of $m \times n$ real matrices
$\mathbb{R}^n$	real vector space
$\varrho(A)$	spectral radius of A
$\sigma_{\max}(A)$	maximum singular value of A
$\sigma_{\min}(A)$	minimum singular value of A

1.2.2 Specific notations

Notations marked in red are important and should be memorized.

$\mathbf{A}$	interval matrix
$\lvert A\rvert$	absolute value of a matrix ($\lvert A\rvert = (\lvert a_{ij}\rvert)$ for $A = (a_{ij})$)
$\underline{A}$	lower bound of an interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$
$\overline{A}$	upper bound of an interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$
A_c	midpoint matrix of an interval matrix $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$
$\mathbf{A}_s$	$= [(\underline{A} + \underline{A}^T)/2, (\overline{A} + \overline{A}^T)/2]$ for $\mathbf{A} = [\underline{A}, \overline{A}]$ (symmetrization)
$\mathbf{A}^T$	$= [\underline{A}^T, \overline{A}^T]$ for $\mathbf{A} = [\underline{A}, \overline{A}]$ (transpose)
A_{yz}	$= A_c - T_y \Delta T_z$
$\frac{a}{0}$	$= 0$ for $a = 0$, $= \infty$ for $a > 0$ (case $a < 0$ does not occur)
$\mathbf{b}$	interval vector
$\underline{b}$	lower bound of an interval vector $\mathbf{b} = [\underline{b}, \overline{b}]$
$\overline{b}$	upper bound of an interval vector $\mathbf{b} = [\underline{b}, \overline{b}]$
b_c	midpoint vector of an interval vector $\mathbf{b} = [b_c - \delta, b_c + \delta]$
b_y	$= b_c + T_y \Delta$
δ	radius vector of an interval vector $\mathbf{b} = [b_c - \delta, b_c + \delta]$
Δ	radius matrix of an interval matrix $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$
$f(A, b, c)$	optimal value of a linear programming problem
$\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c})$	lower bound of the range of the optimal value of an interval linear programming problem
$\overline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c})$	upper bound of the range of the optimal value of an interval linear programming problem
$\varrho_0(A)$	real spectral radius of A (maximum of moduli of *real* eigenvalues)
$\mathbb{R}^n_z$	$= \{x \in \mathbb{R}^n\,;\, T_z x \geq 0\}$ (z-orthant, $z \in Y_n$)
$\operatorname{sgn} A$	sign matrix of a matrix A ($(\operatorname{sgn} A)_{ij} = 1$ if $A_{ij} \geq 0$, $(\operatorname{sgn} A)_{ij} = -1$ otherwise)
$\operatorname{sgn} x$	sign vector of a vector x ($(\operatorname{sgn} x)_i = 1$ if $x_i \geq 0$, $(\operatorname{sgn} x)_i = -1$ otherwise)
T_y	the diagonal matrix with diagonal vector y
X	the solution set of $\mathbf{A}x = \mathbf{b}$
$[X]$	the interval hull of X
$\lvert x\rvert$	absolute value of a vector ($\lvert x\rvert = (\lvert x_i\rvert)$ for $x = (x_i)$)
$[\underline{x}, \overline{x}]$	the interval hull of the solution set X
$[\underline{\underline{x}}, \overline{\overline{x}}]$	enclosure of X (in particular, that by Hansen-Bliek-Rohn)
x_y	mostly used for the unique solution of the equation $A_c x - T_y \Delta \lvert x\rvert = b_y$
Y_m	the set of all ± 1-vectors in $\mathbb{R}^m$

1.3 Auxiliary results

1.3.1 An algorithm for generating plus/minus-one vectors

It proves helpful at a later stage to generate all the ± 1-vectors forming the set Y_m systematically one-by-one in such a way that any two successive vectors differ in exactly one entry. We describe here an algorithm for performing this task, formulated in terms of generating the whole set Y_m; in later applications the last-but-one line "$Y = Y \cup \{y\}$" is replaced by the respective action on the current vector y. The algorithm employs an auxiliary $(0, 1)$-vector $z \in \mathbb{R}^m$ used for determining the index k for which the current value of y_k should be changed to $-y_k$, and its description is as follows (throughout, we describe algorithms in the form of MATLAB-like functions):

function $Y = \textbf{yset}(m)$
$z = 0 \in \mathbb{R}^m$; $y = e \in Y_m$; $Y = \{y\}$;
while $z \neq e$
    $k = \min\{i\,;\, z_i = 0\}$;
    for $i = 1 : k-1$, $z_i = 0$; **end**
    $z_k = 1$;
    $y_k = -y_k$;
    $Y = Y \cup \{y\}$;
end

Figure 1.1: An algorithm for generating the set Y_m.

Theorem 1. *For each $m \geq 1$ the algorithm at the output yields the set $Y = Y_m$.*

Proof: [98]. □

Proof: We shall prove the assertion by induction on m. For $m = 1$ it is a matter of simple computation to verify that the algorithm, if started from $y = 1$, generates $Y = \{1, -1\}$, and if started from $y = -1$, generates $Y = \{-1, 1\}$, in both cases $Y = Y_1$. Thus let the assertion hold for some $m - 1 \geq 1$ and let the algorithm be run for m. To see what is being done in the course of the algorithm, let us notice that in the main loop the initial string of the form

$$(1, 1, \ldots, 1, 0, \ldots)^T$$

of the current vector z is being found, where 0 is at the kth position, and it is being changed to

$$(0, 0, \ldots, 0, 1, \ldots)^T$$

until the vector z of all ones is reached (the last vector preceding it is $(0, 1, \ldots, 1, 1)^T$). Hence if we start the algorithm for m, then the sequence of vectors z and y, restricted to their first $m-1$ entries, is the same as if the algorithm were run for $m-1$, until vector z of the form

$$(1, 1, \ldots, 1, 0)^T \tag{1.2}$$

is reached. By that time, according to the induction hypothesis, the algorithm has constructed all the vectors $y \in Y_m$ with y_m being fixed throughout at its initial value. In the next step the vector (1.2) is switched to

$$(0, 0, \ldots, 0, 1)^T$$

and y_m is switched to $-y_m$. Now, from the point of view of the first $m-1$ entries, the algorithm again starts from zero vector z and due to the induction hypothesis it again generates all the $(m-1)$-dimensional ± 1-vectors in the first $m-1$ entries, this time with the opposite value of y_m. This implies that at the end (when vector z of all ones is reached) the whole set Y_m is generated, which completes the proof by induction. □

The performance of the algorithm for $m = 3$ is illustrated in the following table. The algorithm starts from $z = 0$, $y = e$ (the first row) and the current values of z, y at the end of each pass through the **while** loop are given in the next seven rows of the table.

z^T	y^T
(0, 0, 0)	(1, 1, 1)
(1, 0, 0)	(-1, 1, 1)
(0, 1, 0)	(-1, -1, 1)
(1, 1, 0)	(1, -1, 1)
(0, 0, 1)	(1, -1, -1)
(1, 0, 1)	(-1, -1, -1)
(0, 1, 1)	(-1, 1, -1)
(1, 1, 1)	(1, 1, -1)

1.3.2 Norms

Vector norms in $\mathbb{R}^n$:

$$\begin{aligned}
\|x\|_1 &= e^T|x| = \sum_{i=1}^{n} |x_i|, \\
\|x\|_2 &= \sqrt{x^T x}, \\
\|x\|_\infty &= \max_{i=1,\ldots,n} |x_i|.
\end{aligned}$$

Matrix norms in $\mathbb{R}^{m\times n}$:

$$\begin{aligned}
\|A\|_1 &= \max_{j=1,\ldots,n} \sum_{i=1}^{m} |a_{ij}|, \\
\|A\|_2 &= \sqrt{\lambda_{\max}(A^T A)}, \\
\|A\|_\infty &= \max_{i=1,\ldots,m} \sum_{j=1}^{n} |a_{ij}|, \\
\|A\|_{1,\infty} &= \max_{ij} |a_{ij}|, \\
\|A\|_{\infty,1} &= \max_{y\in Y_n} \|Ay\|_1 = \max_{z\in Y_m,\, y\in Y_n} z^T A y
\end{aligned} \tag{1.3}$$

(see Horn and Johnson [22] for a general treatment and [97] for (1.3)). The last norm $\|A\|_{\infty,1}$ plays a particular role in interval analysis, for two reasons. First, it arises quite naturally with many interval linear problems (see e.g. Theorem 38). Second, its computation is NP-hard (Theorem 21), a fact which shreds bad light on all these problems, making them also NP-hard. The following algorithm for computing $\|A\|_{\infty,1}$ uses the first formula in (1.3) in conjunction with the algorithm for computing the set Y_n (Fig. 1.1):

Given two vector norms $\|x\|_\alpha$ and $\|x\|_\beta$ in $\mathbb{R}^n$, a subordinate matrix norm $\|A\|_{\alpha,\beta}$ for $A \in \mathbb{R}^{n\times n}$ is defined by

$$\|A\|_{\alpha,\beta} = \max_{\|x\|_\alpha=1} \|Ax\|_\beta \tag{1.4}$$

(see Higham [20], p. 121). If we use the norms $\|x\|_1 = e^T|x| = \sum_i |x_i|$, $\|x\|_\infty = \max_i |x_i|$, then from (1.4) we obtain $\|A\|_{1,1} = \max_j \sum_i |a_{ij}|$, $\|A\|_{\infty,\infty} = \max_i \sum_j |a_{ij}|$, and $\|A\|_{1,\infty} = \max_{ij} |a_{ij}|$, so that all three norms are easy to compute. This, however, is no more true for the fourth norm $\|A\|_{\infty,1}$. In [97] it is proved that

$$\|A\|_{\infty,1} = \max_{y\in Y_n} \|Ay\|_1 = \max_{z,y\in Y_n} z^T A y, \tag{1.5}$$

where the set Y_n consists of 2^n vectors. One might hope to find an essentially better formula for $\|A\|_{\infty,1}$, but such an attempt is not likely to succeed due to the following complexity result proved again in [97]:

```
function ν = norminfone (A)
 y = e ∈ ℝ^n; z = 0 ∈ ℝ^{n−1};
 x = Ay;
 ν = ‖x‖_1;
 while z ≠ e
    k = min{i ; z_i = 0};
    x = x − 2 y_k A_{•k};
    ν = max{ν, ‖x‖_1};
    for  i = 1 : k − 1, z_i = 0; end
    z_k = 1;
    y_k = −y_k;
 end
```

Figure 1.2: An algorithm for computing the norm $\|A\|_{\infty,1}$.

1.3.3 The Sherman-Morrison formula

Theorem 2. (Sherman-Morrison) *Let $A \in \mathbb{R}^{n\times n}$ be nonsingular, $b, c \in \mathbb{R}^n$, and let $\alpha = 1 + c^T A^{-1} b$. Then we have:*

(i) $\det(A + bc^T) = \alpha \det A$,
(ii) if $\alpha = 0$, then $A + bc^T$ is singular,
(iii) if $\alpha \neq 0$, then

$$(A + bc^T)^{-1} = A^{-1} - \tfrac{1}{\alpha} A^{-1} bc^T A^{-1}.$$

Proof: [15]. □

The Sherman-Morrison formula comes in question as soon as we are to deal with a matrix of the form $A + bc^T$ (a so-called rank one update of A). It is used e.g. in the proof of one of our basic results, Theorem 30.

Proof: (i) From the identities

$$\begin{pmatrix} I + A^{-1}bc^T & 0 \\ -c^T & 1 \end{pmatrix} = \begin{pmatrix} I & -A^{-1}b \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} I & A^{-1}b \\ -c^T & 1 \end{pmatrix},$$

$$\begin{pmatrix} I & A^{-1}b \\ 0^T & \alpha \end{pmatrix} = \begin{pmatrix} I & 0 \\ c^T & 1 \end{pmatrix} \begin{pmatrix} I & A^{-1}b \\ -c^T & 1 \end{pmatrix}$$

it follows that

$$\det(I + A^{-1}bc^T) = \det \begin{pmatrix} I & A^{-1}b \\ -c^T & 1 \end{pmatrix} = \det \begin{pmatrix} I & A^{-1}b \\ 0^T & \alpha \end{pmatrix} = \alpha,$$

hence

$$\det(A + bc^T) = \det A \cdot \det(I + A^{-1}bc^T) = \alpha \det A.$$

(ii) If $\alpha = 0$, then $\det(A + bc^T) = 0$ by (i).

(iii) If $\alpha \neq 0$, then direct computation shows that

$$\begin{aligned} & (A + bc^T)(A^{-1} - \tfrac{1}{1+c^TA^{-1}b}A^{-1}bc^TA^{-1}) \\ = \;& I - \tfrac{1}{1+c^TA^{-1}b}bc^TA^{-1} + bc^TA^{-1} - \tfrac{1}{1+c^TA^{-1}b}b(c^TA^{-1}b)c^TA^{-1} \\ = \;& I + (-\tfrac{1}{1+c^TA^{-1}b} + 1 - \tfrac{c^TA^{-1}b}{1+c^TA^{-1}b})bc^TA^{-1} = I, \end{aligned}$$

since the last term in parentheses equals zero. This implies that

$$(A + bc^T)^{-1} = A^{-1} - \tfrac{1}{1+c^TA^{-1}b}A^{-1}bc^TA^{-1} = A^{-1} - \tfrac{1}{\alpha}A^{-1}bc^TA^{-1}.$$

□

1.3.4 Spectral radius and related results

Definition. The number

$$\varrho(A) = \max\{|\lambda|;\ \lambda \text{ is an eigenvalue of } A\}$$

is called the spectral radius of a square matrix A.

Theorem 3. *For each $A \in \mathbb{R}^{n\times n}$ and each $\varepsilon > 0$ there exists a matrix norm $\|\cdot\|$ (depending on A and ε) such that*

$$\varrho(A) \le \|A\| < \varrho(A) + \varepsilon.$$

Proof: See [22], p. 297. □

We utilize the following properties of the spectral radius; all the proofs can be found in Horn and Johnson [22].

Theorem 4. *Let $A, B \in \mathbb{R}^{n\times n}$. Then we have:*

(a) $\varrho(AB) = \varrho(BA)$,
(b) if $|A| \le B$, *then* $\varrho(A) \le \varrho(|A|) \le \varrho(B)$.

Theorem 5. *For a matrix $A \in \mathbb{R}^{n\times n}$, the following assertions are equivalent:*

(i) $\varrho(A) < 1$,
(ii) $A^j \to 0$ *as* $j \to \infty$,
(iii) $I - A$ *is nonsingular,* $\sum_{j=0}^{\infty} A^j$ *converges and* $(I - A)^{-1} = \sum_{j=0}^{\infty} A^j$ *holds.*

Proof: We shall prove (i)⇒(ii)⇒(iii)⇒(ii)⇒(i).

(i)⇒(ii): If $\varrho(A) < 1$, then there exists a matrix norm satisfying $\|A\| < 1$ (Theorem 3), and equivalence of norms implies existence of a constant $c > 0$ such that $\|X\|_1 \le c\|X\|$ for each $X \in \mathbb{R}^{n\times n}$. Then for each $j \ge 1$ we have by induction

$$\|A^j\|_1 \le c\|A^j\| \le c\|A\|^j,$$

where $\|A\|^j \to 0$ because of $\|A\| < 1$, hence $\|A^j\|_1 \to 0$, which implies that $A^j \to 0$ (componentwise).

(ii)⇒(iii): If $I - A$ were singular, we would have $Ax = x$ for some $x \ne 0$ and by induction $A^j x = x$ for each $j \ge 1$. Since $A^j \to 0$, this would imply $A^j x \to 0$, hence $x = 0$, a contradiction. Thus $I - A$ is nonsingular. Then from the identity

$$(I - A)\sum_{j=0}^{k} A^j = I - A^{k+1} \qquad (k = 0, 1, 2, \ldots)$$

we have

$$\sum_{j=0}^{k} A^j = (I - A)^{-1}(I - A^{k+1})$$

and since $A^{k+1} \to 0$, this implies that $\sum_{j=0}^{\infty} A^j$ converges and

$$\sum_{j=0}^{\infty} A^j = (I - A)^{-1}$$

holds.

(iii)⇒(ii): If $\sum_{j=0}^{\infty} A^j$ converges, then $A^j \to 0$.

(ii)⇒(i): Let $\lambda \in \mathbb{C}$ be an arbitrary eigenvalue of A, i.e., $Ax = \lambda x$ for some $0 \neq x \in \mathbb{C}^{n \times n}$. Then $A^j x = \lambda^j x$ for each $j \geq 1$. Since $A^j \to 0$, we have $\lambda^j x \to 0$, and in view of $x \neq 0$ it must be $\lambda^j \to 0$, implying $|\lambda| < 1$. Hence $\varrho(A) < 1$. □

Theorem 6. (Perron) *If A is square nonnegative, then there holds*

$$Ax = \varrho(A)x$$

for some $x \geq 0$, $x \neq 0$.

Theorem 7. *For a nonnegative square matrix A, the following assertions are equivalent:*

(i) $\varrho(A) < 1$,

(ii) $I - A$ *is nonsingular and* $(I - A)^{-1} \geq I$,

(iii) $Ax < x$ *for some* $x > 0$.

Proof: We prove (i)⇒(ii)⇒(iii)⇒(i).

(i)⇒(ii): If $\varrho(A) < 1$, then by Theorem 5 in view of nonnegativity of A we have $(I - A)^{-1} = \sum_{j=0}^{\infty} A^j \geq I$.

(ii)⇒(iii): Take an arbitrary $y > 0$ and put $x = (I - A)^{-1}y$. Then $x \geq Iy = y > 0$ and $x = Ax + y > Ax$.

(iii)⇒(i): Since A^T is again nonnegative and has the same eigenvalues as A, by the Perron theorem there holds $A^T y = \varrho(A)y$ for some $y \geq 0$, $y \neq 0$. If $Ax < x$ for some $x > 0$, then $x^T y > 0$ and $(Ax)^T y < x^T y$, hence premultiplying the equality $A^T y = \varrho(A)y$ by x^T yields

$$\varrho(A) = \frac{x^T A^T y}{x^T y} = \frac{(Ax)^T y}{x^T y} < \frac{x^T y}{x^T y} = 1.$$

□

Theorem 8. *For each $a, b \in \mathbb{R}^n$ there holds*

$$\varrho(ab^T) = |b^T a|.$$

1.3.5 Eigenvalues of symmetric matrices

It is well known that a symmetric matrix $A \in \mathbb{R}^{n\times n}$ has all eigenvalues real. They are usually ordered in a nonincreasing sequence as

$$\lambda_1(A) \geq \ldots \geq \lambda_n(A).$$

The Courant-Fischer minimax theorem [15] gives explicit formulae for the eigenvalues of a symmetric matrix A by

$$\lambda_k(A) = \max_{\dim W=k} \min_{0\neq x\in W} \frac{x^T Ax}{x^T x} = \max_{\dim W=k} \min_{\substack{x\in W \\ \|x\|_2=1}} x^T Ax \qquad (k = 1, \ldots, n), \tag{1.6}$$

where the maximum is taken over all subspaces W of $\mathbb{R}^n$ of dimension k. In particular, for $k = 1$ and $k = n$ we obtain

$$\begin{aligned} \lambda_1(A) &= \max_{\|x\|_2=1} x^T Ax, \\ \lambda_n(A) &= \min_{\|x\|_2=1} x^T Ax. \end{aligned}$$

It is customary to write also $\lambda_{\max}(A)$, $\lambda_{\min}(A)$ instead of $\lambda_1(A)$, $\lambda_n(A)$.

Theorem 9. (Wielandt-Hoffman) *For symmetric matrices $A, B \in \mathbb{R}^{n\times n}$ there holds*

$$|\lambda_k(A) - \lambda_k(B)| \leq \varrho(A - B)$$

for $k = 1, \ldots, n$.

Proof: [15]. □

This shows that for each k, $\lambda_k(A)$ is a continuous function of A provided the argument stays within the set of symmetric matrices.

Eigenvalues of symmetric matrices can be computed very effectively by the symmetric version of the QR algorithm (see Golub and van Loan [15] or Watkins [118]).

1.3.6 Singular values

Each *rectangular* matrix $A \in \mathbb{R}^{m\times n}$ has $q = \min\{m, n\}$ uniquely determined singular values $\sigma_1(A), \ldots, \sigma_q(A)$. Although they are usually introduced by means of the singular value decomposition (see Horn and Johnson [22], [23] or Golub and van Loan [15]), they can also be expressed in the following way. Let $r \in \{0, \ldots, n\}$ be the number of positive eigenvalues of the symmetric matrix $A^T A$. Then the singular values of A satisfy

$$\sigma_i(A) = \sqrt{\lambda_i(A^T A)} \qquad (i = 1, \ldots, r), \tag{1.7}$$

$$\sigma_i(A) = 0 \qquad (i = r+1, \ldots, q) \tag{1.8}$$

(the matrix $A^T A$ is positive semidefinite, so that all its eigenvalues are nonnegative). Hence there holds

$$\sigma_1(A) \geq \ldots \geq \sigma_q(A) \geq 0.$$

Again it is customary to write $\sigma_{\max}(A)$, $\sigma_{\min}(A)$ instead of $\sigma_1(A)$ and $\sigma_q(A)$. There is also another characterization of singular values which turns out to be better suited for our purposes because it avoids computation of $A^T A$:

Theorem 10. (Jordan-Wielandt) *If a matrix $A \in \mathbb{R}^{m\times n}$ has singular values $\sigma_1(A) \geq \ldots \geq \sigma_q(A) \geq 0$, $q = \min\{m, n\}$, then the symmetric $(m+n) \times (m+n)$ matrix*

$$\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} \tag{1.9}$$

has eigenvalues

$$\sigma_1(A) \geq \ldots \geq \sigma_q(A) \geq 0 = \ldots = 0 \geq -\sigma_q(A) \geq \ldots - \geq \sigma_1(A).$$

Proof: [114]. □

Hence, we can obtain the singular values of A simply by taking the positive eigenvalues of the symmetric matrix (1.9) and adding zeros to the overall number $q = \min\{m, n\}$.

1.3.7 The Moore-Penrose inverse

Theorem 11. (Moore-Penrose) *For each* $A \in \mathbb{R}^{m\times n}$ *there exists exactly one matrix* $A^+ \in \mathbb{R}^{n\times m}$ *satisfying*

1. $AA^+A = A$,
2. $A^+AA^+ = A^+$,
3. $(AA^+)^T = AA^+$,
4. $(A^+A)^T = A^+A$.

Proof: [114]. □

Definition. The matrix A^+ is called the Moore-Penrose inverse (or, pseudoinverse) of A.

We have the following special cases:

(a) $A^+ = (A^TA)^{-1}A^T$ if A has linearly independent columns,
(b) $A^+ = A^T(AA^T)^{-1}$ if A has linearly independent rows,
(c) $A^+ = A^{-1}$ if A is square nonsingular.

This can be proved by direct verification that in each of the three cases A^+ has the above properties 1.-4.

For a general A, its Moore-Penrose inverse can be computed either from the SVD decomposition of A (Stewart and Sun [114]), or by Greville's algorithm [16], [17].

1.3.8 Least squares problem

Definition. Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. A vector $x \in \mathbb{R}^n$ satisfying

$$\|Ax - b\|_2 = \min\{\|Ay - b\|_2\,;\, y \in \mathbb{R}^n\}$$

is called the least squares solution of the system $Ax = b$.

Theorem 12. *The set X of the least squares solutions of a system $Ax = b$ is described by*

$$X = \{A^+b + (I - A^+A)y\,;\, y \in \mathbb{R}^n\}.$$

Proof: [114]. □

This implies that a least squares solution *always exists* since $x = A^+b$ is one of them. If A has linearly independent columns, then $A^+A = I$ (Subsection 1.3.7, (a)) and A^+b is the unique least squares solution of $Ax = b$.

We shall later employ another useful characterization:

Theorem 13. *A vector $x \in \mathbb{R}^n$ is a least squares solution of $Ax = b$ if and only if it satisfies*

$$\begin{pmatrix} A & -I \\ 0 & A^T \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix} \tag{1.10}$$

for some $y \in \mathbb{R}^m$.

Proof: [114]. □

The advantage of (1.10) is that the system matrix is square and does not contain terms of the form A^+A or A^TA whose use for systems with interval data is not recommendable.

1.3.9 P-matrices and the linear complementarity problem

Definition. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be a P-matrix (or, to have the P-property) if all its principal minors are positive; principal minors are determinants of square submatrices formed from rows and columns with the same indices (there are $2^n - 1$ of them).

Theorem 14. (Fiedler and Pták) *A matrix $A \in \mathbb{R}^{n\times n}$ is a P-matrix if and only if for each $x \neq 0$ there exists an $i \in \{1, \ldots, n\}$ such that $x_i(Ax)_i > 0$.*

Proof: [10]. □

For us, P-matrices are important mainly due to their essential property:

Theorem 15. (Samelson, Thrall and Wesler; Ingleton; Murty) *A square matrix A is a P-matrix if and only if for each right-hand side b the linear complementarity problem*

$$x^+ = Ax^- + b \tag{1.11}$$

has a unique solution.

Comment. For $x = (x_i)$, $x^+ = (\max\{x_i, 0\})$ and $x^- = (\max\{-x_i, 0\})$, see p. 10.

Proof: Samelson, Thrall and Wesler [104]; independently Ingleton [24]; independently Murty [51]. □

The problem (1.11) is called the linear complementarity problem. We use it in this form which is most suitable for our purposes. Another equivalent formulation, used in most textbooks, is

$$y = Az + b,$$

$$y \geq 0,\ z \geq 0,$$

$$y^T z = 0.$$

1.3.10 Farkas lemma

In this section we consider systems of linear equations $Ax = b$ or systems of linear inequalities $Ax \leq b$. It is assumed that $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, where m and n are arbitrary positive integers.

Definition. A system of linear equations $Ax = b$ is called *solvable* if it has a solution, and *feasible* if it has a nonnegative solution.

Throughout this text the reader is kindly asked to bear in mind that *feasibility means nonnegative solvability*. The basic result concerning feasibility of linear equations was proved by Farkas [9] in 1902.

Theorem 16. (Farkas)[1] *A system*

$$Ax = b \tag{1.12}$$

is feasible if and only if each p with $A^T p \geq 0$ satisfies $b^T p \geq 0$.

Proof: Farkas [9]; also available in [98]. □

Proof: (a) If the system (5.2) has a solution $x \geq 0$ and if $A^T p \geq 0$ holds for some $p \in \mathbb{R}^m$, then $b^T p = (Ax)^T p = x^T(A^T p) \geq 0$. This proves the "only if" part of the theorem.

(b) We shall prove the "if" part by contradiction, proving that if the system (5.2) does not possess a nonnegative solution, then there exists a $p \in \mathbb{R}^m$ satisfying $A^T p \geq 0$ and $b^T p < 0$; for the purposes of the proof it is advantageous to write down this system in the column form

$$p^T A_{\cdot j} \geq 0 \qquad (j = 1, \ldots, n), \tag{1.13}$$

$$p^T b < 0. \tag{1.14}$$

We shall prove this assertion by induction on n.

(b1) If $n = 1$, then A consists of a single column a. Let $W = \{\alpha a;\ \alpha \in \mathbb{R}\}$ be the subspace spanned by a. According to the orthogonal decomposition theorem (Meyer [43], p. 405), b can be written in the form

$$b = b_W + b_{W^\perp},$$

where $b_W \in W$ and $b_{W^\perp} \in W^\perp$, $W^\perp$ being the orthogonal complement of W. We shall consider two cases. If $b_{W^\perp} = 0$, then $b \in W$, so that $b = \alpha a$ for some $\alpha \in \mathbb{R}$. Since $Ax = b$ does not possess a nonnegative solution due to the assumption, it must be $\alpha < 0$ and $a \neq 0$, so that if we put $p = a$, then $p^T a = \|a\|_2^2 \geq 0$ and

[1] Also called "Farkas lemma".

$p^Tb = \alpha\|a\|_2^2 < 0$, hence p satisfies (5.3), (5.4). If $b_{W^\perp} \neq 0$, put $p = -b_{W^\perp}$, then $p^Ta = 0$ and $p^Tb = -\|b_{W^\perp}\|_2^2 < 0$, so that p again satisfies (5.3), (5.4).

(b2) Let the induction hypothesis hold for $n-1 \geq 1$ and let a system (5.2), where $A \in \mathbb{R}^{m\times n}$, does not possess a nonnegative solution. Then neither does the system

$$\sum_{j=1}^{n-1} A_{\cdot j}x_j = b$$

(otherwise for $x_n = 0$ we would get a nonnegative solution of (5.2)), hence according to the induction hypothesis there exists a $\overline{p} \in \mathbb{R}^m$ satisfying

$$\overline{p}^TA_{\cdot j} \geq 0 \qquad (j = 1,\ldots,n-1), \tag{1.15}$$

$$\overline{p}^Tb < 0. \tag{1.16}$$

If $\overline{p}^TA_{\cdot n} \geq 0$, then p satisfies (5.3), (5.4) and we are done. Thus assume that

$$\overline{p}^TA_{\cdot n} < 0. \tag{1.17}$$

Put

$$\begin{aligned} \alpha_j &= \overline{p}^TA_{\cdot j} \qquad (j = 1,\ldots,n), \\ \beta &= \overline{p}^Tb, \end{aligned}$$

then $\alpha_1 \geq 0,\ldots,\alpha_{n-1} \geq 0$, $\alpha_n < 0$ and $\beta < 0$. Consider the system

$$\sum_{j=1}^{n-1}(\alpha_nA_{\cdot j} - \alpha_jA_{\cdot n})x_j = \alpha_nb - \beta A_{\cdot n}. \tag{1.18}$$

If it had a nonnegative solution $x_1,\ldots,x_{n-1}$, then we could rearrange it to the form

$$\sum_{j=1}^{n-1} A_{\cdot j}x_j + A_{\cdot n}x_n = b, \tag{1.19}$$

where

$$x_n = \frac{\beta - \sum_{j=1}^{n-1}\alpha_jx_j}{\alpha_n} > 0$$

due to (5.5), (5.6), (5.7), so that the system (5.9), and thus also (5.2), would have a nonnegative solution $x_1,\ldots,x_n$ contrary to the assumption. Therefore the system (5.8) does not possess a nonnegative solution and thus according to the induction hypothesis there exists a $\tilde{p}$ such that

$$\tilde{p}^T(\alpha_nA_{\cdot j} - \alpha_jA_{\cdot n}) \geq 0 \qquad (j = 1,\ldots,n-1), \tag{1.20}$$

$$\tilde{p}^T(\alpha_nb - \beta A_{\cdot n}) < 0. \tag{1.21}$$

Now we set
$$p = \alpha_n \tilde{p} - (\tilde{p}^T A_{\cdot n})\overline{p}$$
and we shall show that p satisfies (5.3), (5.4). For $j = 1, \ldots, n-1$ we have according to (5.10)
$$p^T A_{\cdot j} = \alpha_n \tilde{p}^T A_{\cdot j} - (\tilde{p}^T A_{\cdot n})\overline{p}^T A_{\cdot j} \geq \alpha_j \tilde{p}^T A_{\cdot n} - (\tilde{p}^T A_{\cdot n})\alpha_j = 0, \tag{1.22}$$
for $j = n$ we get
$$p^T A_{\cdot n} = \alpha_n \tilde{p}^T A_{\cdot n} - (\tilde{p}^T A_{\cdot n})\overline{p}^T A_{\cdot n} = \alpha_n \tilde{p}^T A_{\cdot n} - (\tilde{p}^T A_{\cdot n})\alpha_n = 0, \tag{1.23}$$
and finally from (5.11)
$$p^T b = \alpha_n \tilde{p}^T b - (\tilde{p}^T A_{\cdot n})\overline{p}^T b < \beta \tilde{p}^T A_{\cdot n} - (\tilde{p}^T A_{\cdot n})\beta = 0, \tag{1.24}$$
so that (5.12), (5.13), (5.14) imply (5.3) and (5.4), hence p is a vector having the asserted properties, which completes the proof by induction. □

With the help of Farkas theorem we can characterize solvability of systems of linear equations:

Corollary 17. *A system $Ax = b$ is solvable if and only if each p with $A^T p = 0$ satisfies $b^T p = 0$.*

Proof: [98]. □

Proof: If x solves $Ax = b$ and $A^T p = 0$ holds for some p, then $b^T p = p^T b = p^T Ax = (A^T p)^T x = 0$. Conversely, let the condition hold. Then for each p such that $A^T p \geq 0$ and $A^T p \leq 0$ we have $b^T p \geq 0$. But this, according to the Farkas theorem, is just the sufficient condition for the system
$$Ax_1 - Ax_2 = b \tag{1.25}$$
to be feasible. Hence (5.15) has a solution $x_1 \geq 0$, $x_2 \geq 0$, thus $A(x_1 - x_2) = b$ and $Ax = b$ is solvable. □

For systems of linear inequalities we introduce the notions of solvability and feasibility in the same way:

Definition. A system $Ax \leq b$ is called *solvable* if it has a solution, and *feasible* if it has a nonnegative solution.

Again, we can use Farkas theorem for characterizing solvability and feasibility:

Corollary 18. *A system $Ax \leq b$ is solvable if and only if each $p \geq 0$ with $A^T p = 0$ satisfies $b^T p \geq 0$.*

Proof: [98]. □

Proof: If x solves $Ax \leq b$ and $A^T p = 0$ holds for some $p \geq 0$, then $b^T p = p^T b \geq p^T Ax = 0$. Conversely, let the condition hold, so that each $p \geq 0$ with $A^T p \geq 0$, $A^T p \leq 0$ satisfies $b^T p \geq 0$. This, however, in view of the Farkas theorem means that the system

$$Ax_1 - Ax_2 + x_3 = b$$

is feasible. Hence due to the nonnegativity of x_3 we have $A(x_1 - x_2) \leq b$, and the system $Ax \leq b$ is solvable. □

Corollary 19. *A system $Ax \leq b$ is feasible if and only if each $p \geq 0$ with $A^T p \geq 0$ satisfies $b^T p \geq 0$.*

Proof: [98]. □

Proof: If $x \geq 0$ solves $Ax \leq b$ and $A^T p \geq 0$ holds for some $p \geq 0$, then $b^T p = p^T b = p^T Ax = (A^T p)^T x \geq 0$. Conversely, let the condition hold; then it is exactly the Farkas condition for the system

$$Ax_1 + x_2 = b \tag{1.26}$$

to be feasible. Hence (5.16) has a solution $x_1 \geq 0$, $x_2 \geq 0$, which implies $Ax_1 \leq b$, so that the system $Ax \leq b$ is feasible. □

We sum up the results in the form of a table which reveals similarities and differences among the four necessary and sufficient conditions:

Problem	**Condition**
solvability of $Ax = b$	$(\forall p)(A^T p = 0 \Rightarrow b^T p = 0)$
feasibility of $Ax = b$	$(\forall p)(A^T p \geq 0 \Rightarrow b^T p \geq 0)$
solvability of $Ax \leq b$	$(\forall p \geq 0)(A^T p = 0 \Rightarrow b^T p \geq 0)$
feasibility of $Ax \leq b$	$(\forall p \geq 0)(A^T p \geq 0 \Rightarrow b^T p \geq 0)$

An important result published by Khachiyan [34] in 1979 says that feasibility of a system of linear equations can be checked (and a solution to it, if it exists, found) in polynomial time. Since all three other problems can be reduced to this one, it follows that all four problems can be solved in polynomial time.

1.3.11 Existence lemma

Lemma 20 (Existence lemma) *Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and let for each $y \in Y$ the inequality*

$$T_y Ax \geq T_y b \tag{1.27}$$

have a solution x_y. Then the equation

$$Ax = b$$

has a solution in the set

$$\mathrm{Conv}\{x_y;\ y \in Y\}. \tag{1.28}$$

Proof: We shall prove that the system of linear equations

$$\sum_{y\in Y} \lambda_y A x_y = b, \tag{1.29}$$

$$\sum_{y\in Y} \lambda_y = 1 \tag{1.30}$$

has a solution $\lambda_y \geq 0$, $y \in Y$. In view of the Farkas theorem, it suffices to show that for each $p \in \mathbb{R}^m$ and for each $p_0 \in \mathbb{R}$,

$$p^T A x_y + p_0 \geq 0 \text{ for each } y \in Y \tag{1.31}$$

implies

$$p^T b + p_0 \geq 0. \tag{1.32}$$

Thus let p, p_0 satisfy (5.21). Put $y = -\mathrm{sgn}\, p$, then $p = -T_y|p|$ and from (5.17), (5.21) we have

$$p^T b + p_0 = -|p|^T T_y b + p_0 \geq -|p|^T T_y A x_y + p_0 = p^T A x_y + p_0 \geq 0,$$

which proves (5.22). Hence the system (5.19), (5.20) has a solution $\lambda_y \geq 0$, $y \in Y$. Put $x = \sum_{y\in Y} \lambda_y x_y$, then $Ax = b$ by (5.19) and x belongs to the set (5.18) by (5.20). □

In the "if" part of the proof we proved that for each $A \in \mathbf{A}$ and $b \in \mathbf{b}$ the equation $Ax = b$ has a solution in the set $\mathrm{Conv}\{x_y^1 - x_y^2;\ y \in Y_m\}$. The proof, relying on Farkas lemma, was purely existential. It is worth noting, however, that such a solution can be found in a constructive way when using an algorithm described in [85]. For its description we need a special order of elements of Y_m defined inductively via the sets Y_j, $j = 1, \ldots, m-1$, in the following way:

(i) the order of Y_1 is $-1, 1$,

(ii) if $y_1, \ldots, y_{2^j}$ is the order of Y_j, then $(y_1, -1), \ldots, (y_{2^j}, -1), (y_1, 1), \ldots, (y_{2^j}, 1)$ is the order of Y_{j+1}.

Further, for a sequence $z_1, \ldots, z_{2h}$ with an even number of elements, each pair z_j, z_{j+h} is called a conjugate pair, $j = 1, \ldots, h$. As in Theorem ..., let for each $y \in Y_m$, x_y^1 and x_y^2 be a solution to ..., ... Then the algorithm runs as follows:

1. Select $A \in \mathbf{A}$ and $b \in \mathbf{b}$.
2. Form a sequence of vectors $((x_{-y}^1 - x_{-y}^2)^T, (A(x_{-y}^1 - x_{-y}^2) - b)^T)^T$ ordered in the order of the y's in Y_m.
3. For each conjugate pair x, x' in the current sequence compute

$$\lambda = \begin{cases} x'_k/(x'_k - x_k) & \text{if } x'_k \neq x_k, \\ 1 & \text{otherwise,} \end{cases}$$

   where k is the index of the current last entry, and set

$$x := \lambda x + (1 - \lambda)x'.$$

4. Cancel the second part of the sequence and in the remaining part delete the last entry of each vector.
5. If there remains a single vector x, terminate: x solves $Ax = b$ and $x \in \text{Conv}\{x_y^1 - x_y^2;\ y \in Y_m\}$. Otherwise go to Step 3.

The algorithm starts with 2^m vectors $((x_{-y}^1 - x_{-y}^2)^T, (A(x_{-y}^1 - x_{-y}^2) - b)^T)^T \in \mathbb{R}^{n+m}$, $y \in Y_m$, and proceeds by halving the sequence and deleting the last entry, hence it is finite and at the end it produces a single vector $x \in \mathbb{R}^n$. The assertion made in Step 5 is a consequence of Theorem 2 in [85] because we have

$$T_y A x_y \geq T_y b$$

for each $y \in Y_m$, hence also

$$T_y A x_{-y} \leq T_y b$$

for each $y \in Y_m$, which is the form used in [85].

1.3.12 Complexity

1.3.12.1 Basic notions of the complexity theory

An algorithm is called a polynomial–time algorithm if there exists a polynomial p such that for each instance (input data) of length ℓ the number of steps of the algorithm is $\leq p(\ell)$. Length: number of bits of the input. Consequence: only rational data allowed (usually represented by pairs of integers). Example: modified Gaussian elimination[2] [2].

Decision ("yes or no") problems are considered in complexity theory. (So that problem setting consists of an *instance*, i.e. the set of allowed data, and a *question*, see e.g. Theorem 21 below.) A problem belongs to the class P if it is solvable by a polynomial–time algorithm, and to the class NP if a *guessed*[3] candidate for a solution can be verified by a polynomial–time algorithm.

A problem I can be reduced in polynomial time to problem J, which we denote by $I \to J$, if there exists a polynomial–time algorithm π which transforms each instance i of I to an instance $\pi(i)$ of J so that the answer to i is "yes" if and only if the answer to $\pi(i)$ is "yes" (or, the answer to i is "yes" if and only if the answer to $\pi(i)$ is "no"). Hence, if $I \to J$, then each algorithm for solving J may be employed for solving I; consequently, J is "at least as difficult" as I.

A problem J is called NP–hard if $I \to J$ for each $I \in NP$. If, moreover, J itself belongs to NP, then it is called NP–complete. An NP–complete problem exists (Cook [6]; thousands of them have been found since). If J is NP-hard and the problem J' formed from J by *negating* its question is in NP, then J is called co-NP-complete.[4]

Method of proving NP–hardness/NP–completeness: if J is NP–hard and $J \to K$, then K is NP–hard; if, additionally, $K \in NP$, then K is NP-complete. (Hence, one must have an "appropriate" problem NP-hard/NP–complete problem J at hand; two such problems are presented below.)

Computing the value of

$$\max_{x \in X} f(x)$$

is said to be NP–hard if the decision problem

"is $f(x) \geq r$ for some $x \in X$?"

[2] Gaussian elimination itself is not a polynomial-time algorithm; it must be modified. But this *can* be done, so that computing determinants, solving linear systems and inverting matrices can be performed in polynomial time.

[3] You may view it as presented by an "oracle".

[4] For explanation, look at the problem formulation in Theorem 41. Assume we have already proved that the problem is NP-hard. The negated question is "does $[A_c - E, A_c + E]$ contain a singular matrix?". If we are given a (guessed candidate) matrix $A \in [A_c - E, A_c + E]$, we can verify its singularity in polynomial time by checking $\det(A) = 0$. Hence the problem J' is in NP and thus the original problem of Theorem 41 is co-NP-complete, which is a less loose - and thus preferable - property than NP-hardness.

is NP–hard (r rational).

If *some* NP–hard problem can be solved by a polynomial–time algorithm, then *all* problems in NP are solvable by polynomial–time algorithms (due to the definition of NP-completeness). This would imply $P = NP$. However, *no* such problem (or algorithm) is known to date, and it is widely believed (but not proved) that

$$P \neq NP.$$

Hence, if this conjecture is true, then no NP–hard/NP–complete problem can be solved by a polynomial–time algorithm.

This abridged description comes from [94]. For more details, see Garey and Johnson [13].

1.3.12.2 Two basic NP-complete problems

Definition. A square matrix $A = (a_{ij})$ is called an M-matrix if $a_{ij} \leq 0$ for $i \neq j$ and $A^{-1} \geq 0$.

Theorem 21. *The following decision problem is NP-complete:*
Instance. *A symmetric rational M-matrix $A \in \mathbb{R}^{n\times n}$ with $\|A\|_1 \leq 2n-1$.*
Question. *Is $\|A\|_{\infty,1} \geq 1$?*

Proof: [90], [97]. □

Theorem 22. *The problem of checking whether a system of inequalities*

$$-e \leq Ax \leq e, \tag{1.33}$$

$$e^T|x| \geq 1 \tag{1.34}$$

has a solution is NP-complete in the set of nonnegative positive definite rational matrices.

Comment. Clearly, $e^T|x| = \|x\|_1$, so that the inequality (1.34) could be equivalently written as $\|x\|_1 \geq 1$. We prefer, however, the formulation given because terms of the form $e^T|x|$ arise quite naturally in the analysis of complexity of interval linear problems.

Proof: Given a symmetric rational M-matrix $A \in \mathbb{R}^{n\times n}$, consider the system

$$-e \leq A^{-1}x \leq e, \tag{1.35}$$

$$e^T|x| \geq 1 \tag{1.36}$$

which can be constructed in polynomial time since the same is true for A^{-1} (see Bareiss [2]). Since A is positive definite ([23], p. 114, assertion 2.5.3.3), A^{-1} is rational nonnegative positive definite. Obviously, the system (1.35), (1.36) has a solution if and only if

$$\begin{aligned} 1 &\leq \max\{e^T|x|;\, -e \leq A^{-1}x \leq e\} = \max\{e^T|Ax'|;\, -e \leq x' \leq e\} \\ &= \max\{\|Ax'\|_1;\, -e \leq x' \leq e\} = \max\{\|Ay\|_1;\, y \in Y_n\} = \|A\|_{\infty,1} \end{aligned}$$

holds, since the function $\|Ax'\|_1$ is convex over the unit cube $\{x';\, -e \leq x' \leq e\}$ and therefore its maximum is attained at one of its vertices which are just the vectors in Y_n. Summing up, we have shown that $\|A\|_{\infty,1} \geq 1$ holds if and only if the system (1.35), (1.36) has a solution. Since the former problem is NP-complete (Theorem **??**), the latter one is NP-hard, hence also the problem (1.33), (1.34) is NP-hard. Moreover, if (1.33), (1.34) has a solution, then, as we have seen, it also has a rational solution of the form $x = A^{-1}y$ for some $y \in Y_n$, and verification whether x solves (1.33), (1.34) can be performed in polynomial time. Hence the problem of checking solvability of (1.33), (1.34) belongs to the class NP and therefore it is NP-complete. □

1.3.13 Interval arithmetic

Operations over closed real intervals are defined by the general rule

$$[\underline{a}, \overline{a}] \circ [\underline{b}, \overline{b}] = \{\alpha \circ \beta;\ \alpha \in [\underline{a}, \overline{a}],\ \beta \in [\underline{b}, \overline{b}]\},$$

where

$$\circ \in \{+, -, \cdot, /\}$$

and division is defined only if $0 \notin [\underline{b}, \overline{b}]$.

Explicitly:

$$\begin{aligned} [\underline{a}, \overline{a}] + [\underline{b}, \overline{b}] &= [\underline{a} + \underline{b}, \overline{a} + \overline{b}], \\ [\underline{a}, \overline{a}] - [\underline{b}, \overline{b}] &= [\underline{a} - \overline{b}, \overline{a} - \underline{b}], \\ [\underline{a}, \overline{a}] \cdot [\underline{b}, \overline{b}] &= [\min M, \max M], \\ [\underline{a}, \overline{a}] / [\underline{b}, \overline{b}] &= [\min N, \max N], \end{aligned}$$

where

$$\begin{aligned} M &= \{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}, \\ N &= \{\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b}\}. \end{aligned}$$

The basic property of interval arithmetic consists in the fact that if $\alpha \in [\underline{a}, \overline{a}]$ and $\beta \in [\underline{b}, \overline{b}]$, then $\alpha \circ \beta \in [\underline{a}, \overline{a}] \circ [\underline{b}, \overline{b}]$.

See Alefeld and Herzberger [1] or Neumaier [55] for further properties of the interval arithmetic.

Chapter 2

Basic tools

2.1 Introduction

2.2 Existence lemma

Lemma 23. *Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and let for each $y \in Y_m$ the inequality*

$$T_y A x \geq T_y b \tag{2.1}$$

have a solution x_y. Then the equation

$$Ax = b$$

has a solution in the set

$$\mathrm{Conv}\{x_y;\, y \in Y_m\}. \tag{2.2}$$

Proof: First proved in [73]. Journal publications: existencial proof (from Farkas lemma) in [83], constructive proof in [85]. □

The ith inequality in (5.17) has the form $(Ax)_i \geq b_i$ if $y_i = 1$ and is of the form $(Ax)_i \leq b_i$ if $y_i = -1$. Thus, if we start from a system $Ax = b$ and if we "relax" each equation $(Ax)_i = b_i$ to either $(Ax)_i \geq b_i$ or $(Ax)_i \leq b_i$, we obtain as a result just the family of all systems (5.17) for $y \in Y_m$ (hence, 2^m of them). The lemma says that (a) if each of these systems of inequalities has a solution, then $Ax = b$ has a solution, (b) at least one of solutions of $Ax = b$ belongs to the set (5.18).

The existence lemma plays for interval linear systems a similar role as the Farkas lemma does for noninterval ones. It is vital for the proof of the crucial convex hull theorem 154, and is also used in the proofs of Theorems 34 (several of the assertions) and 209.

Proof: We shall prove that the system of linear equations

$$\sum_{y\in Y_m} \lambda_y A x_y = b, \tag{2.3}$$

$$\sum_{y\in Y_m} \lambda_y = 1 \tag{2.4}$$

has a solution $\lambda_y \geq 0$, $y \in Y_m$. In view of the Farkas lemma, it suffices to show that for each $p \in \mathbb{R}^m$ and each $p_0 \in \mathbb{R}$,

$$p^T A x_y + p_0 \geq 0 \text{ for each } y \in Y_m \tag{2.5}$$

implies

$$p^T b + p_0 \geq 0. \tag{2.6}$$

Thus let p and p_0 satisfy (5.53). Put $y = -\operatorname{sgn} p$, then $p = -T_y|p|$ and from (5.17), (5.53) we have

$$p^T b + p_0 = -|p|^T T_y b + p_0 \geq -|p|^T T_y A x_y + p_0 = p^T A x_y + p_0 \geq 0,$$

which proves (5.54). Hence the system (5.51), (5.52) has a solution $\lambda_y \geq 0$, $y \in Y_m$. Put $x = \sum_{y \in Y_m} \lambda_y x_y$, then $Ax = b$ by (5.51) and x belongs to the set $\mathrm{Conv}\{x_y;\, y \in Y_m\} = \mathrm{Conv}\{x_y^1 - x_y^2;\, y \in Y_m\}$ by (5.52). □

2.3 Regularity of interval matrices

Definition. A square interval matrix $\mathbf{A}$ is called *regular* if each $A \in \mathbf{A}$ is nonsingular, and *singular* in the opposite case (i.e., if $\mathbf{A}$ contains a singular matrix).

Regularity is a very strong property and implies rich consequences; the whole long Section 3.3 is dedicated to it. Here we present only a basic characterization used in the proofs of theorems of this chapter. For these purposes it is better to formulate it in terms of singularity instead of regularity:

Theorem 24. *A square interval matrix* $\mathbf{A}$ *is singular if and only if the inequality*

$$|A_c x| \le \Delta |x|$$

has a nontrivial solution.

Proof: A consequence of the Oettli-Prager theorem 139 for zero right-hand side; for a direct elementary proof see e.g. [92]. □

2.4 Regularity lemma

Lemma 25. *If* $\mathbf{A}$ *is regular, then for each* $A_1, A_2 \in \mathbf{A}$ *both* $A_1^{-1}A_2$ *and* $A_1A_2^{-1}$ *are* P*-matrices.*

Proof: First proved in [70], by elementary means; first journal publication in [81] (another proof using the Fiedler-Pták theorem 14); also [92] (still another proof, also using the Fiedler-Pták theorem; preferable). □

This lemma forms a bridge between interval analysis, linear algebra and linear complementarity theory and as such, as all the bridge-type theorems, is important because it yields a deeper insight. It is used in one of the two proofs of the key Theorem 26 - that one which is shorter, more evident, but only existential.

Proof: Assume to the contrary that $A_1^{-1}A_2$ is not a P–matrix for some $A_1, A_2 \in \mathbf{A} = [A_c - \Delta, A_c + \Delta]$. Then according to the Fiedler–Pták theorem (Theorem 14) there exists an $x \neq 0$ such that $x_i(A_1^{-1}A_2x)_i \leq 0$ for each i. Put $x' = A_1^{-1}A_2x$, then

$$x_i x_i' \leq 0 \quad (i = 1, \ldots, n) \tag{2.7}$$

and

$$x \neq x' \tag{2.8}$$

holds. In fact, since $x \neq 0$, there exists a j with $x_j \neq 0$; then $x_j^2 > 0$ whereas (2.7) implies $x_j x_j' \leq 0$, hence $x_j \neq x_j'$. Now we have

$$|A_c(x' - x)| = |(A_c - A_1)x' + (A_2 - A_c)x| \leq \Delta|x'| + \Delta|x| = \Delta|x' - x| \tag{2.9}$$

since $|x'| + |x| = |x' - x|$ due to (2.7). Hence Theorem 24, (ii) in the light of (2.9) and (2.8) implies that $\mathbf{A}$ is singular, which is a contradiction. □

2.5 The equation $Ax + B|x| = b$: The key theorem

Theorem 26. *Let $A, B \in \mathbb{R}^{n\times n}$. If the interval matrix $[A - |B|, A + |B|]$ is regular, then for each right-hand side $b \in \mathbb{R}^n$ the equation*

$$Ax + B|x| = b \tag{2.10}$$

has a unique solution.

This is a fundamental result which because of its nature (it asserts existence of some uniquely defined objects) plays a key role throughout the text, mainly in the proofs of the convex hull theorem 154, existence and uniqueness of the matrices Q_z (Theorem 57), several assertions of Theorem 34, etc. So far it has been published in interval setting only (in the form of Theorem 154; first published in [70], journal publication in [81]). But the present setting should be preferred, since in this form it is another "bridge-type" result. We give here a short proof relying heavily on previous theorems; a constructive form of it will be given in the next section.

Proof: Using $x = x^+ - x^-$, $|x| = x^+ + x^-$ (see p. 10), we can write (2.10) in the form

$$(A + B)x^+ - (A - B)x^- = b. \tag{2.11}$$

Since both $A + B$ and $A - B$ belong to $[A - |B|, A + |B|]$, we can rewrite (2.11) as

$$x^+ = (A + B)^{-1}(A - B)x^- + (A + B)^{-1}b,$$

thereby obtaining a linear complementarity problem with a P-matrix $(A+B)^{-1}(A-B)$ (Lemma 25), which has a unique solution by Theorem 15. □

Corollary 27. *If $[A - |B|, A + |B|]$ is regular, then for each C with $|C| \le |B|$ and for each $b \in \mathbb{R}^n$ the equation*

$$Ax + C|x| = b \tag{2.12}$$

has a unique solution.

Proof: $[A - |C|, A + |C|] \subseteq [A - |B|, A + |B|]$; then Theorem 26 applies. □

There is a natural question what can be said if the matrix $[A - |B|, A + |B|]$ in Theorem 26 is singular. In this case we can state the following (cf. Subsection 3.3.10):

Theorem 28. *If $[A - |B|, A + |B|]$ is singular, then there exist $d \in [0, 1]$ and[1] $y \in Y_n$ such that the equation*

$$Ax + dT_y|B||x| = 0 \tag{2.13}$$

has a nontrivial solution (and, consequently, infinitely many of them).

[1]For T_y, see p. 13.

Proof: [86]. □

Observe that $|dT_y|B|| \leq |B|$, so that (2.13) is of the form (2.12).

2.6 The sign accord algorithm

Lemma 29. *Let $[A-|B|, A+|B|]$ be regular and let*

$$(A + BT_{z'})x' = (A + BT_{z''})x'' \tag{2.14}$$

hold for some $z', z'' \in Y_n$ and $x' \neq x''$. Then there exists a j satisfying $z'_j z''_j = -1$ and $x'_j x''_j > 0$.

Proof: Assume to the contrary that for each j, $z'_j z''_j = -1$ implies $x'_j x''_j \leq 0$ and hence also $|x'_j - x''_j| = |x'_j| + |x''_j|$. We shall prove that in this case

$$|T_{z'}x' - T_{z''}x''| \leq |x' - x''|, \tag{2.15}$$

i.e. that

$$|z'_j x'_j - z''_j x''_j| \leq |x'_j - x''_j|$$

holds for each j. Since $|z'_j x'_j - z''_j x''_j| = |z'_j(x'_j - z'_j z''_j x''_j)| = |x'_j - z'_j z''_j x''_j|$, this fact is obvious for $z'_j z''_j = 1$. If $z'_j z''_j = -1$, then

$$|z'_j x'_j - z''_j x''_j| = |x'_j + x''_j| \leq |x'_j| + |x''_j| = |x'_j - x''_j|,$$

which together proves (2.15). Now, from (2.14) we have

$$|A(x' - x'')| = |B(T_{z'}x' - T_{z''}x'')| \leq |B||T_{z'}x' - T_{z''}x''| \leq |B||x' - x''|$$

due to (2.15), where $x' - x'' \neq 0$, hence $[A - |B|, A + |B|]$ is singular by Theorem 24, a contradiction. □

Theorem 30. *For each $A, B \in \mathbb{R}^{n\times n}$ and each $b \in \mathbb{R}^n$, the sign accord algorithm (Fig. 2.2) in a finite number of steps either finds a solution of the equation*

$$Ax + B|x| = b, \tag{2.16}$$

or states singularity of the interval matrix $[A - |B|, A + |B|]$ (and, in certain cases, finds a singular matrix $A_s \in [A - |B|, A + |B|]$).

Comment. For better understandability, we shall describe the basic idea behind the sign accord algorithm (Fig. 2.2). If we knew the sign vector $z = \operatorname{sgn} x$ of the solution x of (2.16), we could rewrite (2.16) as $(A + BT_z)x = b$ and solve it for x as $x = (A + BT_z)^{-1}b$. The problem is, we know neither x, nor z; but we do know that they should satisfy $T_z x = |x| \geq 0$, i.e., $z_j x_j \geq 0$ for each j (a situation we call a sign accord of z and x). In its kernel form the sign accord algorithm computes the z's and x's repeatedly until a sign accord occurs: A combinatorial argument (parts 3.1 and 3.2 of the proof) based on Lemma 29 is used to prove that in case of regularity of

$z = \text{sgn}\,(A^{-1}b)$;
$x = (A + BT_z)^{-1}b$;
while $z_j x_j < 0$ for some j
  $k = \min\{j\,;\, z_j x_j < 0\}$;
  $z_k = -z_k$;
  $x = (A + BT_z)^{-1}b$;
end

Figure 2.1: The kernel of the sign accord algorithm.

$[A - |B|, A + |B|]$, a sign accord is achieved within a specified finite number of steps, so that crossing this number indicates singularity of $[A - |B|, A + |B|]$.

Proof: The proof consists of several steps.

1. Finiteness. The algorithm starts with the vector $p = 0$ and during each pass through the **while** loop it increases some p_k by 1. This means that after a finite number of steps p_k will become greater than 2^{n-k} for some k, and the algorithm will terminate in the fourth **if** statement[2] (if not earlier).

2. Simplification. Next we shall simplify the description of the algorithm by proving by induction that after each updating of C, the current values of z, x and C satisfy

$$x = (A + BT_z)^{-1}b, \tag{2.17}$$
$$C = -(A + BT_z)^{-1}B. \tag{2.18}$$

This is obviously so for the initial values of z, x and C. Thus let (2.17), (2.18) hold true at some step. Then for each real t the matrix

$$A + B(T_z - 2tz_k e_k e_k^T) = A + BT_z - (2tz_k Be_k)e_k^T$$

is a rank one update of the matrix $A + BT_z$, which is nonsingular by the induction hypothesis because (2.17) holds, hence by the well-known formula we have

$$\begin{aligned}\det(A + B(T_z - 2tz_k e_k e_k^T)) &= (1 - 2tz_k e_k^T (A + BT_z)^{-1} Be_k) \det(A + BT_z) \\ &= (1 + 2tz_k C_{kk}) \det(A + BT_z). \qquad (2.19)\end{aligned}$$

Now two possibilities may occur.

2.1. The case of $1 + 2z_k C_{kk} \leq 0$. Then the real function $\varphi(t) = 1 + 2tz_k C_{kk}$ satisfies $\varphi(0)\varphi(1) = 1 + 2z_k C_{kk} \leq 0$, hence $\varphi(\tau) = 0$ for $\tau = (-1)/(2z_k C_{kk}) \in [0, 1]$ and

$$\det(A + B(T_z - 2\tau z_k e_k e_k^T)) = 0.$$

[2] We prefer to write the condition as $\log_2 p_k > n - k$ instead of $p_k > 2^{n-k}$ to avoid a possibly large number 2^{n-k}.

function $[x, flag, A_s] =$ **signaccord** (A, B, b)
% Finds a solution to $Ax + B|x| = b$ or states
% singularity of $[A - |B|, A + |B|]$.
$x = [\,]$; $flag = \text{'}singular\text{'}$; $A_s = [\,]$;
if A is singular, $A_s = A$; **return**, **end**
$p = 0 \in \mathbb{R}^n$;
$x = A^{-1}b$;
$z = \operatorname{sgn} x$;
if $A + BT_z$ is singular, $A_s = A + BT_z$; **return**, **end**
$x = (A + BT_z)^{-1}b$;
$C = -(A + BT_z)^{-1}B$;
while $z_j x_j < 0$ for some j
    $k = \min\{j\,;\, z_j x_j < 0\}$;
    if $1 + 2z_k C_{kk} \le 0$
        $\tau = (-1)/(2z_k C_{kk})$;
        $A_s = A + B(T_z - 2\tau z_k e_k e_k^T)$;
        $x = [\,]$;
        return
    end
    $p_k = p_k + 1$;
    $z_k = -z_k$;
    if $\log_2 p_k > n - k$, $x = [\,]$; **return**, **end**
    $\alpha = 2z_k/(1 - 2z_k C_{kk})$;
    $x = x + \alpha x_k C_{\bullet k}$;
    $C = C + \alpha C_{\bullet k} C_{k \bullet}$;
end
$flag = \text{'}solution\ found\text{'}$;

Figure 2.2: The sign accord algorithm.

Because of $\tau \in [0,1]$ we have $|T_z - 2\tau z_k e_k e_k^T| \le I$, so that the matrix $A + B(T_z - 2\tau z_k e_k e_k^T)$ belongs to $[A - |B|, A + |B|]$ and is singular. This is the case of the first **if** statement in the **while** loop. In this case the algorithm terminates with a singular matrix $A_s = A + B(T_z - 2\tau z_k e_k e_k^T) \in [A - |B|, A + |B|]$.

2.2. The case of $1 + 2z_k C_{kk} > 0$. Here the first **if** statement of the **while** loop is not in effect and provided this is also the case of the second one, the algorithm constructs the updated values $\widetilde{z}$, $\widetilde{x}$ and $\widetilde{C}$ along the formulae

$$\begin{aligned} \widetilde{z}_k &= -z_k, \\ \alpha &= 2\widetilde{z}_k/(1 - 2\widetilde{z}_k C_{kk}) = -2z_k/(1 + 2z_k C_{kk}), \\ \widetilde{x} &= x + \alpha x_k C_{\bullet k}, \\ \widetilde{C} &= C + \alpha C_{\bullet k} C_{k \bullet}. \end{aligned}$$

Then the matrix

$$A + BT_{\widetilde{z}} = A + B(T_z - 2z_k e_k e_k^T) = A + BT_z - (2z_k B e_k) e_k^T$$

is nonsingular due to (2.19) (with $t = 1$), hence by the Sherman-Morrison formula there holds

$$\begin{aligned} (A + BT_{\widetilde{z}})^{-1} &= (A + BT_z)^{-1} + \frac{(A + BT_z)^{-1} 2z_k B e_k e_k^T (A + BT_z)^{-1}}{1 + 2z_k C_{kk}} \\ &= (A + BT_z)^{-1} + \alpha C_{\bullet k} e_k^T (A + BT_z)^{-1}. \end{aligned}$$

Then we have

$$(A + BT_{\widetilde{z}})^{-1} b = (A + BT_z)^{-1} b + \alpha C_{\bullet k} e_k^T (A + BT_z)^{-1} b = x + \alpha x_k C_{\bullet k} = \widetilde{x}$$

and

$$-(A + BT_{\widetilde{z}})^{-1} B = -(A + BT_z)^{-1} B - \alpha C_{\bullet k} e_k^T (A + BT_z)^{-1} B = C + \alpha C_{\bullet k} C_{k \bullet} = \widetilde{C},$$

which proves (2.17), (2.18) by induction. Hence we can see that the matrix C plays a purely auxiliary role, helping to avoid an explicit computation of $x = (A + BT_z)^{-1} b$ at each step.

3. Termination. If the condition of the **while** loop is not satisfied at some step, then $z_j x_j \ge 0$ for each j, hence $T_z x \ge 0$, so that $T_z x = |x|$. Because $x = (A + BT_z)^{-1} b$ by (2.17), we have that $Ax + B|x| = (A + BT_z)x = b$, so that x solves the equation (2.16). Next there are four possible terminations in the four **if** statements. In the first three of them singularity is clearly detected (this is obvious with the first two of them, and the fact that the matrix A_s constructed in the third **if** statement is singular has been proved in part 2.1). Thus it remains to be shown that if the condition of the fourth **if** statement is satisfied, i.e., if $\log_2 p_k > n - k$ for some k, then $[A - |B|, A + |B|]$

is singular. This will be proved if we demonstrate that if $[A-|B|, A+|B|]$ is regular, then

$$p_k \leq 2^{n-k} \tag{2.20}$$

holds throughout the algorithm for each k, which will exclude the possibility of $\log_2 p_k > n-k$. Thus let $[A-|B|, A+|B|]$ be regular, and consider the sequence of k's generated in the **while** loop of the algorithm. We shall prove by induction that each k can appear there at most 2^{n-k} times ($k = n, \ldots, 1$).

3.1. Case $k = n$. Assume that n appears at least twice in the sequence, and let z', x' and z'', x'' correspond to any two nearest occurrences of it (i.e., there is no other occurrence of n between them). Then $z'_j x'_j \geq 0$, $z''_j x''_j \geq 0$ for $j = 1, \ldots, n-1$, and $z'_n x'_n < 0$, $z''_n x''_n < 0$, $z'_n z''_n = -1$, which implies $z'_n x'_n z''_n x''_n > 0$ and $x'_n x''_n < 0$. Hence, $z'_j x'_j z''_j x''_j \geq 0$ for each j. But since

$$(A + BT_{z'})x' = b = (A + BT_{z''})x'' \tag{2.21}$$

holds due to (2.17) and $x' \neq x''$ (because $x'_n x''_n < 0$), it follows from Lemma 29 that there exists a j with $z'_j z''_j = -1$ and $x'_j x''_j > 0$, implying $z'_j x'_j z''_j x''_j < 0$, a contradiction; hence n occurs at most once in the sequence.

3.2. Case $k < n$. Again, let z', x' and z'', x'' correspond to any two nearest occurrences of k, so that $z'_j x'_j \geq 0$, $z''_j x''_j \geq 0$ for $j = 1, \ldots, k-1$, $z'_k x'_k < 0$, $z''_k x''_k < 0$ and $z'_k z''_k = -1$. This implies that $z'_j x'_j z''_j x''_j \geq 0$ for $j = 1, \ldots, k-1$, $z'_k x'_k z''_k x''_k > 0$ and $x'_k x''_k < 0$. Since (2.21) holds due to (2.17), and $x' \neq x''$ because of $x'_k x''_k < 0$, Lemma 29 implies existence of a j with $z'_j z''_j = -1$ and $x'_j x''_j > 0$, hence $z'_j x'_j z''_j x''_j < 0$, so that $j > k$. Since $z'_j z''_j = -1$, j must have entered the sequence between the two occurrences of k. Hence between any two nearest occurrences of k there is an occurrence of some $j > k$ in the sequence; this means by induction hypothesis that k cannot occur there more than $(2^{n-k-1} + \ldots + 2 + 1) + 1 = 2^{n-k}$ times.

3.3. Conclusion. We have proved that in case of regularity (2.20) holds for each k, hence a situation of $\log_2 p_k > n-k$ indicates singularity of $[A-|B|, A+|B|]$. This justifies the last possible termination, and thereby also the whole algorithm. □

Theorem 31. *If the interval matrix $[A-|B|, A+|B|]$ is regular, then for each right-hand side b the unique solution (Theorem 26) of the equation*

$$Ax + B|x| = b$$

can be found by the sign accord algorithm (Fig. 2.2) in a finite number of steps.

Proof: Let $b \in \mathbb{R}^n$. Since $[A-|B|, A+|B|]$ is regular, the sign accord algorithm cannot state singularity, hence according to Theorem 30 it finds in a finite number of steps a solution of the equation (2.10). To prove uniqueness, assume to the contrary that (2.10) has solutions x' and x'', $x' \neq x''$. Put $z' = \operatorname{sgn} x'$, $z'' = \operatorname{sgn} x''$, then

$T_{z'}x' \geq 0$, $T_{z''}x'' \geq 0$ and $(A + BT_{z'})x' = b = (A + BT_{z''})x''$ holds, hence by Lemma 29 there exists a j with $z'_j z''_j = -1$ and $x'_j x''_j > 0$, implying $z'_j x'_j z''_j x''_j < 0$ contrary to $z'_j x'_j \geq 0$ and $z''_j x''_j \geq 0$, a contradiction. Hence the solution of (2.10) is unique. $\square$

Chapter 3

Interval matrices

3.1 Introduction

3.2 Interval matrices

3.2.1 Basic notations for interval matrices

There are several ways how to express inexactness of the data. One of them, which has particularly nice properties from the point of view of a user, employs the so-called interval matrices which we are going to define in this section.

If $\underline{A}$, $\overline{A}$ are two matrices in $\mathbb{R}^{m\times n}$, $\underline{A} \le \overline{A}$, then the set of matrices

$$\mathbf{A} = [\underline{A}, \overline{A}] = \{A;\ \underline{A} \le A \le \overline{A}\}$$

is called an interval matrix, and the matrices $\underline{A}$, $\overline{A}$ are called its bounds. Hence, if $\underline{A} = (\underline{a}_{ij})$ and $\overline{A} = (\overline{a}_{ij})$, then $\mathbf{A}$ is the set of all matrices $A = (a_{ij})$ satisfying

$$\underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij} \tag{3.1}$$

for $i = 1, \ldots, m$, $j = 1, \ldots, n$. It is worth noting that each coefficient may attain any value in its interval (3.1) independently of the values taken on by other coefficients. Introducing additional relations among different coefficients makes interval problems much more difficult to solve and we shall not follow this line in this chapter.

As it will be seen later, in many cases it is more advantageous to express the data in terms of the center matrix

$$A_c = \tfrac{1}{2}(\underline{A} + \overline{A}) \tag{3.2}$$

and of the radius matrix

$$\Delta = \tfrac{1}{2}(\overline{A} - \underline{A}), \tag{3.3}$$

which is always nonnegative. From (3.2), (3.3) we easily obtain that

$$\underline{A} = A_c - \Delta,$$

$$\overline{A} = A_c + \Delta,$$

so that $\mathbf{A}$ can be given either as $[\underline{A}, \overline{A}]$, or as $[A_c - \Delta, A_c + \Delta]$, and consequently we can also write

$$\mathbf{A} = \{A;\ |A - A_c| \le \Delta\}.$$

In the sequel *we shall employ both forms and we shall switch freely between them* according to which one will be more useful in the current context. The following proposition is the first example of usefulness of the center-radius notation:

Proposition 32 *Let* $\tilde{\mathbf{A}} = [\tilde{A}_c - \tilde{\Delta}, \tilde{A}_c + \tilde{\Delta}]$ *and* $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ *be interval matrices of the same size. Then* $\tilde{\mathbf{A}} \subseteq \mathbf{A}$ *if and only if*

$$|A_c - \tilde{A}_c| \le \Delta - \tilde{\Delta}$$

holds.

Proof: If $\tilde{\mathbf{A}} \subseteq \mathbf{A}$, then from

$$A_c - \Delta \le \tilde{A}_c - \tilde{\Delta} \le \tilde{A}_c + \tilde{\Delta} \le A_c + \Delta \tag{3.4}$$

we obtain

$$-(\Delta - \tilde{\Delta}) \le A_c - \tilde{A}_c \le \Delta - \tilde{\Delta}, \tag{3.5}$$

which gives

$$|A_c - \tilde{A}_c| \le \Delta - \tilde{\Delta}. \tag{3.6}$$

Conversely, (3.6) implies (3.5) and (3.4), hence $\tilde{\mathbf{A}} \subseteq \mathbf{A}$. □

For an interval matrix $\mathbf{A} = [\underline{A}, \overline{A}] = [A_c - \Delta, A_c + \Delta]$, its transpose is defined by $\mathbf{A}^T = \{A^T;\ A \in \mathbf{A}\}$. Obviously, $\mathbf{A}^T = [\underline{A}^T, \overline{A}^T] = [A_c^T - \Delta^T, A_c^T + \Delta^T]$.

A special case of an interval matrix is an interval vector which is a one-column interval matrix

$$\mathbf{b} = \{b;\ \underline{b} \le b \le \overline{b}\},$$

where $\underline{b}, \overline{b} \in \mathbb{R}^m$. We shall again use the center vector

$$b_c = \tfrac{1}{2}(\underline{b} + \overline{b})$$

and the nonnegative radius vector

$$\delta = \tfrac{1}{2}(\overline{b} - \underline{b}),$$

and we shall employ both forms $\mathbf{b} = [\underline{b}, \overline{b}] = [b_c - \delta, b_c + \delta]$. Notice that interval matrices and vectors are typeset in boldface letters.

3.2.2 The matrices A_{yz}

Given an $m \times n$ interval matrix $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$, we define matrices

$$A_{yz} = A_c - T_y \Delta T_z \tag{3.7}$$

for each $y \in Y_m$ and $z \in Y_n$ (T_y is given by (1.1)). The definition implies that

$$(A_{yz})_{ij} = (A_c)_{ij} - y_i \Delta_{ij} z_j = \begin{cases} \overline{a}_{ij} & \text{if } y_i z_j = -1, \\ \underline{a}_{ij} & \text{if } y_i z_j = 1 \end{cases}$$

($i = 1, \ldots, m$, $j = 1, \ldots, n$), so that $A_{yz} \in \mathbf{A}$ for each $y \in Y_m$, $z \in Y_n$. This finite set of matrices from $\mathbf{A}$ (of cardinality at most 2^{m+n-1} because $A_{yz} = A_{-y,-z}$ for each $y \in Y_m$, $z \in Y_n$), introduced in [81], plays an important role because it turns out that many problems with interval-valued data can be characterized in terms of these matrices, thereby obtaining finite characterizations of problems involving infinitely many sets of data. In theorems to follow we shall see several examples of this approach, the most striking one being Theorem 209. We shall write A_{-yz} instead of $A_{-y,z}$. In particular, we have $A_{ye} = A_c - T_y \Delta$, $A_{ez} = A_c - \Delta T_z$, $A_{ee} = \underline{A}$ and $A_{-ee} = \overline{A}$.

Theorem 33. *For an $m \times n$ interval matrix $\mathbf{A}$, there are at most 2^{m+n-1} mutually different matrices A_{yz}, and this upper bound is attained if $\Delta > 0$.*

For an m-dimensional interval vector $\mathbf{b} = [b_c - \delta, b_c + \delta]$, in analogy with matrices A_{yz} we define vectors

$$b_y = b_c + T_y \delta$$

for each $y \in Y_m$. Then for each such a y we have

$$(b_y)_i = (b_c)_i + y_i \delta_i = \begin{cases} \underline{b}_i & \text{if } y_i = -1, \\ \overline{b}_i & \text{if } y_i = 1 \end{cases}$$

($i = 1, \ldots, m$), so that $b_y \in \mathbf{b}$ for each $y \in Y_m$. In particular, $b_{-e} = \underline{b}$ and $b_e = \overline{b}$. Together with matrices A_{yz}, vectors b_y are used in finite characterizations of interval problems having right-hand sides.

3.3 Regularity

3.3.1 Definition

Definition. A square interval matrix $\mathbf{A}$ is called regular if each $A \in \mathbf{A}$ is nonsingular, and it is said to be singular otherwise (i.e., if it contains a singular matrix).

3.3.2 Necessary and sufficient conditions

Theorem 34. *For a square interval matrix* $\mathbf{A}$*, the following assertions are equivalent:*

(i) $\mathbf{A}$ *is regular,*

(ii) the inequality

$$|A_c x| \le \Delta |x|$$

has only the trivial solution $x = 0$*,*

(iii) $(\det A_c)(\det A_{yz}) > 0$ *for each* $y, z \in Y_n$*,*

(iv) $(\det A_{yz})(\det A_{y'z'}) > 0$ *for each* $y, z, y', z' \in Y_n$*,*

(v) $(\det A_{yz})(\det A_{yz'}) > 0$ *for each* $y, z, z' \in Y_n$ *such that* z *and* z' *differ in exactly one entry,*

(vi) A_c *is nonsingular and*

$$\max_{y,z \in Y_n} \varrho_0(A_c^{-1} T_y \Delta T_z) < 1$$

holds,

(vii) for each $y \in Y_n$ *the system*

$$A_{ye} x_1 - A_{-ye} x_2 = y,$$

$$x_1 \ge 0, \; x_2 \ge 0$$

has a solution,

(viii) for each $y \in Y_n$ *the equation*

$$A_c x - T_y \Delta |x| = y$$

has a solution,

(ix) for each $y \in Y_n$ *the equation*

$$A_c x - T_y \Delta |x| = y$$

has a unique solution,

(x) for each $z \in Y_n$ *the matrix equation*

$$Q A_c - |Q| \Delta T_z = I$$

has a solution,

(xi) for each $z \in Y_n$ *the matrix equation*

$$Q A_c - |Q| \Delta T_z = I$$

has a unique solution,

(xii) for each $y \in Y_n$ *the equation*

$$|A_c x| = \Delta |x| + e$$

has a solution x_y *satisfying* $A_c x_y \in \mathbb{R}^n_y$*,* *(xiii)* *for each* $y \in Y_n$ *the equation*

$$|A_c x| = \Delta|x| + e$$

has a unique solution x_y *satisfying* $A_c x_y \in \mathbb{R}^n_y$,

(xiv) *for each* $y \in Y_n$ *the inequality*

$$\Delta|x| < |A_c x|$$

has a solution satisfying $A_c x \in \mathbb{R}^n_y$,

(xv) A_c *is nonsingular and the equation*

$$|x| = \Delta|A_c^{-1} x| + e$$

has a solution in each orthant,

(xvi) A_c *is nonsingular and the equation*

$$|x| = \Delta|A_c^{-1} x| + e$$

has a unique solution in each orthant,

(xvii) A_c *is nonsingular and the inequality*

$$\Delta|A_c^{-1} x| < |x|$$

has a solution in each orthant,

(xviii) *there exists an* $R \in \mathbb{R}^{n \times n}$ *such that the inequality*

$$|(I - A_c R)x| + \Delta|Rx| < |x|$$

has a solution in each orthant,

(xix) *for each* $y \in Y_n$, A_{ye} *is nonsingular and* $A_{ye}^{-1} A_{-ye}$ *is a* P*-matrix,*

(xx) *for each* $y \in Y_n$, A_{ye} *is nonsingular and the system*

$$A_{ye}^{-1} A_{-ye} x > 0, \ \ x > 0$$

has a solution,

(xxi) *for each* $y \in Y_n$, A_{ye} *and* A_{-ye} *are nonsingular and the system*

$$A_{ye}^{-1} x > 0, \ \ A_{-ye}^{-1} x > 0$$

has a solution,

(xxii) *for each* $y \in Y_n$, A_{ye} *is nonsingular and the system*

$$|A_c^{-1} T_y \Delta x| < x$$

has a solution,

(xxiii) *for each* $y, z \in Y_n$, A_{yz} *is nonsingular and*

$$(A_c A_{yz}^{-1})_{ii} > \tfrac{1}{2}$$

holds for each $i \in \{1, \ldots, n\}$.

3.3.3 Nontrivial consequences of regularity

Lemma 35. (Regularity lemma) *Let* $\mathbf{A}$ *be regular and let* $A'x' = A''x''$ *hold for some* $A', A'' \in \mathbf{A}$ *and* $x' \neq x''$*. Then there exists a* j *such that* $A'_{.j} \neq A''_{.j}$ *and* $x'_j x''_j > 0$*.*

Proof: Assume to the contrary that $A'x' = A''x''$ holds for some $A', A'' \in \mathbf{A}$ and $x' \neq x''$ such that for each j, either $A'_{.j} = A''_{.j}$, or $x'_j x''_j \leq 0$. Put $J = \{j;\ x'_j x''_j \leq 0\}$, then for each $j \in J$ we have $|x'_j| + |x''_j| = |x'_j - x''_j|$, and

$$\begin{aligned} A_c(x' - x'') &= (A_c - A')x' + (A'' - A_c)x'' = \sum_{j \notin J} (A_c - A')_{.j}(x'_j - x''_j) \\ &+ \sum_{j \in J} (A_c - A')_{.j} x'_j + \sum_{j \in J} (A'' - A_c)_{.j} x''_j, \end{aligned}$$

which implies

$$\begin{aligned} |A_c(x' - x'')| &\leq \sum_{j \notin J} \Delta_{.j} |x'_j - x''_j| + \sum_{j \in J} \Delta_{.j} (|x'_j| + |x''_j|) \\ &= \sum_{j \notin J} \Delta_{.j} |x'_j - x''_j| + \sum_{j \in J} \Delta_{.j} |x'_j - x''_j| \\ &= \Delta |x' - x''|, \end{aligned}$$

hence $\mathbf{A}$ is singular due to the Oettli-Prager theorem, which is a contradiction. □

Corollary 36 *Let* $\mathbf{A}$ *be regular and let*

$$A_{yz'} x' = A_{yz''} x''$$

hold for some $y, z', z'' \in Y$ *and* $x' \neq x''$*. Then there exists a* j *satisfying*

$$z'_j z''_j = -1$$

and

$$x'_j x''_j > 0.$$

Proof: Under the assumptions it follows from Lemma 35 that there exists a j with $(A_{yz'})_{.j} \neq (A_{yz''})_{.j}$ and $x'_j x''_j > 0$. Since $(A_{yz'})_{.j} - (A_{yz''})_{.j} = (z''_j - z'_j) T_y \Delta_{.j} \neq 0$, it must be $z'_j \neq z''_j$, hence $z'_j z''_j = -1$. □

Theorem 37. *Let* $\mathbf{A}$ *be regular. Then:*

(i) $A_1^{-1} A_2$ *and* $A_1 A_2^{-1}$ *are P-matrices for each* $A_1, A_2 \in \mathbf{A}$*,*

(ii) *for each* $A_1, A_2 \in \mathbf{A}$ *and for each* $y \in Y_n$ *there exist vectors* x_1, x_2 *satisfying* $A_1x_1 = A_2x_2$, $T_yx_1 > 0$ *and* $T_yx_2 > 0$,

(iii) $(A_cA^{-1})_{ii} > \frac{1}{2}$ *for each* $A \in \mathbf{A}$ *and each* $i \in \{1, \ldots, n\}$,

(iv) $\mathbf{A}x_1 \not\subseteq \mathbf{A}x_2$ *for each* $x_1 \neq x_2$,

(v) *for each* $a > 0$ *the equation*

$$|x| = \Delta|A_c^{-1}x| + a$$

has a unique solution in each orthant.

Comment. In (iv), $\mathbf{A}x = \{Ax\,;\, A \in \mathbf{A}\}$.

3.3.4 Special case: Rank one radius matrix

Theorem 38. *Let $A_c \in \mathbb{R}^{n\times n}$ be nonsingular and let p, q be nonnegative vectors in $\mathbb{R}^n$. Then the interval matrix*

$$[A_c - pq^T, A_c + pq^T]$$

is regular if and only if

$$\|T_q A_c^{-1} T_p\|_{\infty,1} < 1$$

holds.

Theorem 39. *Let $\mathbf{A} = [A_c - pq^T, A_c + pq^T]$, where A_c is nonsingular, $p \geq 0$ and $q \geq 0$. If $\mathbf{A}$ is singular, then for each $z, y \in Y_n$ satisfying*

$$z^T T_q A_c^{-1} T_p y \geq 1 \tag{3.8}$$

(which exist due to Theorem 38 and the formula (1.3)) the matrix

$$A = A_c - \frac{T_y p q^T T_z}{z^T T_q A_c^{-1} T_p y}$$

is a singular matrix in $\mathbf{A}$.

Proof: Because of (3.8) we have

$$|A - A_c| \leq |T_y p q^T T_z| = pq^T,$$

so that $A \in \mathbf{A}$. Next,

$$A(A_c^{-1} T_y p) = T_y p - \frac{T_y p (q^T T_z A_c^{-1} T_y p)}{z^T T_q A_c^{-1} T_p y} = T_y p - T_y p = 0,$$

hence A is singular since $A_c^{-1} T_y p \neq 0$ in view of $p \neq 0$. □

3.3.5 Co-NP-completeness

Proposition 40. *If a rational interval matrix* $\mathbf{A}$ *is singular, then it contains a rational singular matrix.*

Theorem 41. *The following problem is co-NP-complete:*
Instance. *A nonnegative symmetric positive definite rational matrix* A_c.
Question. *Is* $[A_c - E, A_c + E]$ *regular?*

Proof: [92], pp. 9-10. □

3.3.6 Sufficient regularity conditions I: Strong regularity

Theorem 42. *If A_c is nonsingular and*

$$\varrho(|A_c^{-1}|\Delta) < 1 \tag{3.9}$$

holds, then $\mathbf{A}$ *is regular.*

Definition. An interval matrix $\mathbf{A}$ having nonsingular A_c and satisfying the condition (3.9) is called strongly regular.

Proof: For each $A \in \mathbf{A}$ we have

$$\varrho(A_c^{-1}(A_c - A)) \leq \varrho(|A_c^{-1}(A_c - A)|) \leq \varrho(|A_c^{-1}|\Delta) < 1.$$

Hence by Theorem 5 the matrix

$$I - A_c^{-1}(A_c - A) = A_c^{-1}A$$

is invertible and thus nonsingular. Then A is nonsingular, and $\mathbf{A}$ is regular. □

The condition (3.9) can be verified in polynomial time since it is equivalent to

$$(I - |A_c^{-1}|\Delta)^{-1} \geq 0.$$

In his recent papers [103], [102], Rump proved that each regular $n \times n$ interval matrix $[A_c - \Delta, A_c + \Delta]$ satisfies

$$\varrho(|A_c^{-1}|\Delta) < (3 + 2\sqrt{2})n,$$

and that for each $n \geq 1$ there exists a regular $n \times n$ interval matrix such that

$$\varrho(|A_c^{-1}|\Delta) > n - 1.$$

These facts help to clarify the strength of the sufficient condition (3.9).

Theorem 43. *For an interval matrix* $\mathbf{A}$*, the following assertions are equivalent:*

(i) $\mathbf{A}$ *is strongly regular,*
(ii) $(I - |A_c^{-1}|\Delta)^{-1} \geq I$*,*
(iii) there exists a matrix R satisfying

$$\varrho(|I - RA_c| + |R|\Delta) < 1, \tag{3.10}$$

(iv) there exist matrices $M \geq 0$ and R satisfying

$$M(I - |I - RA_c| - |R|\Delta) \geq I. \tag{3.11}$$

Moreover, in cases (ii), (iii) R is nonsingular and in case (iii) we have

$$A^{-1} \in [R - (M - I)|R|,\ R + (M - I)|R|]$$

for each $A \in \mathbf{A}$.

Proof: We shall prove (i)⇒(ii)⇒(iii)⇒(ii)⇒(i), and in the frame of the proof we shall utilize several times the properties of interval matrices with spectral radius less than one stated in Theorems **??** and **??**. Denote $G = |I - RA_c| + |R|\Delta$, so that $G \geq 0$.

(i)⇒(ii): If (3.33) holds, then (3.34) is satisfied with $R = A_c^{-1}$.

(ii)⇒(iii): If (3.34) holds, then $\varrho(G) < 1$, hence $M = (I - G)^{-1} \geq 0$ and with this M, (5.76) is satisfied as equation.

(iii)⇒(ii): Let (5.76) hold. Then $M(I - G) \geq I$, which can be written as

$$I + MG \leq M.$$

Postmultiplying this inequality by G and adding I to both sides, we obtain

$$I + G + MG^2 \leq I + MG \leq M$$

and by induction

$$\sum_{j=0}^{k} G^j + MG^{k+1} \leq M$$

for $k = 0, 1, \ldots$. This shows that the nonnegative matrix series $\sum_{j=0}^{\infty} G^j$ satisfies

$$\sum_{j=0}^{\infty} G^j \leq M, \tag{3.12}$$

hence it is convergent and consequently $\varrho(G) < 1$, which is (3.34).

(ii)⇒(i): Let (3.34) hold for some R. Then we have

$$I - RA_c \leq |I - RA_c| \leq G,$$

hence

$$\varrho(I - RA_c) \leq \varrho(|I - RA_c|) \leq \varrho(G) < 1. \tag{3.13}$$

Since $\varrho(I - RA_c) < 1$, the matrix

$$RA_c = I - (I - RA_c) \tag{3.14}$$

is nonsingular, which gives that both A_c and R are nonsingular. Moreover, (5.78) implies that

$$A_c^{-1}R^{-1} = (RA_c)^{-1} = \sum_{j=0}^{\infty}(I - RA_c)^j,$$

hence

$$A_c^{-1} = \sum_{j=0}^{\infty} (I - RA_c)^j R,$$

and thus also

$$|A_c^{-1}| \le \sum_{j=0}^{\infty} |I - RA_c|^j |R| = (I - |I - RA_c|)^{-1} |R|$$

(because $\sum_{j=0}^{\infty} |I - RA_c|^j$ is again convergent by (5.77)), and

$$|A_c^{-1}|\Delta \le (I - |I - RA_c|)^{-1} |R| \Delta. \tag{3.15}$$

Since $\varrho(G) < 1$, Theorem **??** implies existence of an $x > 0$ satisfying $Gx < x$, i.e.,

$$|I - RA_c|x + |R|\Delta x < x,$$

hence

$$|R|\Delta x < (I - |I - RA_c|)x$$

and

$$(I - |I - RA_c|)^{-1} |R| \Delta x < x \tag{3.16}$$

in view of (5.77). Now, from (5.79) and (3.38) we finally obtain

$$|A_c^{-1}|\Delta x \le (I - |I - RA_c|)^{-1} |R| \Delta x < x,$$

where $x > 0$, hence $\varrho(|A_c^{-1}|\Delta) < 1$ by Theorem **??**, which completes the proof of (ii)⇒(i), and thus also of the mutual equivalence of (i), (ii) and (iii).

To prove the remaining two assertions, take an $A \in \mathbf{A}$. Then it satisfies the identity

$$RA = I - (I - RA_c + R(A_c - A)) \tag{3.17}$$

and since

$$I - RA_c + R(A_c - A) \le |I - RA_c| + |R|\Delta = G,$$

there holds

$$\varrho(I - RA_c + R(A_c - A)) < 1,$$

so that (3.17) shows that RA is nonsingular, hence A is nonsingular. This proves that $\mathbf{A}$ is regular. Moreover, from (3.17) it follows

$$A^{-1}R^{-1} = \sum_{j=0}^{\infty} (I - RA_c + R(A_c - A))^j,$$

hence

$$A^{-1} = \sum_{j=0}^{\infty} (I - RA_c + R(A_c - A))^j R$$

and

$$|A^{-1} - R| \leq \sum_{j=1}^{\infty}(|I - RA_c| + |R|\Delta)^j|R| = (\sum_{j=0}^{\infty} G^j - I)|R| \leq (M - I)|R|$$

by (3.12), which proves (**??**). □

Theorem 44. *If* $\mathbf{A}$ *is strongly regular and* $\tilde{\mathbf{A}} \subseteq \mathbf{A}$*, then* $\tilde{\mathbf{A}}$ *is strongly regular as well.*

Proposition 45 *Let* $T_z A_c^{-1} T_y \geq 0$ *for some* $z, y \in Y_n$*. Then* $\mathbf{A}$ *is regular if and only if (3.9) holds.*

Corollary 46. *If* $A_c^{-1} \geq 0$*, then* $\mathbf{A}$ *is regular if and only if* $\varrho(A_c^{-1}\Delta) < 1$*.*

3.3.7 How strong is strong regularity?

$$|A_c^{-1}T_y\Delta x| < x \tag{3.18}$$

Theorem 47. *Let* $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ *be a square interval matrix with* A_c *nonsingular. Then* $\mathbf{A}$ *is*

- *regular if and only if for each* $y \in Y_n$*,* A_{ye} *is nonsingular and the inequality (3.18) has a solution,*
- *strongly regular if and only if all the inequalities (3.18),* $y \in Y_n$*, have a solution in common.*

3.3.8 Sufficient regularity conditions II: Other conditions

Theorem 48. *If the matrix*

$$A_c^T A_c - \sigma_{\max}(\Delta) I$$

is positive definite, then $\mathbf{A}$ *is regular.*

Theorem 49. *If an interval matrix* $\mathbf{A}$ *satisfies*

$$\sigma_{\max}(\Delta) < \sigma_{\min}(A_c),$$

then it is regular.

3.3.9 Sufficient singularity conditions

Theorem 50. *If*

$$\max_j (|A_c^{-1}|\Delta)_{jj} \geq 1,$$

then $\mathbf{A}$ *is singular.*

Theorem 51. *Let*

$$(I + |I - RA_c|)_{i\cdot} \leq (|R|\Delta)_{i\cdot}$$

hold for some R *and some* $i \in \{1, \ldots, n\}$. *Then* $\mathbf{A}$ *is singular.*

Theorem 52. *If the matrix*

$$\Delta^T \Delta - A_c^T A_c$$

is positive semidefinite, then $\mathbf{A}$ *is singular.*

Theorem 53. *If an interval matrix* $\mathbf{A}$ *satisfies*

$$\sigma_{\max}(A_c) \leq \sigma_{\min}(\Delta),$$

then it is singular.

3.3.10 Normal forms of a singular matrix

Theorem 54. *Let* $\mathbf{A}$ *be singular. Then there exist* $y, z \in Y_n$ *and* $x \neq 0$, $p \neq 0$ *such that*

$$\begin{aligned}(A_c - dT_y \Delta T_z)x &= 0,\\ (A_c - dT_y \Delta T_z)^T p &= 0,\\ T_z x &\geq 0,\\ T_y p &\geq 0\end{aligned}$$

hold, where

$$d = \min\{\varepsilon \geq 0;\ [A_c - \varepsilon\Delta, A_c + \varepsilon\Delta] \text{ is singular}\} \in [0, 1].$$

Corollary 55. *If* $\mathbf{A}$ *is singular, then it contains a singular matrix of the form*

$$A_c - dT_y \Delta T_z$$

for some $d \in [0, 1]$ *and* $y, z \in Y_n$.

Theorem 56. *If* $\mathbf{A}$ *is singular, then it contains a singular matrix* A *of the form*

$$A_{ij} \in \begin{cases} \{\underline{A}_{ij}, \overline{A}_{ij}\} & \text{if } (i,j) \neq (k,\ell),\\ [\underline{A}_{ij}, \overline{A}_{ij}] & \text{if } (i,j) = (k,\ell) \end{cases}$$

for some k, ℓ.

3.3.11 The matrices Q_z

Theorem 57. *Let* $\mathbf{A}$ *be regular. Then for each* $z \in Y_n$ *the equation*

$$QA_c - |Q|\Delta T_z = I$$

has a unique matrix solution Q_z.

```
function [Q_z, flag] = qzmatrix (A, z)
for i = 1 : n
    [x, flag] = signaccord (A_c^T, -T_z Δ^T, e_i);
    if flag = 'singular', Q_z = [ ]; return
    end
    (Q_z)_{i•} = x^T;
end
flag = 'Q_z computed';
```

Figure 3.1: An algorithm for computing Q_z.

Proof: If $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ is regular, then it transpose $\mathbf{A}^T = [A_c^T - \Delta^T, A_c^T + \Delta^T]$ is also regular, hence the sign accord algorithm when applied to it is finite and the procedure described in the theorem yields for each $z \in Y_n$ a matrix Q_z satisfying

$$A_c^T (Q_z^T)_{\cdot i} - T_z \Delta^T |(Q_z^T)_{\cdot i}| = e_i \tag{3.19}$$

for each i, hence

$$A_c^T Q_z^T - T_z \Delta^T |Q_z^T| = I$$

and

$$Q_z A_c - |Q_z| \Delta T_z = I.$$

Uniqueness of Q_z follows from the uniqueness of solution of the equation (4.38) stated in Theorem 154. □

3.3.12 An iterative method for computing Q_z

Theorem 58. *Let $M \geq 0$ and R satisfy the strong regularity condition*

$$M(I - |I - RA_c| - |R|\Delta) \geq I.$$

Then for each $z \in Y_n$ the sequence generated by $Q^0 = R$ and

$$Q^{i+1} = Q^i(I - A_cR) + |Q^i|\Delta T_z R + R$$

$(i = 0, 1, 2, \ldots)$ *converges to Q_z and for each $i \geq 0$ we have*

...

$$|\overline{x}_z - (Q^i b_c + |Q^i|\delta)| \leq |Q^{i+1} - Q^i| q,$$

where ⇒

$$q = M(|b_c| + \delta).$$

3.3.13 An explicit formula for Q_z for the case $A_c = I$

Theorem 59. *An interval matrix* $[I - \Delta, I + \Delta]$ *is regular if and only if* $\varrho(\Delta) < 1$.

Theorem 60. *Let* $A_c = I$ *and* $\varrho(\Delta) < 1$. *Then for each* $z \in Y_n$ *the matrix* Q_z *is given by*

$$Q_z = T_z M T_z + (I - T_z) T_\lambda (M - I) T_z,$$

where

$$M = (I - \Delta)^{-1} = (m_{ij})$$

and

$$\lambda_i = \frac{m_{ii}}{2m_{ii} - 1} \qquad (i = 1, \dots, n),$$

or componentwise

$$(Q_z)_{ij} = \frac{(z_j + (1 - z_j) I_{ij}) m_{ij}}{z_i + (1 - z_i) m_{ii}} = \begin{cases} m_{ij} z_j & \text{if } z_i = 1, \\ (2\lambda_i - 1) m_{ij} z_j & \text{if } z_i = -1 \text{ and } j \neq i, \\ \lambda_i & \text{if } z_i = -1 \text{ and } j = i. \end{cases}$$

$(i, j = 1, \dots, n)$.

Proposition 61. *Under the assumptions and notations of the previous theorem, we have for each* $z \in Y_n$,

$$|Q_z| = (T_z + (I - T_z) T_\lambda) M,$$

or componentwise

$$|Q_z|_{ij} = (z_i + (1 - z_i)\lambda_i) m_{ij} = \begin{cases} m_{ij} & \text{if } z_i = 1, \\ (2\lambda_i - 1) m_{ij} & \text{if } z_i = -1 \end{cases}$$

$(i, j = 1, \dots, n)$.

3.3.14 A general algorithm for checking regularity

function $flag =$ **regularity** $(\mathbf{A})$
 if A_c is singular, $flag =$ *'singular'*; **return**, **end**
 $R = A_c^{-1}$;
 if $\varrho(|R|\Delta) < 1$, $flag =$ *'regular'*; **return**, **end**
 if $\max_j(\Delta|R|)_{jj} \geq 1$, $flag =$ *'singular'*; **return**, **end**
 $b = e$; $\gamma = \min_k |Rb|_k$;
 for $i = 1 : n$
  for $j = 1 : n$
   $b' = b$; $b'_j = -b'_j$;
   if $\min_k |Rb'|_k > \gamma$, $\gamma = \min_k |Rb'|_k$; $b = b'$; **end**
  end
 end
 $[\underline{x}, \overline{x}, flag] =$ **hull** $(\mathbf{A}, [b, b])$;
 if $flag =$ *'hull computed'*, $flag =$ *'regular'*; **return**
 end

Figure 3.2: An algorithm for checking regularity.

3.3.15 Inverse matrix representation

Theorem 62. *Let* $\mathbf{A}$ *be regular. Then for each* $A \in \mathbf{A}$ *there exist nonnegative diagonal matrices* L_{yz}, $y, z \in Y_n$ *satisfying* $\sum_{y,z \in Y_n} L_{yz} = I$ *such that*

$$A^{-1} = \sum_{y,z \in Y_n} L_{yz} A_{yz}^{-1}$$

holds.

Proof: [69], p. 47; [87], p. 865 (there the result is given as $\sum_{y,z \in Y_n} A_{yz}^{-1} L_{yz}$; the present order[1] is obtained by applying the result to $\mathbf{A}^T$). □

Theorem 63. *Let* $\mathbf{A}$ *be regular. Then for each* $A \in \mathbf{A}$ *there exist nonnegative diagonal matrices* L_z, $z \in Y_n$ *satisfying* $\sum_{z \in Y_n} L_z = I$ *such that*

$$A^{-1} = \sum_{y,z \in Y_n} L_z Q_z$$

holds.

Proof: Unpublished, but proved in a similar way as Theorem 62 using the matrices Q_z instead of A_{yz}^{-1}. □

[1] Which seems more proper to me, indicating a kind of a convex combination.

3.3.16 Inverse stability

Definition. A regular interval matrix $\mathbf{A}$ is called inverse stable[2] if $|A^{-1}| > 0$ for each $A \in \mathbf{A}$.

Due the continuity of the determinant, this means that for each ij, either $(A^{-1})_{ij} < 0$ for each $A \in \mathbf{A}$, or $(A^{-1})_{ij} > 0$ for each $A \in \mathbf{A}$. Thus we can also say that inverse stability is equivalent to existence of a matrix Z, $|Z| = E$, such that $Z \circ A^{-1} > 0$ for each $A \in \mathbf{A}$.

Theorem 64. *$\mathbf{A}$ is inverse stable if and only if there exists a matrix Z, $|Z| = E$, such that $Z \circ A_{yz}^{-1} > 0$ for each $y, z \in Y_n$.*

Proof: [87], pp. 866-867. □

Theorem 65. *$\mathbf{A}$ is inverse stable if and only if there exists a matrix Z, $|Z| = E$, such that $Z \circ Q_z > 0$ for each $z \in Y_n$.*

Proof: Unpublished, but proved in a similar manner as Theorem 64. □

[2]Meant: inverse sign stable.

3.3.17 Inverse interval matrix

Definition. For a regular $\mathbf{A}$ we define $\mathbf{A}^{-1} = [\underline{B}, \overline{B}]$, where

$$\underline{B}_{ij} = \min\{(A^{-1})_{ij} \,;\, A \in \mathbf{A}\},$$

$$\overline{B}_{ij} = \max\{(A^{-1})_{ij} \,;\, A \in \mathbf{A}\}$$

$(i, j = 1, \ldots, n)$.

Theorem 66. *Let* $\mathbf{A}$ *be regular. Then for its inverse* $\mathbf{A}^{-1} = [\underline{B}, \overline{B}]$ *we have*

$$\underline{B} = \min_{z \in Y_n} Q_z = \min_{y,z \in Y_n} A_{yz}^{-1},$$

$$\overline{B} = \max_{z \in Y_n} Q_z = \max_{y,z \in Y_n} A_{yz}^{-1}.$$

Theorem 67. *Computing the inverse interval matrix is NP-hard in the class of symmetric strongly regular interval matrices with rational bounds.*

3.3.18 Explicit formulae for the special case of inverse stability

Definition. A regular interval matrix $\mathbf{A}$ is called inverse stable if for each $i, j \in \{1, \ldots, n\}$, either $(A^{-1})_{ij} > 0$ for each $A \in \mathbf{A}$, or $(A^{-1})_{ij} < 0$ for each $A \in \mathbf{A}$.

Theorem 68. *Let $\mathbf{A}$ be inverse stable. Then the coefficients of its inverse $\mathbf{A}^{-1} = [\underline{B}, \overline{B}]$ are given by the explicit formulae*

$$\begin{aligned} \underline{B}_{ij} &= (A^{-1}_{-y(i),z(j)})_{ij} \\ \overline{B}_{ij} &= (A^{-1}_{y(i)z(j)})_{ij} \end{aligned}$$

$(i, j = 1, \ldots, n)$, *where* $y(i) = \operatorname{sgn}(A_c^{-1})_{i\cdot}$ *and* $z(j) = \operatorname{sgn}(A_c^{-1})_{\cdot j}$ *for each* i, j.

Proof: [70], p. 28; [87], p. 868. □

3.3.19 Explicit formulae for the special case $A_c = I$

Theorem 69. *Let* $\mathbf{A} = [I - \Delta, I + \Delta]$ *with* $\varrho(\Delta) < 1$*. Then the inverse interval matrix* $(\mathbf{A})^{-1} = [\underline{B}, \overline{B}]$ *is given by*

$$\underline{B} = -M + 2T_\kappa, \tag{3.20}$$

$$\overline{B} = M, \tag{3.21}$$

where

$$\kappa_j = \frac{m_{jj}^2}{2m_{jj} - 1}$$

$(j = 1, \ldots, n)$*, or componentwise*

$$\underline{B}_{ij} = \begin{cases} -m_{ij} & \text{if } i \neq j, \\ \mu_j & \text{if } i = j \end{cases} \tag{3.22}$$

$$\overline{B}_{ij} = m_{ij} \tag{3.23}$$

$(i, j = 1, \ldots, n)$,

$$\mu_j = \frac{m_{jj}}{2m_{jj} - 1}.$$

3.3.20 Inverse sign pattern

Definition. $\mathbf{A}$ is said to be of inverse sign pattern (z, y) if there exist $z, y \in Y_n$ such that $T_z A^{-1} T_y \geq 0$ holds for each $A \in \mathbf{A}$. If $\mathbf{A}$ is of inverse sign pattern (e, e), then it is called inverse nonnegative.

Theorem 70. $\mathbf{A}$ *is of inverse sign pattern* (z, y) *if and only if*

$$T_z A_{yz}^{-1} T_y \geq 0, \tag{3.24}$$

$$T_z A_{-yz}^{-1} T_y \geq 0 \tag{3.25}$$

hold.

3.3.21 Nonnegative invertibility and M-matrices

Theorem 71. *For an interval matrix* $\mathbf{A}$*, the following assertions are equivalent:*

(i) $\mathbf{A}$ *is nonnegative invertible,*
(ii) $\underline{A}^{-1} \geq 0$ *and* $\overline{A}^{-1} \geq 0$*,*
(iii) $\overline{A}^{-1} \geq 0$ *and* $\varrho(\overline{A}^{-1}(\overline{A} - \underline{A})) < 1$*,*
(iv) $\overline{A}^{-1} \geq 0$ *and* $\mathbf{A}$ *is regular.*

Proposition 72. *If* $\mathbf{A}$ *is inverse nonnegative, then for each* $A \in \mathbf{A}$ *we have*

$$A^{-1} = \Big(\sum_{j=0}^{\infty} (\overline{A}^{-1}(\overline{A} - A))^j\Big)\overline{A}^{-1}.$$

3.3.22 The Hansen-Bliek-Rohn-type enclosure for inverse interval matrix

Theorem 73. *Let $M \geq 0$ and R be arbitrary matrices satisfying the strong regularity condition*

$$M(I - |I - RA_c| - |R|\Delta) \geq I.$$

Then for each $A \in \mathbf{A}$ we have

$$\min\{\underset{\sim}{B}, T_\nu \underset{\sim}{B}\} \leq A^{-1} \leq \max\{\widetilde{B}, T_\nu \widetilde{B}\},$$

where M, μ and T_ν are as in Theorem 189 and

$$\begin{aligned} \underset{\sim}{B} &= -M|R| + T_\mu(R + |R|), \\ \widetilde{B} &= \quad M|R| + T_\mu(R - |R|). \end{aligned}$$

Theorem 74. *Let $\mathbf{A}$ be strongly regular. Then for each $A \in \mathbf{A}$ we have*

$$\min\{\underset{\sim}{B}, T_\nu \underset{\sim}{B}\} \leq A^{-1} \leq \max\{\widetilde{B}, T_\nu \widetilde{B}\},$$

where M, μ and T_ν are as in Theorem 190 and

$$\begin{aligned} \underset{\sim}{B} &= -M|A_c^{-1}| + T_\mu(A_c^{-1} + |A_c^{-1}|), \\ \widetilde{B} &= \quad M|A_c^{-1}| + T_\mu(A_c^{-1} - |A_c^{-1}|). \end{aligned}$$

Proof: Since $(A^{-1})_{\cdot j}$ is the solution of the system $Ax = e_j$, we obtain the result simply by applying Theorem 190 to interval linear systems $\mathbf{A}x = [e_j, e_j]$ for $j = 1, \ldots, n$. $\square$

3.3.23 P-property

Theorem 75. *If* $\mathbf{A}$ *is regular, then* $A_1^{-1}A_2$ *is a* P*–matrix for each* $A_1, A_2 \in \mathbf{A}$*.*

Proof: Assume to the contrary that $A_1^{-1}A_2$ is not a P-matrix for some $A_1, A_2 \in \mathbf{A}$. Then according to the Fiedler-Pták theorem there exists an $x \neq 0$ such that $x_i(A_1^{-1}A_2x)_i \leq 0$ for each i. Take $x' = A_1^{-1}A_2x$, then $x_ix'_i \leq 0$ holds for each i, which implies that

$$|x'| + |x| = |x' - x|. \tag{3.26}$$

Now we have

$$|A_c(x' - x)| = |(A_c - A_1)x' + (A_2 - A_c)x| \leq \Delta|x'| + \Delta|x| = \Delta|x' - x|$$

due to (3.26) which also gives that $x' \neq x$ since $x' = x$ would imply $x = 0$ contrary to $x \neq 0$. Hence by the Oettli-Prager theorem there exists an $A \in \mathbf{A}$ with $A(x' - x) = 0$ which means that A is singular, a contradiction. □

An interval matrix $\mathbf{A}$ is called a P–matrix if each $A \in \mathbf{A}$ is a P–matrix. In this section we show that due to a close relationship between P–property and positive definiteness, the problem of checking P–property of interval matrices is NP–hard even in the symmetric case.

Theorem 76. $\mathbf{A}$ *is a* P*-matrix if and only if each* $A_{zz}, z \in Z$*, is a* P*–matrix.*

Proof: If $\mathbf{A}$ is a P–matrix, then each $A_{zz} \in \mathbf{A}$ is obviously also a P–matrix. Conversely, let each $A_{zz}, z \in Z$, be a P–matrix. Let $A \in \mathbf{A}$, $x \in \mathbb{R}^n$, $x \neq 0$, and let $z = \operatorname{sgn} x$. Since A_{zz} is a P–matrix, there exists an i with $x_i(A_{zz}x)_i > 0$, then we have $x_i(Ax)_i \geq x_i(A_{zz}x)_i > 0$ so that A is a P-matrix by the Fiedler-Pták theorem, hence $\mathbf{A}$ is a P–matrix. □

As quoted above, a symmetric matrix A is a P–matrix if and only if it is positive definite. The following result, although it sounds verbally alike, is not a trivial consequence of the previous statement since here nonsymmetric matrices may be involved.

Proposition 77. *A symmetric interval matrix* $\mathbf{A}$ *is a* P*–matrix if and only if it is positive definite.*

Proof: All the matrices A_{zz}, $z \in Z$ defined by (**??**) are symmetric for a symmetric interval matrix $\mathbf{A}$. Hence, $\mathbf{A}$ is a P–matrix if and only if each A_{zz}, $z \in Z$ is a P–matrix, which is the case if and only if each A_{zz}, $z \in Z$ is positive definite, and this is equivalent to positive definiteness of $\mathbf{A}$ (Theorem **??**). □

In the introduction to this section we explained that checking a symmetric matrix for P–property can be performed in polynomial time. Unless P≠NP, this is not more true for symmetric interval matrices):

Theorem 78. *The following problem is NP–hard:*

Instance. *A nonnegative symmetric rational P–matrix A.*

Question. *Is $[A-E, A+E]$ a P–matrix?*

Proof: [92], p. 20. □

Proof: Since A is symmetric positive definite, $[A-E, A+E]$ is a P–matrix if and only if it is positive definite (Proposition 77). Checking positive definiteness of this class of interval matrices was proved to be NP–hard in Theorem 107. □

3.3.24 Radius of nonsingularity: Definition and basic formulae

Definition. $d(A_c, \Delta) = \inf\{\varepsilon \geq 0; [A_c - \varepsilon\Delta, A_c + \varepsilon\Delta] \text{ is singular}\}$.
Convention $\frac{0}{0} = 0$, $\frac{a}{0} = \infty$ for $a > 0$.

Theorem 79. *For each square A_c, $\Delta \geq 0$ we have*

$$d(A_c, \Delta) = \inf_{x \neq 0} \max_i \frac{|A_c x|_i}{(\Delta|x|)_i} = \frac{1}{\max_{y,z \in Y_n} \varrho_0(A_c^{-1} T_y \Delta T_z)}, \tag{3.27}$$

the second formula assuming nonsingularity of A_c.

$$d(A) = d(A, E)$$

Proposition 80. *The following problem is NP–hard:*
Instance. *A nonnegative symmetric positive definite rational matrix A.*
Question. *Is $d(A) \leq 1$?*

Proof: [92], p. 14. □

Theorem 81. *Computing the radius of nonsingularity is NP-hard (even in the special case $\Delta = E$).*

Proof: [92], p. 14. □

Theorem 82. *Suppose there exists a polynomial–time algorithm which for each nonnegative symmetric positive definite rational matrix A computes a rational approximation $d'(A)$ of $d(A)$ satisfying*

$$\left| \frac{d'(A) - d(A)}{d(A)} \right| \leq \frac{1}{4n^2},$$

where n is the size of A. Then P=NP.

3.3.25 Radius of nonsingularity: Properties

Theorem 83. *The radius of nonsingularity has the following properties:*

(i) $0 \le d(A_c, \Delta) \le \infty$,
$d(A_c, \Delta) = 0$ *if and only if* A_c *is singular,*

(ii) $d(A_c, \Delta) > 0$ *if* $\Delta e > 0$,
$d(A_c, \Delta) < \infty$ *if* $\Delta e > 0$,

(iii) $d(A_c, \Delta) = \infty$ *if* A_c *is nonsingular and* $\Delta = 0$,

(iv) $d(A_c + B_c, \Delta) \le d(A_c, \Delta) + d(B_c, \Delta)$,

(v) $0 \le \Delta \le \Delta'$ *implies* $d(A_c, \Delta') \le d(A_c, \Delta)$,

(vi) $d(\alpha A_c, \beta\Delta) = \frac{|\alpha|}{\beta} d(A_c, \Delta)$ *for* $\alpha \in \mathbb{R}$ *and* $\beta > 0$,

(vii) $d(A_c, \Delta) = 1/\varrho(|A_c^{-1}|\Delta)$ *if* A_c *is nonsingular and* $\mathbf{A}$ *is of some inverse sign pattern* (z, y),

(viii) $d(A_c, pq^T) = 1/\|T_q A_c^{-1} T_p\|_{\infty,1}$ *if* A_c *is nonsingular and* $p \ge 0, p \ne 0, q \ge 0, q \ne 0$,
$[A_c - \Delta, A_c + \Delta]$ *is regular if and only if* $d(A_c, \Delta) > 1$.

Theorem 84. *If* $\mathbf{A}$ *is nonsingular, then*

$$\frac{1}{\varrho(|A_c^{-1}|\Delta)} \le d(A_c, \Delta) \le \frac{1}{\max_j (|A_c^{-1}|\Delta)_{jj}}.$$

3.3.26

3.3.27 Rump's results

3.4 Eigenvalues

3.4.1 Real eigenvalues

Theorem 85. *Let A_c have n simple real eigenvalues*

$$\lambda_1(A_c) < \lambda_2(A_c) < \ldots < \lambda_n(A_c)$$

and let there exist real numbers $\mu_0, \ldots, \mu_n$ satisfying

$$\mu_0 < \lambda_1(A_c) < \mu_1 < \lambda_2(A_c) < \mu_2 < \ldots < \lambda_n(A_c) < \mu_n \tag{3.28}$$

such that the interval matrix

$$[(A_c - \mu_j I) - \Delta, (A_c - \mu_j I) + \Delta] \tag{3.29}$$

is regular for $j = 0, \ldots, n$. Then each $A \in \mathbf{A}$ has n simple real eigenvalues satisfying

$$\mu_0 < \lambda_1(A) < \mu_1 < \lambda_2(A) < \mu_2 < \ldots < \lambda_n(A) < \mu_n. \tag{3.30}$$

Proof: For an $A \in \mathbf{A}$, let

$$p(\lambda) = \det(A - \lambda I)$$

denote its characteristic polynomial and let

$$p_c(\lambda) = \det(A_c - \lambda I)$$

be the characteristic polynomial of A_c. Then for each $j \in \{0, \ldots, n\}$ we have $|(A - \mu_j I) - (A_c - \mu_j I)| = |A - A_c| \leq \Delta$, hence

$$A - \mu_j I \in [A_c - \mu_j I - \Delta, A_c - \mu_j I + \Delta],$$

and regularity of (3.29) implies

$$p(\mu_j) p_c(\mu_j) > 0 \tag{3.31}$$

since $p(\mu_j) p_c(\mu_j) \leq 0$ would imply, by continuity of the determinant, existence of a singular matrix in (3.29), a contradiction. Now, since all eigenvalues of A_c are real and simple, (3.28) gives

$$p_c(\mu_j) p_c(\mu_{j+1}) < 0 \tag{3.32}$$

for $j = 0, \ldots, n-1$. For each such j we have from (3.31)

$$p(\mu_j) p_c(\mu_j) p(\mu_{j+1}) p_c(\mu_{j+1}) > 0,$$

which in view of (3.32) implies

$$p(\mu_j) p(\mu_{j+1}) < 0,$$

hence the characteristic polynomial of A has a root in each of the open intervals (μ_j, μ_{j+1}), $j = 0, \ldots, n-1$. This proves that A has exactly n simple real eigenvalues satisfying (3.30). □

Assumptions

(A1) Each $A \in \mathbf{A}$ has exactly m real eigenvalues $\lambda_1(A) < \ldots < \lambda_m(A)$, where $1 \leq m \leq n$. Hence we can define the sets $L_i = \{\lambda_i(A);\ A \in \mathbf{A}\}$, $i = 1, \ldots, m$.

(A2) $\overline{L_i} \cap \overline{L_j} = \emptyset$ for $i \neq j$, $i, j \in \{1, \ldots, m\}$.

(A3) For each $i \in \{1, \ldots, m\}$ there exist vectors $y_i, z_i \in Y_n$ such that each eigenvector x (left eigenvector p) pertaining to the ith real eigenvalue of some $A \in \mathbf{A}$ satisfies either $T_{z_i}x > 0$ or $T_{z_i}x < 0$ ($T_{y_i}p > 0$ or $T_{y_i}p < 0$, respectively).

Theorem 86. *Let an interval matrix* $\mathbf{A}$ *satisfy Assumptions (A1)-(A3). Then for each* $i \in \{1, \ldots, m\}$ *we have*

$$L_i = [\underline{\lambda}_i, \overline{\lambda}_i],$$

where

$$\underline{\lambda}_i = \min\{\lambda_i(A_{y_i z_i}), \lambda_i(A_{-y_i z_i})\},$$
$$\overline{\lambda}_i = \max\{\lambda_i(A_{y_i z_i}), \lambda_i(A_{-y_i z_i})\}.$$

Theorem 87. *Let* $\mathbf{A}$ *satisfy Assumptions (A1)-(A3) and let* $i \in \{1, \ldots, n\}$. *Then each* $\lambda \in L_i$ *is the* i*th real eigenvalue of some matrix belonging to the segment connecting* $A_{y_i z_i}$ *with* $A_{-y_i z_i}$.

Theorem 88. *Let a symmetric* $\mathbf{A}$ *satisfy Assumptions (A1)-(A3). Then each* $\lambda \in \bigcup_{i=1}^{m} L_i$ *is an eigenvalue of some symmetric matrix in* $\mathbf{A}$.

Theorem 89. *A nonzero real vector* x *is an eigenvector of some matrix in* $\mathbf{A}$ *if and only if it satisfies*

$$T_z A_{zz} x x^T T_z \leq (T_z A_{-zz} x x^T T_z)^T,$$

where $z = \operatorname{sgn} x$.

3.4.2 Real eigenvectors

3.4.3 Real eigenpairs

3.4.4 Perron vectors

In this paper we consider only square $n \times n$ interval matrices. Such a matrix A is called nonnegative if all its coefficients are nonnegative. A nonnegative matrix $A \in \mathbb{R}^{n\times n}$ is said to be reducible if there exists a permutation matrix P such that

$$P^T AP = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

where B and D are square matrices (i.e., at least of size 1×1), and it is called irreducible if it is not reducible. The basic eigenvalue properties of irreducible nonnegative matrices are summed up in the Perron-Frobenius theorem (see Horn and Johnson [22], p. 508). We formulate here only a portion of it relevant to the scope of this paper; $\varrho(A)$ denotes the spectral radius of A, $e = (1, 1, \dots, 1)^T \in \mathbb{R}^n$, and $x > 0$ means that all entries of x are positive.

Theorem 90. *For each irreducible nonnegative matrix A there exists a unique vector x satisfying*

$$Ax = \varrho(A)x, \quad (3.33)$$
$$e^T x = 1, \quad (3.34)$$
$$x > 0, \quad (3.35)$$

and no eigenvalue $\lambda \neq \varrho(A)$ has a positive eigenvector.

The positive eigenvector determined uniquely by (3.33)–(5.76) is called the Perron vector of A; we shall denote it by $x(A)$.

Given $\underline{A}$, $\overline{A} \in \mathbb{R}^{n\times n}$ with $\underline{A} \leq \overline{A}$, the set

$$\mathbf{A} = [\underline{A}, \overline{A}] = \{ A \mid \underline{A} \leq A \leq \overline{A} \}$$

is called an interval matrix with the bounds $\underline{A}$ and $\overline{A}$. $\mathbf{A}$ is said to be nonnegative if $\underline{A} \geq 0$, which is the same as to say that all matrices in $\mathbf{A}$ are nonnegative. A nonnegative interval matrix $\mathbf{A}$ is called irreducible if each $A \in \mathbf{A}$ is irreducible. It turns out that checking irreducibility of $\mathbf{A} = [\underline{A}, \overline{A}]$ reduces to checking this property for $\underline{A}$ only.

Proposition 91. *A nonnegative interval matrix $[\underline{A}, \overline{A}]$ is irreducible if and only if $\underline{A}$ is irreducible.*

Proof: If each $A \in [\underline{A}, \overline{A}]$ is irreducible, then so is $\underline{A}$. Conversely, assume that $\underline{A}$ is irreducible and that some $A \in [\underline{A}, \overline{A}]$ is reducible, so that there exists a permutation matrix P such that

$$P^T AP = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

where 0 is of size at least 1×1. Then from $0 \le \underline{A} \le A$ it follows

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \le P^T \underline{A} P = \begin{pmatrix} B_1 & C_1 \\ E_1 & D_1 \end{pmatrix} \le P^T A P = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

which implies that $E_1 = 0$, hence $\underline{A}$ is reducible. This contradiction shows that each $A \in [\underline{A}, \overline{A}]$ is irreducible, and the proof is complete. □

The set of spectral radii of all the matrices contained in an irreducible nonnegative interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$ is easy to describe:

$$\{ \varrho(A) \mid A \in \mathbf{A} \} = [\varrho(\underline{A}), \varrho(\overline{A})],$$

because the spectral radius is a continuous function of A (Horn and Johnson [22], p. 313), hence the real function $\psi(t) = \varrho(\underline{A} + t(\overline{A} - \underline{A}))$ is continuous in $[0, 1]$, so that it attains all the intermediate values between the endpoint values $\varrho(\underline{A})$ and $\varrho(\overline{A})$, and no spectral radius can exceed this interval because $0 \le \underline{A} \le A \le \overline{A}$ implies that $\varrho(\underline{A}) \le \varrho(A) \le \varrho(\overline{A})$ (Horn and Johnson [22], p. 491).

The following main result of this paper presents a description of the set $\{ x(A) \mid A \in \mathbf{A} \}$ of the Perron vectors of all matrices contained in a given irreducible nonnegative interval matrix $\mathbf{A}$.

Theorem 92. *Let $\mathbf{A} = [\underline{A}, \overline{A}]$ be an irreducible nonnegative interval matrix. Then a vector $x \in \mathbb{R}^n$ is the Perron vector of some matrix $A \in \mathbf{A}$ if and only if it satisfies*

$$\underline{A} x x^T \le x x^T \overline{A}^T, \tag{3.36}$$
$$e^T x = 1, \tag{3.37}$$
$$x > 0. \tag{3.38}$$

Proof: Let x be the Perron vector of some matrix $A \in [\underline{A}, \overline{A}]$, so that (3.33)–(5.76) hold. Then from $\underline{A} \le A \le \overline{A}$ in view of positivity of x we obtain

$$\underline{A} x \le A x = \varrho(A) x \le \overline{A} x,$$

hence for each $i, j = 1, \dots, n$ we have

$$\frac{(\underline{A} x)_i}{x_i} \le \varrho(A) \le \frac{(\overline{A} x)_j}{x_j}$$

and thus also

$$(\underline{A} x x^T)_{ij} = (\underline{A} x)_i x_j \le x_i (\overline{A} x)_j = (x x^T \overline{A}^T)_{ij},$$

which proves (5.78); (5.79) and (3.38) are given by (3.34), (5.76).

Conversely, let x satisfy (5.78)–(3.38). Then for each i, j we have

$$(\underline{A}x)_i x_j = (\underline{A}xx^T)_{ij} \leq (xx^T\overline{A}^T)_{ij} = x_i(\overline{A}x)_j,$$

hence

$$\frac{(\underline{A}x)_i}{x_i} \leq \frac{(\overline{A}x)_j}{x_j},$$

which implies that

$$\max_i \frac{(\underline{A}x)_i}{x_i} \leq \min_j \frac{(\overline{A}x)_j}{x_j}.$$

Let us choose any λ satisfying

$$\max_i \frac{(\underline{A}x)_i}{x_i} \leq \lambda \leq \min_j \frac{(\overline{A}x)_j}{x_j}.$$

Then from the first inequality it follows that $\underline{A}x \leq \lambda x$, whereas the second one gives $\lambda x \leq \overline{A}x$, together

$$\underline{A}x \leq \lambda x \leq \overline{A}x. \tag{3.39}$$

For each $i = 1, \ldots, n$ define a real function of one real variable t by

$$\varphi_i(t) = ((\underline{A} + t(\overline{A} - \underline{A}))x - \lambda x)_i.$$

Then $\varphi_i(0) = (\underline{A}x - \lambda x)_i \leq 0$ and $\varphi_i(1) = (\overline{A}x - \lambda x)_i \geq 0$ by (3.39), hence by continuity of φ_i there exists a $t_i \in [0, 1]$ such that $\varphi_i(t_i) = 0$. Now put

$$A = \underline{A} + \operatorname{diag}(t_1, \ldots, t_n)(\overline{A} - \underline{A})$$

(where $\operatorname{diag}(t_1, \ldots, t_n)$ denotes the diagonal matrix with diagonal entries $t_1, \ldots, t_n$), then $A \in [\underline{A}, \overline{A}]$ because $t_i \in [0, 1]$ for each i, and we have $(Ax - \lambda x)_i = \varphi_i(t_i) = 0$ for each i, hence

$$Ax = \lambda x.$$

Since $e^T x = 1$ and $x > 0$ by (5.79), (3.38), Theorem 90 gives that $\lambda = \varrho(A)$ and $x = x(A)$, hence x is the Perron vector of A, which proves the second implication. □

The inequality (5.78) could also be written in a more "symmetric" form

$$\underline{A}xx^T \leq (\overline{A}xx^T)^T,$$

but we prefer the form (5.78) which, as we have seen, arises naturally in the proof.

The construction given in the second part of the proof is worth summarizing as a separate assertion.

Theorem 93. *Let x satisfy (5.78)–(3.38). Then*

$$\max_i \frac{(\underline{A}x)_i}{x_i} \le \min_j \frac{(\overline{A}x)_j}{x_j} \tag{3.40}$$

and for each λ with

$$\max_i \frac{(\underline{A}x)_i}{x_i} \le \lambda \le \min_j \frac{(\overline{A}x)_j}{x_j} \tag{3.41}$$

there holds $\lambda = \varrho(A)$ and $x = x(A)$, where the matrix $A \in [\underline{A}, \overline{A}]$ is given by

$$A = \underline{A} + \operatorname{diag}(t_1, \ldots, t_n)(\overline{A} - \underline{A}), \tag{3.42}$$

with

$$t_i = \begin{cases} (\lambda x - \underline{A}x)_i / ((\overline{A} - \underline{A})x)_i & \text{if } ((\overline{A} - \underline{A})x)_i > 0, \\ 1 & \text{if } ((\overline{A} - \underline{A})x)_i = 0 \end{cases} \qquad (i = 1, \ldots, n). \tag{3.43}$$

Proof: As everything else has been stated in the proof of Theorem 92, it remains to explain the formula (3.43) for t_i only. This value is a solution of the equation $\varphi_i(t_i) = 0$, i.e., it satisfies

$$t_i((\overline{A} - \underline{A})x)_i = (\lambda x - \underline{A}x)_i. \tag{3.44}$$

If $((\overline{A} - \underline{A})x)_i > 0$, then this equation has the unique solution

$$t_i = \frac{(\lambda x - \underline{A}x)_i}{((\overline{A} - \underline{A})x)_i}.$$

If $((\overline{A}-\underline{A})x)_i = 0$, then, since we know from the proof of Theorem 92 that the equation (3.44) has a solution, it must be $(\lambda x - \underline{A}x)_i = 0$, hence the equation is satisfied for any $t_i \in \mathbb{R}$, thus also for our choice $t_i = 1$. □

In accordance with the construction made in (3.42), denote

$$\mathbf{A}_* = \{\, \underline{A} + T(\overline{A} - \underline{A}) \mid 0 \le T \le I \,\},$$

so that $\mathbf{A}_*$ is a subset of $\mathbf{A}$. Let us compare it with the description of $\mathbf{A}$ which can also be written as

$$\mathbf{A} = \{\, \underline{A} + T(\overline{A} - \underline{A}) \mid 0 \le T \le ee^T \}.$$

We can see that the description of $\mathbf{A}_*$ involves n "parameters" $t_{ii} \in [0, 1]$ $(i = 1, \ldots, n)$, whereas that of $\mathbf{A}$ contains n^2 "parameters" $t_{ij} \in [0, 1]$ $(i, j = 1, \ldots, n)$. Nevertheless, the following consequence of Theorem 93 shows that all the spectral radii and Perron vectors of $\mathbf{A}$ are attained over its subset $\mathbf{A}_*$.

Theorem 94. *Let* $\mathbf{A}$ *be an irreducible nonnegative interval matrix. Then for each* $A \in \mathbf{A}$ *there exists an* $A' \in \mathbf{A}_*$ *such that* $\varrho(A) = \varrho(A')$ *and* $x(A) = x(A')$.

Proof: Let $A \in \mathbf{A}$. Then $x = x(A)$ satisfies (5.78)–(3.38) by Theorem 92 and there holds

$$\varrho(A) = \frac{(Ax)_k}{x_k}$$

for each k, so that from $\underline{A} \leq A \leq \overline{A}$ it follows

$$\frac{(\underline{A}x)_k}{x_k} \leq \varrho(A) \leq \frac{(\overline{A}x)_k}{x_k}$$

for each k, hence $\lambda = \varrho(A)$ satisfies (3.41) and a direct application of Theorem 93 gives that $\varrho(A) = \varrho(A')$ and $x(A) = x(A')$, where A' is given by (3.42), (3.43) and thus belongs to $\mathbf{A}_*$. □

3.4.5 Symmetric matrices

Definition. An interval matrix $\mathbf{A}$ is called symmetric if $\mathbf{A}^T = \mathbf{A}$.

Proposition 95. *For an interval matrix $\mathbf{A}$, the following assertions are equivalent:*

(i) $\mathbf{A}$ *is symmetric,*
(ii) both $\underline{A}$ *and* $\overline{A}$ *are symmetric,*
(iii) both A_c *and* Δ *are symmetric.*

Proof: Unpublished, but evident. □

Theorem 96. *Let $\mathbf{A}$ be symmetric. Then for each symmetric $A \in \mathbf{A}$ there holds*

$$\lambda_i(A) \in [\lambda_i(A_c) - \varrho(\Delta), \lambda_i(A_c) + \varrho(\Delta)] \qquad (i = 1, \dots, n).$$

Proof: From the Wielandt-Hoffman theorem, see [15], pp. 395-396 (the proof of it reveals that the norm can be replaced by the spectral radius). □

Definition. For a symmetric interval matrix $\mathbf{A}$, define

$$\lambda_{\min}(\mathbf{A}) = \min\{\lambda\,;\, \lambda \text{ is an eigenvalue of some symmetric } A \in \mathbf{A}\},$$

$$\lambda_{\max}(\mathbf{A}) = \max\{\lambda\,;\, \lambda \text{ is an eigenvalue of some symmetric } A \in \mathbf{A}\}.$$

Theorem 97. *For a symmetric interval matrix $\mathbf{A}$ there holds*

$$\lambda_{\min}(\mathbf{A}) = \min_{\|x\|_2=1} (x^T A_c x - |x|^T \Delta |x|) = \min_{y \in Y_n} \lambda_{\min}(A_{yy}),$$

$$\lambda_{\max}(\mathbf{A}) = \max_{\|x\|_2=1} (x^T A_c x + |x|^T \Delta |x|) = \max_{y \in Y_n} \lambda_{\max}(A_{yy}).$$

Proof: [95], p. S1049; [91], pp. 5-6. □

Proposition 98. *For a symmetric interval matrix $\mathbf{A}$, the set*

$$\{\lambda_{\max}(A)\,;\, A \text{ symmetric}, A \in \mathbf{A}\}$$

is a compact interval; the same holds for the minimal eigenvalue.

Proof: [92], p. 25-26. □

Proof: Let

$$\begin{aligned}\underline{\lambda}(A^I) &= \min\{\lambda_{\max}(A);\ A \text{ symmetric}, A \in \mathbf{A}\},\\ \overline{\lambda}(A^I) &= \max\{\lambda_{\max}(A);\ A \text{ symmetric}, A \in \mathbf{A}\}.\end{aligned}$$

By continuity argument, both bounds are achieved, hence

$$\begin{aligned}\underline{\lambda}(A^I) &= \lambda_{\max}(A_1),\\ \overline{\lambda}(A^I) &= \lambda_{\max}(A_2)\end{aligned}$$

for some symmetric $A_1, A_2 \in \mathbf{A}$. Define a real function φ of one real variable by

$$\varphi(t) = f(A_1 + t(A_2 - A_1)), \quad t \in [0,1],$$

where

$$f(A) = \max_{\|x\|_2=1} x^T Ax.$$

φ is continuous since $f(A)$ is continuous [?], and $\varphi(0) = f(A_1) = \lambda_{\max}(A_1) = \underline{\lambda}(A^I)$, $\varphi(1) = f(A_2) = \lambda_{\max}(A_2) = \overline{\lambda}(A^I)$, hence for each $\lambda \in [\underline{\lambda}(A^I), \overline{\lambda}(A^I)]$ there exists a $t_\lambda \in [0,1]$ such that

$$\lambda = \varphi(t_\lambda) = f(A_1 + t_\lambda(A_2 - A_1)) = \lambda_{\max}(A_1 + t_\lambda(A_2 - A_1)).$$

Hence each $\lambda \in [\underline{\lambda}(A^I), \overline{\lambda}(A^I)]$ is the maximal eigenvalue of some symmetric matrix in $\mathbf{A}$, and we have

$$\lambda^I_{\max}(\mathbf{A}) = [\underline{\lambda}(A^I), \overline{\lambda}(A^I)].$$

□

Proposition 99. *If* $\mathbf{A}$ *is symmetric, then*

$$|\mathrm{Im}\,\lambda| \le \lambda_{\max}\begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}$$

for each eigenvalue λ *of each* $A \in \mathbf{A}$.

Proof: [91], p. 10. □

3.4.6 NP-hardness results for eigenvalues

Theorem 100. *The following problem is NP-hard:*

Instance. *A nonnegative symmetric positive definite rational matrix A and a rational number λ.*

Question. *Is λ an eigenvalue of some symmetric matrix in $[A-E, A+E]$?*

Proof: [92], p. 24. □

Proof: $[A-E, A+E]$ is singular if and only if 0 is an eigenvalue of some symmetric matrix in $[A-E, A+E]$ (Proposition **??**). Hence the NP–hard problem of Theorem **??** can be reduced in polynomial time to the current problem, which is thereby NP–hard. □

Theorem 101. *Suppose there exists a polynomial-time algorithm which for each interval matrix of the form $\mathbf{A} = [A-E, A+E]$, A rational nonpositive symmetric stable, computes a rational number $\tilde{\lambda}(\mathbf{A})$ satisfying*

$$\left| \frac{\tilde{\lambda}(\mathbf{A}) - \lambda_{\max}(\mathbf{A})}{\lambda_{\max}(\mathbf{A})} \right| < 1$$

if $\lambda_{\max}(\mathbf{A}) \neq 0$ and $\tilde{\lambda}(\mathbf{A}) \geq 0$ otherwise. Then P=NP.

Proof: [92], p. 25. □

Proof: Under the assumptions, $\tilde{\lambda}(\mathbf{A}) < 0$ if and only if $\overline{\lambda}(A^I) < 0$, and this is equivalent to stability of $\mathbf{A}$. Hence we have a polynomial–time algorithm for solving the NP–hard problem of Theorem 109, which implies P=NP. □

Theorem 102. *The following problem is NP-hard:*

Instance. *A nonpositive symmetric stable rational matrix A, and rational numbers a, b, $a < b$.*

Question. *Is $\{\lambda_{\max}(A') \,;\, A' \text{ symmetric}, A' \in [A-E, A+E]\} \subseteq (a, b)$?*

Proof: [92], p. 26. □

Proof: For each symmetric $A' \in [A-E, A+E]$ we have

$$|\lambda_{\max}(A')| \leq \varrho(A') \leq \|A'\|_1 \leq \|A\|_1 + \|E\|_1 = \|A\|_1 + n < \alpha := \|A\|_1 + n + 1.$$

Hence due to Theorem **??**, $[A - E, A + E]$ is stable if and only if

$$\lambda^I_{\max}([A - E, A + E]) \subset (-\alpha, 0)$$

holds. This shows that the NP–hard problem of checking stability of $[A - E, A + E]$ (Theorem 109) can be reduced in polynomial time to the current problem, which is thus NP–hard. $\square$

3.4.7 A Gerschgorin-disc-type theorem

3.5 Positive (semi)definiteness

3.5.1 Definition

Notation $\mathbf{A}_s = [\frac{1}{2}(\underline{A} + \underline{A}^T), \frac{1}{2}(\overline{A} + \overline{A}^T)]$.

Notation $\mathbf{A}_s = [(\underline{A} + \underline{A}^T)/2, (\overline{A} + \overline{A}^T)/2]$.

3.5.2 Positive semidefiniteness

Theorem 103. *For a square interval matrix* $\mathbf{A}$*, the following assertions are equivalent:*

(i) $\mathbf{A}$ *is positive semidefinite,*
(ii) $\mathbf{A}_s$ *is positive semidefinite,*
(iii) $x^T A_c x \geq |x|^T \Delta |x|$ *for each* x*,*
(iv) A_{yy} *is positive semidefinite for each* $y \in Y_n$*.*

Corollary 104. $\mathbf{A}$ *is positive semidefinite if*

$$\varrho(\Delta + \Delta^T) \leq \lambda_{\min}(A_c + A_c^T)$$

holds.

3.5.3 Positive definiteness

Theorem 105. *For a square interval matrix* $\mathbf{A}$*, the following assertions are equivalent:*

(i) $\mathbf{A}$ *is positive definite,*
(ii) $\mathbf{A}_s$ *is positive definite,*
(iii) $x^T A_c x > |x|^T \Delta |x|$ *for each* $x \neq 0$*,*
(iv) A_{yy} *is positive definite for each* $y \in Y_n$*,*
(v) $\mathbf{A}_s$ *is regular and* A_c *is positive definite.*

Corollary 106. $\mathbf{A}$ *is positive definite if*

$$\varrho(\Delta + \Delta^T) < \lambda_{\min}(A_c + A_c^T)$$

holds.

3.5.4 NP-hardness

Theorem 107. *The following problem is NP–hard:*
Instance. *A nonnegative symmetric positive definite rational matrix* A.
Question. *Is* $[A - E, A + E]$ *positive definite?*

Proof: [92], p. 18. □

3.5.5 Sufficient condition

3.5.6 An algorithm for checking positive definiteness

function $flag =$ **posdefness** $(\mathbf{A})$
$A'_c = (A_c + A_c^T)/2;\ \Delta' = (\Delta + \Delta^T)/2;$
if A'_c is not positive definite
$\quad flag =$ '*not positive definite*'; **return**
end
if $\lambda_{\min}(A'_c) > \varrho(\Delta')$
$\quad flag =$ '*positive definite*'; **return**
end
$flag =$ **regularity** $([A'_c - \Delta', A'_c + \Delta']);$
if $flag =$ '*regular*', $flag =$ '*positive definite*'; **return**
else $flag =$ '*not positive definite*'; **return**
end

Figure 3.3: An algorithm for checking positive definiteness.

3.6 Hurwitz stability

3.6.1 Definition

Definition. A is called Hurwitz stable if $\operatorname{Re}\lambda < 0$ for each eigenvalue λ of A. $\mathbf{A}$ is called Hurwitz stable if each $A \in \mathbf{A}$ is Hurwitz stable.

3.6.2 A negative result

3.6.3 The symmetric case

Theorem 108. *For a symmetric interval matrix* $\mathbf{A}$*, the following assertions are equivalent:*

(i) $\mathbf{A}$ *is Hurwitz stable,*

(ii) A_{-yy} *is Hurwitz stable for each* $y \in Y_n$*,*

(iii) $\mathbf{A}$ *is regular and* A_c *is Hurwitz stable.*

3.6.4 NP-hardness

Theorem 109. *The following problem is NP-hard:*

Instance. *A nonpositive symmetric Hurwitz stable rational matrix A.*

Question. *Is $[A - E, A + E]$ Hurwitz stable?*

Proof: [92], p. 22. □

Proof: By Proposition **??**, $[A - E, A + E]$ is stable if and only if $[-A - E, -A + E]$ is positive definite, where $-A$ is a nonnegative symmetric positive definite rational matrix. Hence the result follows from Theorem 107. □

Nemirovskii [53] proved NP–hardness of checking stability for general (nonsymmetric) interval matrices.

3.6.5 Sufficient condition for the general case

Theorem 110. *If* $\mathbf{A}_s$ *is stable, then* $\mathbf{A}$ *is stable.*

For practical purposes we may use the following sufficient condition valid for the nonsymmetric case [63], [25]:

Theorem 111. *An interval matrix* $[A_c - \Delta, A_c + \Delta]$ *is stable if*

$$\lambda_{\max}(A'_c) + \varrho(\Delta') < 0 \tag{3.45}$$

holds, where $A'_c = \frac{1}{2}(A_c + A_c^T)$ *and* $\Delta' = \frac{1}{2}(\Delta + \Delta^T)$.

Proof: If (3.45) holds, then $\varrho(\Delta') < \lambda_{\min}(-A'_c)$, hence $[-A'_c - \Delta', -A'_c + \Delta']$ is positive definite by Theorem **??** and $[A'_c - \Delta', A'_c + \Delta']$ is stable by Proposition **??**. Stability of $[A_c - \Delta, A_c + \Delta]$ then follows by using Bendixson's theorem as in the proof of Proposition **??**. □

3.6.6 An algorithm for checking Hurwitz stability

```
function flag = hurwitzstab (A)
 A'_c = (A_c + A_c^T)/2; Δ' = (Δ + Δ^T)/2;
 flag = posdefness ([−A'_c − Δ', −A'_c + Δ']);
 if flag = 'positive definite'
    flag = 'Hurwitz stable'; return
 else
    if (A'_c = A_c and Δ' = Δ)
       flag = 'not Hurwitz stable'; return
    else
       flag = 'Hurwitz stability not verified'; return
    end
 end
```

Figure 3.4: An algorithm for checking Hurwitz stability.

3.6.7 Many other results

3.7 Schur stability

3.7.1 Definition

Definition. A is called Schur stable if $\varrho(A) < 1$. A *symmetric* $\mathbf{A}$ is called Schur stable if each *symmetric* $A \in \mathbf{A}$ is Schur stable.

3.7.2 The symmetric case

Theorem 112. *For a symmetric interval matrix* $\mathbf{A}$*, the following assertions are equivalent:*

(i) $\mathbf{A}$ *is Schur stable,*

(ii) the interval matrices $[\underline{A} - I, \overline{A} - I]$ *and* $[-\overline{A} - I, -\underline{A} - I]$ *are Hurwitz stable,*

(iii) $\mathbf{A}$ *is regular and* A_c *is Schur stable.*

3.7.3 NP-hardness

Theorem 113. *The following problem is NP-hard:*

Instance. *A symmetric Schur stable rational matrix A with $A \leq I$, and a rational number $\alpha \in [0, 1]$.*

Question. *Is $[A - \alpha E, A + \alpha E]$ Schur stable?*

Proof: [92], p. 23. □

Proof: For a nonpositive symmetric stable rational matrix A, the symmetric interval matrix $[A - E, A + E]$ is stable if and only if $[(I + \alpha A) - \alpha E, (I + \alpha A) + \alpha E]$ is Schur stable, where α is given by (**??**). Here $I + \alpha A$ is a symmetric Schur stable rational matrix with $I + \alpha A \leq I$, and $\alpha \in [0, 1]$. Hence we have a polynomial–time reduction of the NP–hard problem of Theorem 109 to the current problem, which shows that it is NP–hard as well. □

This result differs from those of previous sections where NP–hardness was established for the class of interval matrices of the form $[A - E, A + E]$. This is explained by the fact that regularity, positive definiteness and stability are invariant under multiplication by a positive parameter whereas Schur stability is not.

3.7.4 An algorithm for checking Schur stability

```
function flag = schurstab (A)
 if (A_c^T ≠ A_c or Δ^T ≠ Δ)
    flag = 'Schur stability not verified'; return
 end
 flag = hurwitzstab ([A_c − I − Δ, A_c − I + Δ]);
 if flag = 'not Hurwitz stable'
    flag = 'not Schur stable'; return
 end
 flag = hurwitzstab ([−A_c − I − Δ, −A_c − I + Δ]);
 if flag = 'not Hurwitz stable'
    flag = 'not Schur stable'; return
 end
 flag = 'Schur stable';
```

Figure 3.5: An algorithm for checking Schur stability.

3.8 Summary: Regularity preserves some properties

Theorem 114. *Let* $\mathbf{A}$ *be regular. Then there holds:*

(i) $\mathbf{A}$ *is inverse nonnegative if and only if* $\overline{A}^{-1} \geq 0$,
(ii) $\mathbf{A}$ *is inverse positive if and only if* $\overline{A}^{-1} > 0$,
(iii) $\mathbf{A}$ *is an* M*-matrix if and only if* $\overline{A}$ *is an* M*-matrix,*
(iv) $\mathbf{A}$ *is of inverse sign pattern* (z, y) *if and only if* A_{-yz} *is of inverse sign pattern* (z, y),
(v) $\mathbf{A}$ *is of strict inverse sign pattern* (z, y) *if and only if* A_{-yz} *is of strict inverse sign pattern* (z, y).

Moreover, if $\mathbf{A}$ *is symmetric, then*

(vi) $\mathbf{A}$ *is positive definite if and only if* A_c *is positive definite,*
(vii) $\mathbf{A}$ *is Hurwitz stable if and only if* A_c *is Hurwitz stable*

Proof: [89], p. T688. □

3.9 Determinants

Theorem 115. *Let* $\mathbf{A} = [\underline{A}, \overline{A}]$ *be an interval matrix. Then for each* $A \in \mathbf{A}$ *there exists an* $A' \in \mathbf{A}$ *of the form*

$$A'_{ij} \in \begin{cases} \{\underline{A}_{ij}, \overline{A}_{ij}\} & \text{if } (i,j) \neq (k,m), \\ [\underline{A}_{ij}, \overline{A}_{ij}] & \text{if } (i,j) = (k,m) \end{cases} \tag{3.46}$$

for some (k, m) *such that*

$$\det A = \det A'.$$

Proof: [92], p. 27. □

Proof: For each $\tilde{A} \in \mathbf{A}$ denote by $h(\tilde{A})$ the number of entries with $\tilde{A}_{ij} \notin \{\underline{A}_{ij}, \overline{A}_{ij}\}$, $i, j = 1, \ldots, n$. Given an $A \in \mathbf{A}$, let A' be a matrix satisfying $A' \in \mathbf{A}, \det A' = \det A$ and

$$h(A') = \min\{h(\tilde{A});\ \tilde{A} \in \mathbf{A}, \det \tilde{A} = \det A\}. \tag{3.47}$$

If $h(A') \geq 2$, then there exist indices $(p, q), (r, s), (p, q) \neq (r, s)$ such that $A'_{pq} \in (\underline{A}_{pq}, \overline{A}_{pq}), A'_{rs} \in (\underline{A}_{rs}, \overline{A}_{rs})$. Then we can move these two entries within their intervals in such a way that at least one achieves its bound, and the determinant is kept unchanged. Then the resulting matrix A'' satisfies $h(A'') < h(A')$, which is a contradiction. Hence A' defined by (3.47) satisfies $h(A') \leq 1$, which shows that it is of the form (3.46), and $\det A = \det A'$ holds. □

A matrix of the form (3.46) belongs to an edge of the interval matrix A^I considered a hyperrectangle in R^{n^2}. Hence the theorem says that the range of the determinant over $\mathbf{A}$ is equal to its range over the edges of $\mathbf{A}$. In particular, for zero values of the determinant we have this "normal form" theorem [81].

Theorem 116. *If* $\mathbf{A}$ *is singular, then it contains a singular matrix of the form (3.46).*

As a consequence we obtain that *real* eigenvalues of matrices in $\mathbf{A}$ are attained at the edge matrices of $\mathbf{A}$.

Theorem 117. *If a real number* λ *is an eigenvalue of some* $A \in \mathbf{A}$, *then it is also an eigenvalue of some matrix of the form (3.46).*

Proof: [92], p. 27. □

Proof: If λ is a real eigenvalue of some $A \in \mathbf{A} = [\underline{A}, \overline{A}]$, then $A - \lambda I$ is a singular matrix belonging to $[\underline{A} - \lambda I, \overline{A} - \lambda I]$, which is thus singular, hence by Theorem 116 it contains a singular matrix $A' - \lambda I$, where A' is of the form (3.46). Then λ is an eigenvalue of A'. $\square$

A general "edge theorem" for complex eigenvalues was proved by Hollot and Bartlett in [21].

For an interval matrix $\mathbf{A}$, consider the extremal values of the determinant over $\mathbf{A}$ given by

$$\begin{aligned}\overline{\det}(\mathbf{A}) &= \max\{\det A\,;\, A \in \mathbf{A}\},\\ \underline{\det}(\mathbf{A}) &= \min\{\det A\,;\, A \in \mathbf{A}\}.\end{aligned}$$

Since the determinant is linear in each entry, Theorem 115 implies that the extremal values are attained at some of the 2^{n^2} vertex matrices, i.e. matrices of the form

$$A_{ij} \in \{\underline{A}_{ij}, \overline{A}_{ij}\}, \quad i,j = 1,\ldots,n.$$

We have this result:

Theorem 118. *Computing* $\underline{\det}(\mathbf{A}), \overline{\det}(\mathbf{A})$ *is NP–hard for the class of interval matrices of the form* $\mathbf{A} = [A - E, A + E]$, *$A$ rational nonnegative.*

Proof: [92], p. 28. $\square$

Proof: For an interval matrix of the form $\mathbf{A} = [A-E, A+E]$, where A is a nonnegative symmetric positive definite rational matrix, singularity of $\mathbf{A}$ is equivalent to

$$\overline{\det}(A_0^I) \geq 0, \tag{3.48}$$

where $A_0^I = \mathbf{A}$ if $\det A \leq 0$ and A_0^I is constructed by swapping the first two rows of $\mathbf{A}$ otherwise (which changes the sign of the determinant). Here $A_0^I = [A_0 - E, A_0 + E]$, where A_0 is a nonnegative rational matrix. Hence the NP–hard problem of checking regularity (Theorem **??**) can be reduced in polynomial time to the decision problem (3.48) which shows that computing $\overline{\det}(\mathbf{A})$ is NP–hard in this class of interval matrices. The proof for $\underline{\det}(\mathbf{A})$ is analogous. $\square$

Theorem 119. *Let* $\mathbf{A}$ *be inverse stable. Then* $|\det(A)|$ *attains its unique local minimum over* $\mathbf{A}$ *at the matrix* A^* *given by*

$$(A^*)_{jk} = \begin{cases} \underline{A}_{jk} & \text{if } (A_c^{-1})_{kj} > 0, \\ \overline{A}_{jk} & \text{if } (A_c^{-1})_{kj} < 0 \end{cases} \qquad (j,k = 1,\ldots,n).$$

Proof: [73], p. 114. $\square$

3.10 Rectangular interval matrices

3.10.1 Full rank

As is well known, a real matrix $A \in \mathbb{R}^{m\times n}$ has full column rank if its columns are linearly independent, i.e., if $Ax = 0$ implies $x = 0$.

Definition. An interval matrix $\mathbf{A}$ is said to have full column rank if each $A \in \mathbf{A}$ has full column rank, and it is said to have full row rank if $\mathbf{A}^T$ has full column rank.

We have this characterization:

Theorem 120. *An interval matrix $\mathbf{A}$ has full column rank if and only if the inequality*

$$|A_c x| \le \Delta |x| \tag{3.49}$$

has only the trivial solution $x = 0$.

Proof: $\mathbf{A}$ does not have full column rank if and only if $Ax = 0$ holds for some $A \in \mathbf{A}$ and $x \neq 0$, i.e., if and only if this $x \neq 0$ is a weak solution of the interval linear system $\mathbf{A}x = [0, 0]$, which by the Oettli-Prager theorem is equivalent to existence of a nontrivial solution to (3.49). □

The inequality (3.49) looks simple at the first glance; but unfortunately the right-hand side absolute value turns out to be the source of big computational difficulties as evidenced not only in the following theorem, but also at many places in the subsequent chapters.

Theorem 121. *The following problem is NP-hard:*
Instance. *A nonnegative rational matrix $A \in \mathbb{R}^{m\times n}$.*
Question. *Does $[A - E, A + E]$ have full column rank?*

Since checking full column rank is NP-hard, in practice we must resort to some sufficient conditions (that are not necessary). The following theorem shows a way to this goal.

Theorem 122. *Let for an $m \times n$ interval matrix $\mathbf{A}$ there exist a matrix $R \in \mathbb{R}^{n\times m}$ such that*

$$\varrho(|I - RA_c| + |R|\Delta) < 1. \tag{3.50}$$

Then $\mathbf{A}$ has full column rank.

Proof: [93]. □

Comment. Notice that since $A_c, \Delta \in \mathbb{R}^{m\times n}$ and $R \in \mathbb{R}^{n\times m}$, the matrix $|I - RA_c| + |R|\Delta$ is square $n \times n$, so that we can speak of its spectral radius.

Proof: For each $A \in \mathbf{A}$ we have

$$RA = I - (I - RA) = I - (I - RA_c + R(A_c - A)),$$

where

$$\varrho(I - RA_c + R(A_c - A)) \leq \varrho(|I - RA_c| + |R|\Delta) < 1$$

by (3.50), hence by a well-known theorem (Horn and Johnson [22], p. 301) the matrix RA is invertible and thus nonsingular. Now, if $Ax = 0$, then $RAx = 0$, and nonsingularity of RA implies $x = 0$. In this way we have proved that each $A \in \mathbf{A}$ has full column rank, hence $\mathbf{A}$ has full column rank. □

Theorem 122 does not specify the choice of R. But fortunately, such a choice is at hand:

Theorem 123. *Let A_c have full column rank and let*

$$\varrho(|(A_c^T A_c)^{-1} A_c^T|\Delta) < 1. \tag{3.51}$$

Then $\mathbf{A}$ has full column rank.

Proof: This is a consequence of Theorem 122 for $R = A_c^+$. □

Proof: If A_c has full column rank, then $(A_c^T A_c)^{-1}$ exists and direct substitution shows that the matrix $R = (A_c^T A_c)^{-1} A_c^T$ satisfies (3.50), hence $\mathbf{A}$ has full column rank by Theorem 122. □

Under our assumption, the matrix $(A_c^T A_c)^{-1} A_c^T$ is equal to the Moore-Penrose inverse A_c^+ of A_c (see e.g. Stewart and Sun [114]), hence the condition (3.51) can also be written as

$$\varrho(|A_c^+|\Delta) < 1.$$

For practical purposes, Theorem 122 offers the advantage of possibility of setting R equal to the *computed* (i.e., not necessarily exact) value of $(A_c^T A_c)^{-1} A_c^T$.

The following theorem gives a sufficient full column rank condition in terms of singular values:

Theorem 124. *If*

$$\sigma_{\max}(\Delta) < \sigma_{\min}(A_c), \tag{3.52}$$

then $\mathbf{A}$ has full column rank.

Proof: [102] (for square case; valid for the rectangular case as well). □

Proof: Assume to the contrary that **A** does not have full column rank, so that by Theorem 120 there exists an $x_0 \neq 0$, which may be normalized to achieve $\|x_0\|_2 = 1$, such that

$$|A_c x_0| \leq \Delta |x_0|$$

holds. Then we have

$$|A_c x_0|^T |A_c x_0| \leq (\Delta |x_0|)^T (\Delta |x_0|),$$

which implies

$$\begin{aligned} \sigma_{\min}^2(A_c) &= \lambda_{\min}(A_c^T A_c) = \min_{\|x\|_2 = 1} x^T A_c^T A_c x \leq (A_c x_0)^T (A_c x_0) \\ &\leq |A_c x_0|^T |A_c x_0| \leq (\Delta |x_0|)^T (\Delta |x_0|) = |x_0|^T \Delta^T \Delta |x_0| \\ &\leq \max_{\|x\|_2 = 1} x^T \Delta^T \Delta x = \lambda_{\max}(\Delta^T \Delta) = \sigma_{\max}^2(\Delta), \end{aligned}$$

hence

$$\sigma_{\min}(A_c) \leq \sigma_{\max}(\Delta),$$

which is a contradiction. □

3.10.2 Singular values

3.11 Appendices

3.11.1 Appendix 3A: Theorems of the alternatives

Theorem 125. *Let $A, D \in \mathbb{R}^{n\times n}$, $D \geq 0$. Then exactly one of the following alternatives holds:*

(i) for each $B \in \mathbb{R}^{n\times n}$ with $|B| \leq D$ and for each $b \in \mathbb{R}^n$ the equation

$$Ax + B|x| = b \tag{3.53}$$

has a unique solution,

(ii) there exist $d \in [0, 1]$ and a vector $y \in \mathbb{R}^n$ such that the equation

$$Ax + dT_y D|x| = 0 \tag{3.54}$$

has a nontrivial solution.

Proof: Given $A, D \in \mathbb{R}^{n\times n}$, $D \geq 0$, consider the set

$$\mathbf{A} = \{A' \,;\, |A' - A| \leq \Delta\} = \{A' \,;\, A - D \leq A' \leq A + D\},$$

which is called an interval matrix [81]. $\mathbf{A}$ is said to be regular if each $A' \in \mathbf{A}$ is nonsingular, and it is called singular otherwise (i.e., if it contains a singular matrix). We shall prove that (a) regularity of $\mathbf{A}$ implies (i), (b) singularity of $\mathbf{A}$ implies (ii), and (c) both (i) and (ii) cannot hold simultaneously. This will prove that exactly one of the alternatives (i), (ii) holds.

(a) Let $\mathbf{A}$ be regular and let $|B| \leq D$ and $b \in \mathbb{R}^n$. Then using nonnegative vectors $x^+ = (|x| + x)/2$ and $x^- = (|x| - x)/2$, we have that $x = x^+ - x^-$ and $|x| = x^+ + x^-$, and we may rewrite the equation (3.53) into the equivalent form

$$x^+ = (A + B)^{-1}(A - B)x^- + (A + B)^{-1}b. \tag{3.55}$$

Since $|B| \leq D$, both matrices $A + B$ and $A - B$ belong to $\mathbf{A}$, hence $(A + B)^{-1}$ exists and, moreover, $(A + B)^{-1}(A - B)$ is a P-matrix by Theorem 1.2 in [81]. Hence the linear complementarity problem (3.55) has a unique solution (Murty [51]), and the equivalent equation (3.53) has a unique solution as well.

(b) Let $\mathbf{A}$ be singular. Then the value

$$\lambda = \min\{\varepsilon \geq 0 \,;\, \text{the interval matrix } [A - \varepsilon D, A + \varepsilon D] \text{ is singular}\}$$

belongs to $[0, 1]$ because $\mathbf{A} = [A - D, A + D]$ is singular, and Theorem 2.2 in [86] asserts that there exist ± 1-vectors y, z and an $x \neq 0$ such that

$$(A - \lambda\, \mathrm{diag}\,(y) D\, \mathrm{diag}\,(z))x = 0, \tag{3.56}$$

$$\mathrm{diag}\,(z)\, x \geq 0 \tag{3.57}$$

hold. Then (3.57) implies that $\operatorname{diag}(z)\,x = |x|$, and substituting this quantity into (3.56) we obtain

$$Ax - \lambda \operatorname{diag}(y) D|x| = 0,$$

so that it suffices to put $y := -y$ to conclude that the equation (3.54) has a nontrivial solution.

(c) Finally we show that (i) and (ii) cannot hold simultaneously. For, if (3.54) has a nontrivial solution x for some $\lambda \in [0,1]$ and some ± 1-vector y, and if we put $B = \lambda \operatorname{diag}(y)D$, then $|B| = \lambda D \leq D$, hence (3.54) is of the form (3.53) for some B, but (3.54) has at least two solutions x and 0, which contradicts (i). □ □

Theorem 126. *Let $A, B \in \mathbb{R}^{n\times n}$ and let the inequality*

$$|Ax| \leq |B||x|$$

have the trivial solution only. Then the equation

$$Ax + B|x| = b$$

has a unique solution for each $b \in \mathbb{R}^n$.

Proof: Put $D = |B|$. Then the assertion (γ) of Theorem **??** does not hold, hence neither does (α), which is the assertion (ii) of Theorem 125. Hence (i) holds, which gives that the equation $Ax + C|x| = b$ has a unique solution for each $b \in \mathbb{R}^n$ and each C satisfying $|C| \leq D = |B|$, thus in particular also for $C = B$. □ □

Theorem 127. *Let $A, B \in \mathbb{R}^{n\times n}$. Then exactly one of the following alternatives holds:*

(a) for each $y \in Y_n$ the inequality

$$|Ax| > |B||x|$$

has a solution x_y satisfying $Ax_y \in \mathbb{R}^n_y$,

(b) the inequality

$$|Ax| \leq |B||x|$$

has a nontrivial solution.

Theorem 128. *Let $A, B \in \mathbb{R}^{n\times n}$, A nonsingular. Then exactly one of the following alternatives holds:*

(a) the inequality

$$|x| > |B||Ax|$$

has a solution in each orthant,

(b) the inequality

$$|x| \leq |B||Ax|$$

has a nontrivial solution.

3.11.2 Appendix 3B: Matrix properties under fixed-point rounding

For a real number a and a nonnegative integer d define

$$a_{(d)} = \begin{cases} \lfloor 10^d a + 0.5 \rfloor 10^{-d} & \text{if } \ a \geq 0, \\ -(-a)_{(d)} & \text{if } \ a < 0, \end{cases} \tag{3.58}$$

where

$$\lfloor b \rfloor = \max\{\, c \mid c \leq b,\, c \text{ integer} \,\}.$$

It is obvious that $a_{(d)}$ is the result of rounding a to d decimal places. The following two properties are almost straightforward, but we include them for the sake of completeness because of their repeated use in the sequel. Throughout the paper we denote

$$\delta = 0.5 \cdot 10^{-d}. \tag{3.59}$$

Proposition 129. *If $a \in \mathbb{R}$ and d is a nonnegative integer, then*

$$a_{(d)} - \delta \leq a \leq a_{(d)} + \delta. \tag{3.60}$$

Proof: Let $a \geq 0$. Then (3.58) implies that

$$10^d a_{(d)} = \lfloor 10^d a + 0.5 \rfloor,$$

thus $10^d a_{(d)}$ is the integer part of $10^d a + 0.5$, hence

$$10^d a_{(d)} \leq 10^d a + 0.5 < 10^d a_{(d)} + 1,$$

which gives

$$a_{(d)} - 0.5 \cdot 10^{-d} \leq a < a_{(d)} + 0.5 \cdot 10^{-d},$$

and this in view of (3.59) means that

$$a_{(d)} - \delta \leq a < a_{(d)} + \delta. \tag{3.61}$$

If $a < 0$, then the inequality (3.61) holds for $-a$, hence

$$(-a)_{(d)} - \delta \leq -a < (-a)_{(d)} + \delta$$

and in the light of (3.58) we obtain

$$a_{(d)} - \delta < a \leq a_{(d)} + \delta. \tag{3.62}$$

Hence in both cases (3.61), (3.62) we have (3.60). $\square$

Proposition 130. *If $a \in \mathbb{R}$ and d is a nonnegative integer, then each b with*

$$a_{(d)} - \delta < b < a_{(d)} + \delta \tag{3.63}$$

satisfies

$$b_{(d)} = a_{(d)}.$$

Proof: From (3.63) it follows

$$10^d a_{(d)} < 10^d b + 0.5 < 10^d a_{(d)} + 1,$$

and since $10^d a_{(d)}$ is integer due to (3.58), this implies that

$$10^d a_{(d)} = \lfloor 10^d b + 0.5 \rfloor$$

and

$$a_{(d)} = \lfloor 10^d b + 0.5 \rfloor 10^{-d}.$$

Hence, if $b \geq 0$, then $a_{(d)} = b_{(d)}$ due to (3.58). If $b < 0$, then the result just proved gives $(-a)_{(d)} = (-b)_{(d)}$, hence again $a_{(d)} = b_{(d)}$ by (3.58). □

Now, let $A = (a_{ij})$ be a square matrix (we shall consider only square matrices in the sequel). We define

$$A_{(d)} = ((a_{ij})_{(d)}),$$

hence the matrix $A_{(d)}$ arises from A by rounding off all its coefficients to d decimal places. The main question handled in this paper is the following: assume a real matrix A is not exactly known and we have only its rounded value $A_{(d)}$ at our disposal; if $A_{(d)}$ has some property, under what additional condition(s) can we be sure that the original matrix A possesses this property as well? We shall give answers for the cases of three common properties, namely, nonsingularity, positive definiteness, and positive invertibility. In case of nonsingularity we shall show in Theorem 131 that there exists a real number α computed from $A_{(d)}^{-1}$ such that if $d > \alpha$, then nonsingularity of $A_{(d)}$ implies nonsingularity of A, and if $d < \alpha$, then there exists a singular matrix A' satisfying $A'_{(d)} = A_{(d)}$; hence, in the former case we are done, whereas in the latter one we learn that the original matrix cannot be distinguished, by means of rounding to d decimal places, from a singular matrix. In Theorem 134 we shall show that literally the same result (with the same α) holds for positive definiteness. Both theorems handle the cases $d > \alpha$ and $d < \alpha$ only, but the remaining case $d = \alpha$ occurs with probability 0 because d is integer whereas α is a real number.

Theorem 131. *Let A be square and let $A_{(d)}$ be nonsingular for some integer $d \geq 0$. Then we have:*

(i) if

$$d > \log_{10}(0.5 \cdot \|A_{(d)}^{-1}\|_{\infty,1}), \tag{3.64}$$

then A is nonsingular,

(ii) if

$$d < \log_{10}(0.5 \cdot \|A_{(d)}^{-1}\|_{\infty,1}), \tag{3.65}$$

then there exists a singular matrix A' satisfying $A'_{(d)} = A_{(d)}$.

Proof: (i) If (3.64) holds, then

$$0.5 \cdot 10^{-d}\|A_{(d)}^{-1}\|_{\infty,1} = \delta\|A_{(d)}^{-1}\|_{\infty,1} < 1,$$

hence, by virtue of Proposition **??**, the interval matrix $[A_{(d)} - \delta ee^T, A_{(d)} + \delta ee^T]$ consists of nonsingular matrices only. Since A belongs to this interval matrix by (**??**), it follows that A is nonsingular.

(ii) If (3.65) holds, then

$$\delta\|A_{(d)}^{-1}\|_{\infty,1} > 1.$$

Let us choose a $\delta' \in (0, \delta)$ such that $\delta'\|A_{(d)}^{-1}\|_{\infty,1} > 1$. Then by Proposition **??** there exists a singular matrix $A' \in [A_{(d)} - \delta' ee^T, A_{(d)} + \delta' ee^T]$. Since $\delta' < \delta$, this singular matrix satisfies

$$A_{(d)} - \delta ee^T < A' < A_{(d)} + \delta ee^T,$$

and Proposition 130 gives that $A'_{(d)} = A_{(d)}$, which was to be proved. □

In case (ii) a singular matrix can be given explicitly:

Proposition 132. *Let (3.65) hold and let $z, y \in Y$ be any two vectors satisfying*

$$d < \log_{10}(0.5(z^T A_{(d)}^{-1} y)). \tag{3.66}$$

Then the matrix

$$A' = A_{(d)} - \frac{yz^T}{z^T A_{(d)}^{-1} y} \tag{3.67}$$

is singular and satisfies $A'_{(d)} = A_{(d)}$.

Proof: In fact,

$$A'(A_{(d)}^{-1} y) = y - \frac{y(z^T A_{(d)}^{-1} y)}{z^T A_{(d)}^{-1} y} = 0,$$

hence A' is singular. From (3.66) we have

$$\frac{1}{z^T A_{(d)}^{-1} y} < \delta,$$

which implies $|A' - A_{(d)}| < \delta ee^T$, so that

$$A_{(d)} - \delta ee^T < A' < A_{(d)} + \delta ee^T$$

and $A'_{(d)} = A_{(d)}$ by Proposition 130. □

Corollary 133. *Let A be square and let $A_{(d)}$ be nonsingular for some nonnegative integer d satisfying*

$$d > \log_{10}(0.5 \cdot \||A_{(d)}^{-1}|e\|_1). \tag{3.68}$$

Then A is nonsingular.

Proof: For each $z, y \in Y$ we have

$$z^T A_{(d)}^{-1} y \le |z^T A_{(d)}^{-1} y| \le e^T |A_{(d)}^{-1}| e = \||A_{(d)}^{-1}|e\|_1,$$

hence $\|A_{(d)}^{-1}\|_{\infty,1} \le \||A_{(d)}^{-1}|e\|_1$ and from (3.68) we obtain

$$d > \log_{10}(0.5 \cdot \||A_{(d)}^{-1}|e\|_1) \ge \log_{10}(0.5 \cdot \|A_{(d)}^{-1}\|_{\infty,1}),$$

which is the condition (3.64), and nonsingularity of A is verified. □

Theorem 134. *Let A be symmetric and let $A_{(d)}$ be positive definite for some integer $d \ge 0$. Then we have:*

(i) if

$$d > \log_{10}(0.5 \cdot \|A_{(d)}^{-1}\|_{\infty,1}), \tag{3.69}$$

then A is positive definite,

(ii) if

$$d < \log_{10}(0.5 \cdot \|A_{(d)}^{-1}\|_{\infty,1}), \tag{3.70}$$

then there exists a symmetric matrix A' satisfying $A'_{(d)} = A_{(d)}$ which is not positive definite.

Proof: (i) Since $A_{(d)}$ is positive definite by assumption and since (3.69) guarantees nonsingularity of all matrices contained in $[A_{(d)} - \delta ee^T, A_{(d)} + \delta ee^T]$ (proof of Theorem 131), Proposition **??** gives that each symmetric matrix in $[A_{(d)} - \delta ee^T, A_{(d)} + \delta ee^T]$ is positive definite, thus also A is positive definite.

(ii) If (3.70) holds, then we know from the proof of Theorem 131 that there exists a singular matrix A'' satisfying

$$A_{(d)} - \delta ee^T < A'' < A_{(d)} + \delta ee^T, \tag{3.71}$$

i.e., $A''x = 0$ for some $x \ne 0$. Because both matrices $A_{(d)} - \delta ee^T$ and $A_{(d)} + \delta ee^T$ are symmetric (symmetry of A implies symmetry of $A_{(d)}$), from (3.71) we have

$$A_{(d)} - \delta ee^T < 0.5(A'' + A''^T) < A_{(d)} + \delta ee^T.$$

Then the matrix

$$A' = 0.5(A'' + A''^T)$$

is symmetric, satisfies $A'_{(d)} = A_{(d)}$ by Proposition 130, and

$$x^T A' x = x^T A'' x = 0,$$

so that A' is not positive definite. □

Corollary 135. *Let A be symmetric and let $A_{(d)}$ be positive definite for some nonnegative integer d satisfying*

$$d > \log_{10}(0.5 \cdot \||A_{(d)}^{-1}|e\|_1). \tag{3.72}$$

Then A is positive definite.

Proof: As we have seen in the proof of Corollary 133, (3.72) implies (3.69), hence A is positive definite. □

Theorem 136. *Let A be square and let*

$$(A_{(d)} + 0.5 \cdot 10^{-d} ee^T)^{-1} > 0 \tag{3.73}$$

hold for some integer $d \geq 0$. Then we have:

(i) if

$$d > \log_{10} \|(A_{(d)} + 0.5 \cdot 10^{-d} ee^T)^{-1} e\|_1, \tag{3.74}$$

then A is positive invertible,

(ii) if

$$d < \log_{10} \|(A_{(d)} + 0.5 \cdot 10^{-d} ee^T)^{-1} e\|_1, \tag{3.75}$$

then there exists a matrix A' satisfying $A'_{(d)} = A_{(d)}$ which is not positive invertible.

Proof: Consider again the interval matrix $[\underline{A}, \overline{A}] = [A_{(d)} - \delta ee^T, A_{(d)} + \delta ee^T]$, where $\delta = 0.5 \cdot 10^{-d}$ as before. Then $\overline{A}^{-1} > 0$ by the assumption (3.73), and

$$\varrho(\overline{A}^{-1}(\overline{A} - \underline{A})) = \varrho(2\delta \overline{A}^{-1} ee^T) = 2\delta e^T \overline{A}^{-1} e = 10^{-d} \|\overline{A}^{-1} e\|_1.$$

Hence, $\varrho(\overline{A}^{-1}(\overline{A} - \underline{A})) < 1$ if and only if

$$d > \log_{10} \|\overline{A}^{-1} e\|_1$$

holds. This means that if this condition is met, then, by Proposition **??**, all matrices contained in $[\underline{A}, \overline{A}]$ are positive invertible and thus also A is positive invertible, which proves (i). If (3.75) holds, then

$$2\delta \|(A_{(d)} + \delta ee^T)^{-1} e\|_1 > 1.$$

Because of continuity there exists a $\delta' \in (0, \delta)$ such that $(A_{(d)} + \delta' ee^T)^{-1} > 0$ and

$$2\delta' \|(A_{(d)} + \delta' ee^T)^{-1} e\|_1 > 1.$$

Then, by Proposition **??**, the interval matrix $[A_{(d)} - \delta' ee^T, A_{(d)} + \delta' ee^T]$ contains a matrix A' which is not positive invertible. Since

$$A_{(d)} - \delta ee^T < A_{(d)} - \delta' ee^T \leq A' \leq A_{(d)} + \delta' ee^T < A_{(d)} + \delta ee^T,$$

there holds $A'_{(d)} = A_{(d)}$ by Proposition 130, which concludes the proof of (ii). $\square$

3.11.3 Appendix 3C: Componentwise condition number

$$c_{ij}^{\alpha}(A) = \max\left\{\left|\frac{B_{ij}^{-1} - A_{ij}^{-1}}{A_{ij}^{-1}}\right|; |B - A| \le \alpha|A|\right\}$$

$$c_{ij}(A) = \lim_{\alpha \to 0_+} \frac{c_{ij}^{\alpha}(A)}{\alpha}$$

Theorem 137. *For a nonsingular matrix A we have*

$$c_{ij}(A) = \frac{(|A^{-1}| \cdot |A| \cdot |A^{-1}|)_{ij}}{|A^{-1}|_{ij}}$$

for each i, j with $A_{ij}^{-1} \neq 0$.

Proof: [79], p. 168. □

3.11.4 Appendix 3D: Absolute eigenvalues

Theorem 138. *For each real square matrix A_c there exists a real number $d \geq 0$ and real vectors $x \neq 0, p \neq 0$ satisfying*

$$\begin{aligned} |A_c x| &= d|x|, \\ |A_c^T p| &= d|p|. \end{aligned}$$

3.12 Notes and references

Section 3.10.1. The topic of this section in its full generality (i.e., for rectangular interval matrices) seems to have escaped attention of interval researchers. The only reference known to the author is [93], where Theorems 122 and 123 were proved in equivalent, but less direct formulations. Theorem 124 was proved for the square case by Rump in [102], but his proof goes through without any change for the rectangular case as well. The full column rank problem for the square case (i.e., regularity) has been studied in considerable detail, as evidenced in previous sections.

Chapter 4

Systems of interval linear equations (square case)

4.1 Introduction

4.2 Solution set

4.2.1 Definition

4.2.2 Description: The Oettli-Prager theorem

Theorem 139. (Oettli-Prager) *We have*

$$X = \{x\,;\, |A_c x - b_c| \le \Delta|x| + \delta\}. \tag{4.1}$$

Proof: If $x \in X$, then $Ax = b$ for some $A \in \mathbf{A}$, $b \in \mathbf{b}$, which gives $|A_c x - b_c| = |(A_c - A)x + b - b_c| \le \Delta|x| + \delta$. Conversely, let $|A_c x - b_c| \le \Delta|x| + \delta$ hold for some x. Define $y \in R^n$ by

$$y_i = \begin{cases} \frac{(A_c x - b_c)_i}{(\Delta|x|+\delta)_i} & \text{if } (\Delta|x| + \delta)_i > 0, \\ 1 & \text{if} \end{cases} \qquad (i = 1, \ldots, n), \tag{4.2}$$

then $|y| \le e$ and

$$A_c x - b_c = T_y(\Delta|x| + \delta). \tag{4.3}$$

Put $z = \operatorname{sgn} x$, then $|x| = T_z x$ and from (4.3) we get $A_{yz}x = (A_c - T_y \Delta T_z)x = b_c + T_y\delta = b_y$. Since $|y| \le e$ and $z \in Y_n$, we have $|T_y \Delta T_z| \le \Delta$ and $|T_y\delta| \le \delta$, so that $A_{yz} \in \mathbf{A}$ and $b \in \mathbf{b}$, implying $x \in X$. □

Theorem 140. *We have* $X = \{x\,;\, \underline{A}x^+ - \overline{A}x^- \le \overline{b},\, \overline{A}x^+ - \underline{A}x^- \ge \underline{b}\}$.

Proof: [69], p. 35. □

Theorem 141. *Let* $\mathbf{A}$ *be regular and let* $\delta > 0$*. Then the mapping* $x \mapsto t$ *defined by*

$$t_i = \frac{(Ax - b)_i}{(\Delta|x| + \delta)_i} \qquad (i = 1, \ldots, n),$$

is a continuous one-to-one mapping of the solution set X *onto the interval vector* $[-e, e]$.

Proof: [69], p. 36. □

In the proof of the main result we shall employ the fact that the Oettli-Prager description (5.27) can be reformulated as

$$X = \{x\,;\, \underline{A}x^+ - \overline{A}x^- \le \overline{b},\, \overline{A}x^+ - \underline{A}x^- \ge \underline{b}\}, \tag{4.4}$$

where $x^+ = (|x| + x)/2$ and $x^- = (|x| - x)/2$ ([69], Corollary 1.4). It can be obtained from (5.27) directly by substituting $x = x^+ - x^-$, $|x| = x^+ + x^-$. Notice that the definition of x^+, x^- implies that $x_i^+ = \max\{x_i, 0\}$ and $x_i^- = \max\{-x_i, 0\}$ for $i = 1, \ldots, n$, i.e., it conforms with the standard notation.

Proposition 142. *For each $z \in Y$ we have*

$$X \cap \mathbb{R}^n_z = \{x\,;\, A_{ez} \leq \overline{b}, A_{-ez}x \geq \underline{b}, T_z x \geq 0\}, \tag{4.5}$$

i.e., $X \cap \mathbb{R}^n_z$ is a convex polyhedron. Henceforth, X is a union of at most 2^n convex polyhedra.

Proposition 143. *Let $\mathbf{A}$ be regular and let $b \neq 0$. Then the solution set X of $\mathbf{A}x = [b, b]$ satisfies*

$$X \cap \mathbb{R}^n_z \cap \mathbb{R}^n_{-z} = \emptyset$$

for each $z \in Y_n$, i.e., X cannot simultaneously intersect two opposite orthants.

Proof: [69], p. 44. □

Theorem 144. *If $\mathbf{A}$ is regular, then every two points in X can be connected by a piecewise linear curve lying entirely in X and consisting of at most n segments.*

4.2.3 Normal forms

Proposition 145. *If x solves (4.1), then it satisfies*

$$(A_c - T_y \Delta T_z)x = b_c + T_y \delta,$$

where y is given by

$$y_i = \begin{cases} (A_c x - b_c)_i/(\Delta|x| + \delta)_i & \text{if } (\Delta|x| + \delta)_i > 0, \\ 1 & \text{if } (\Delta|x| + \delta)_i = 0 \end{cases} \qquad (i = 1, \ldots, m),$$

and $z = \operatorname{sgn} x$.

The following result shows that a system $Ax = b$ to be satisfied by a given $x \in X$ can always be chosen in a certain normal form:

Theorem 146 *Let $x \in X$. Then there holds $Ax = b$ for some A and b of the following form: for each $i = 1, \ldots, m$ there exists a $k_i \in \{1, \ldots, n+1\}$ such that*

$$A_{ij} \begin{cases} = \underline{A}_{ij} & \text{if } (k_i - j)x_j < 0 \text{ or } x_j = 0, \\ = \overline{A}_{ij} & \text{if } (k_i - j)x_j > 0, \\ \in [\underline{A}_{ij}, \overline{A}_{ij}] & \text{if } j = k_i \text{ and } x_j \neq 0, \end{cases} \qquad (j = 1, \ldots, n), \tag{4.6}$$

$$b_i \begin{cases} = \overline{b}_i & \text{if } k_i \leq n, \\ \in [\underline{b}_i, \overline{b}_i] & \text{if } k_i = n+1. \end{cases} \tag{4.7}$$

Proof: Let $x \in X$, so that x satisfies

$$\underline{A}x^+ - \overline{A}x^- \leq \overline{b}, \tag{4.8}$$

$$\overline{A}x^+ - \underline{A}x^- \geq \underline{b} \tag{4.9}$$

(see (4.4)). We shall construct a system $Ax = b$ with properties listed equation by equation. Let $i \in \{1, \ldots, m\}$. To construct the ith equation, define a function f of $n+1$ real variables by

$$\begin{aligned} f(t_1, \ldots, t_{n+1}) &= \sum_{j=1}^{n} \Big((\underline{A}_{ij} + t_j(\overline{A}_{ij} - \underline{A}_{ij}))x_j^+ \\ &\quad -(\overline{A}_{ij} + t_j(\underline{A}_{ij} - \overline{A}_{ij}))x_j^- \Big) \\ &\quad -(\overline{b}_i + t_{n+1}(\underline{b}_i - \overline{b}_i)). \end{aligned} \tag{4.10}$$

Then

$$f(0, \ldots, 0)f(1, \ldots, 1) = (\underline{A}x^+ - \overline{A}x^- - \overline{b})_i(\overline{A}x^+ - \underline{A}x^- - \underline{b})_i \leq 0 \tag{4.11}$$

due to (4.8), (4.9). For $k = 1, \dots, n+1$ put

$$\varphi_k = f(1, \dots, 1, 0, 0, \dots, 0) f(1, \dots, 1, 1, 0, \dots, 0),$$

where the two argument vectors differ in the kth position only. Then we have

$$\prod_{k=1}^{n+1} \varphi_k = f(0, \dots, 0) f(1, \dots, 1) \prod_{k=1}^{n} f^2(1, \dots, 1, 0, \dots, 0) \le 0$$

because of (4.11), hence there exists a k for which $\varphi_k \le 0$. Since this k depends on i, let us denote it by k_i. Then

$$f(1, \dots, 1, 0, 0, \dots, 0) f(1, \dots, 1, 1, 0, \dots, 0) = \varphi_{k_i} \le 0,$$

hence by continuity of f there exists a $\tau_i \in [0, 1]$ such that

$$f(1, \dots, 1, \tau_i, 0, \dots, 0) = 0, \tag{4.12}$$

where τ_i stands at the k_ith position. Now, using the vector

$$t = (1, \dots, 1, \tau_i, 0, \dots, 0),$$

define

$$A_{ij} = \begin{cases} \underline{A}_{ij} + t_j(\overline{A}_{ij} - \underline{A}_{ij}) & \text{if } x_j^+ > 0, \\ \overline{A}_{ij} + t_j(\underline{A}_{ij} - \overline{A}_{ij}) & \text{if } x_j^+ = 0, \end{cases} \qquad (j = 1, \dots, n), \tag{4.13}$$

and

$$b_i = \overline{b}_i + t_{n+1}(\underline{b}_i - \overline{b}_i). \tag{4.14}$$

Then, since $x = x^+ - x^-$ and $x_i^+ x_i^- = 0$ for each i, it follows from (4.10) and (4.13), (4.14) that

$$\sum_{j=1}^{n} A_{ij} x_j = b_i \tag{4.15}$$

holds. If $x_j = 0$ for some j, then (4.15) will remain in force if we change A_{ij} to $\underline{A}_{ij}$. After this change A_{ij} is given by

$$A_{ij} = \begin{cases} \underline{A}_{ij} + t_j(\overline{A}_{ij} - \underline{A}_{ij}) & \text{if } x_j^+ > 0, \\ \overline{A}_{ij} + t_j(\underline{A}_{ij} - \overline{A}_{ij}) & \text{if } x_j^- > 0, \\ \underline{A}_{ij} & \text{if } x_j = 0. \end{cases}$$

We shall verify that the quantities A_{ij}, b_i given by (4.16), (4.14) satisfy (4.6), (4.7). If $(k_i - j)x_j < 0$, then either $j > k_i$ and $x_j > 0$, in which case $t_j = 0$, $x_j^+ > 0$, and (4.16) gives $A_{ij} = \underline{A}_{ij}$; or $j < k_i$ and $x_j < 0$, in which case $t_j = 1$, $x_j^- > 0$, and (4.16) again gives $A_{ij} = \underline{A}_{ij}$. If $x_j = 0$, then $A_{ij} = \underline{A}_{ij}$ in accordance with (4.6). Similarly, if $(k_i - j)x_j > 0$, then either $j < k_i$ and $x_j > 0$, hence $t_j = 1$ and $x_j^+ > 0$, implying $A_{ij} = \overline{A}_{ij}$; or $j > k_i$ and $x_j < 0$, hence $t_j = 0$ and $x_j^- > 0$, again implying $A_{ij} = \overline{A}_{ij}$.

If $j = k_i$ and $x_j \neq 0$, then $t_j = \tau_i \in [0,1]$ and (4.16) shows that A_{ij} is a convex combination of $\underline{A}_{ij}$ and $\overline{A}_{ij}$, hence $A_{ij} \in [\underline{A}_{ij}, \overline{A}_{ij}]$. This proves (4.6). If $k_i \leq n$, then $t_{n+1} = 0$ and $b_i = \overline{b}_i$ by (4.14); if $k_i = n+1$, then $t_{n+1} = \tau_i \in [0,1]$ and, again by (4.14), $b_i \in [\underline{b}_i, \overline{b}_i]$. Hence (4.7) holds. We have constructed A_{ij} $(j = 1, \ldots, n)$ and b_i satisfying (4.15), (4.6) and (4.7). Performing the construction for $i = 1, \ldots, m$, we obtain a matrix A satisfying (4.6) and a vector b satisfying (4.7) such that $Ax = b$ holds. This concludes the proof. □

It is worth pointing out that the matrix A given by (4.6) has a very specific pattern:

Corollary 147 *The matrix A constructed in Theorem 146 has the following properties. Given $i_1, i_2 \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, there holds:*

(a) if $(k_{i_1} - j)(k_{i_2} - j) > 0$, then $A_{i_1 j}$, $A_{i_2 j}$ are attained either both at the lower bound, or both at the upper bound,

(b) if $(k_{i_1} - j)(k_{i_2} - j) < 0$ and $x_j \neq 0$, then one of $A_{i_1 j}$, $A_{i_2 j}$ is attained at the lower bound, whereas the other one at the upper bound.

Proof: We use the formula (4.6). (a) If $x_j = 0$, then $A_{i_1 j} = \underline{A}_{i_1 j}$ and $A_{i_2 j} = \underline{A}_{i_2 j}$. If $x_j \neq 0$, then $(k_{i_1} - j)x_j(k_{i_2} - j)x_j = (k_{i_1} - j)(k_{i_2} - j)x_j^2 > 0$, hence $(k_{i_1} - j)x_j$ and $(k_{i_2} - j)x_j$ are of the same sign. If both of them are positive, then $A_{i_1 j} = \overline{A}_{i_1 j}$ and $A_{i_2 j} = \overline{A}_{i_2 j}$; if both of them are negative, then $A_{i_1 j} = \underline{A}_{i_1 j}$ and $A_{i_2 j} = \underline{A}_{i_2 j}$.

(b) If $(k_{i_1} - j)(k_{i_2} - j) < 0$ and $x_j \neq 0$, then $(k_{i_1} - j)x_j(k_{i_2} - j)x_j < 0$, so that $(k_{i_1} - j)x_j$ and $(k_{i_2} - j)x_j$ are of opposite signs, hence one of $A_{i_1 j}$, $A_{i_2 j}$ is attained at its lower bound whereas the other one at its upper bound. □ □

Finally, avoiding the complicated formulae, we can extract the essence of our result in the following simplified statement:

Corollary 148 *Let $x \in X$. Then there holds $Ax = b$ for some $A \in \mathbf{A}$, $b \in \mathbf{b}$ such that for each $i = 1, \ldots, m$ we have $A_{ij} \in \{\underline{A}_{ij}, \overline{A}_{ij}\}$ $(j = 1, \ldots, n)$, $b_i \in \{\underline{b}_i, \overline{b}_i\}$ for all but at most one entry.*

Proof: The result follows immediately from Theorem 146; it is the k_ith entry of the ith row of $(A\ b)$ which makes the exception $(i = 1, \ldots, m)$. □ □

Finally, it should be noted that the normal form of A and b in Theorem 146 can be made unique (for a given x) if for each i first $k_i \in \{1, \ldots, n+1\}$, then $\tau_i \in [0,1]$ are chosen minimal possible (this can be done because $f(1, \ldots, 1, t_{k_i}, 0, \ldots, 0)$ is linear in t_{k_i} and therefore it has a minimum root τ_i).

4.2.4 (Un)boundedness: The Beeck-Jansson theorem

Theorem 149. (Beeck-Jansson) *Let* $\mathbf{A}$ *be square and let* X *be the solution set of* $\mathbf{A}x = \mathbf{b}$*. Then we have:*

(i) if $\mathbf{A}$ *is regular, then* X *is compact and connected,*
(ii) if $\mathbf{A}$ *is singular, then each component of* X *is unbounded.*

Proof: (i) If $\mathbf{A}$ is regular, then X, as the range of the continuous mapping $(A, b) \mapsto A^{-1}b$ of a compact convex set $\mathbf{A} \times \mathbf{b}$, is compact and connected.

(ii) Let $\mathbf{A}$ be singular and let C be a component of X. Since $C \neq \emptyset$ by definition, there exists an $x_0 \in C$ satisfying $A_0 x_0 = b_0$ for some $A_0 \in \mathbf{A}$ and $b_0 \in \mathbf{b}$. If A_0 is singular, then $A_0 \hat{x} = 0$ for some $\hat{x} \neq 0$ and C contains the unbounded set $\{x_0 + \lambda \hat{x};\ \lambda \in R^1\}$, hence C is unbounded. Thus let A_0 be nonsingular. Since $\mathbf{A}$ is singular, it contains a singular matrix A_1. For each $t \in [0, 1]$ denote

$$A_t = A_0 + t(A_1 - A_0), \tag{4.16}$$

and let

$$\tau = \inf\{t \in [0, 1];\ A_t \text{ is singular}\}.$$

In view of continuity of the determinant, the infimum is attained as minimum, hence A_τ is singular and $\tau \in (0, 1]$. For each $t \in [0, \tau)$, A_t is nonsingular, hence

$$x_t = A_t^{-1} b_0$$

is well defined and the mapping $s \mapsto x_s$, $s \in [0, t]$, defines a curve in X connecting x_0 with x_t, hence $x_t \in C$ for each $t \in [0, \tau)$. Consider now the sequence of points $\{x_{t_m}\}$ with

$$t_m = (1 - \tfrac{1}{m})\tau, \tag{4.17}$$

$m = 1, 2, \ldots$. If $\{x_{t_m}\}$ is unbounded, then C is unbounded; if it is bounded, then it contains a convergent subsequence $\{x_{t_{m_k}}\}$, $x_{t_{m_k}} \to x^*$, and $x^* \in C$ since C is closed. As

$$A_{t_{m_k}} x_{t_{m_k}} = b_0$$

holds for each k, taking $k \to \infty$ we obtain in view of (4.16), (4.17) that

$$A_\tau x^* = b_0.$$

But since A_τ is singular, there exists an $\tilde{x} \neq 0$ with $A_\tau \tilde{x} = 0$, hence $A_\tau(x^* + \lambda \tilde{x}) = b_0$ for each $\lambda \in R^1$, which shows that C contains the unbounded set $\{x^* + \lambda \tilde{x};\ \lambda \in R^1\}$; this concludes the second part of the proof. $\square$

Theorem 150. *Given a system* $\mathbf{A}x = \mathbf{b}$*,* $\mathbf{A}$ *square, the following algorithm either detects singularity of* $\mathbf{A}$*, or constructs a subset* $D \subseteq Y_n$ *such that*

$$X \subset \bigcup_{z \in D} \mathbb{R}^n_z$$

(in which case $\mathbf{A}$ *is regular):*

$z := \mathrm{sgn}\,(A_c^{-1} b_c)$; $Z := \{z\}$; $D := \emptyset$;
while $Z \neq \emptyset$
  select $z \in Z$; $Z := Z - \{z\}$; $D := D \cup \{z\}$;
  if $X \cap \mathbb{R}^n_z$ is unbounded
    terminate: $\mathbf{A}$ is singular
  end
  for $j = 1 : n$
    if $x_j = 0$ for some $x \in X \cap \mathbb{R}^n_z$ **and** $z - 2z_j e_j \notin Z \cup D$
      $Z := Z \cup \{z - 2z_j e_j\}$;
    end
  end
end

sing:=false;
if A_c is singular **then** *sing:=true*
else
  $L := \emptyset$; $K := \emptyset$;
  select b; solve $A_c x = b$;
  $z := \mathrm{sgn}\, x$; insert z into L;
  repeat
    remove an item z from L;
    insert z into K;
    if (...) is unbounded **then** *sing:=true*
    else if (...) is feasible **then** $L := L \cup (N(z) - (K \cup L))$
  until (*sing* or $L = \emptyset$);
if *sing* **then** {$\mathbf{A}$ is singular} **else** {$\mathbf{A}$ is regular}.

In order to simplify the proof of the main theorem, we first formulate an auxiliary result concerning the case when singularity was not detected during the algorithm. Denote by C_0 the component of $X(A^I, b)$ containing the point $x_c = A_c^{-1} b$ and let Y_0 be the set of ± 1-vectors that were inserted into L in the course of the algorithm. Then Y_0 has this property:

Lemma 151 *If* $x \in C_0$ *and* $T_{z_0} x \geq 0$ *for some* $z_0 \in Y_0$*, then each* $z \in Z$ *with* $T_z x \geq 0$ *belongs to* Y_0*.*

Proof: For the purpose of the proof, denote the linear programming problem

$$\max\{z^T x;\ (A_c - \Delta T_z)x \le b,\ (A_c + \Delta T_z)x \ge b,\ T_z x \ge 0\} \tag{4.18}$$

by $P(z)$. Then x is a feasible solution of some $P(\overline{z})$. Since $T_{z_0}x = |x| = T_{\overline{z}}x$, we can see from the form of (4.18) that x is a feasible solution of $P(z_0)$. Let $T_z x \ge 0$, $z \ne z_0$. Denote

$$J = \{j;\ z_j \ne (z_0)_j\} = \{j_1, \dots, j_m\}.$$

Since $T_{z_0}x \ge 0$ and $T_z x \ge 0$, it must be $x_j = 0$ for each $j \in J$. Set $z^0 = z_0$ and define vectors $z^k \in Z$, $k = 1, \dots, m$, in the following way:

$$(z^k)_j = \begin{cases} (z^{k-1})_j & \text{if } j \ne j_k, \\ -(z^{k-1})_j & \text{if } j = j_k \end{cases} \tag{4.19}$$

($k = 1, \dots, m$, $j = 1, \dots, n$). We shall prove by induction on $k = 0, \dots, m$ that $z^k \in Y_0$ and x is a feasible solution of $P(z^k)$. This is obvious for $k = 0$. If the assumption is true for some $k-1 \ge 0$, then $z^{k-1} \in Y_0$ and $P(z^{k-1})$ is feasible, hence $N(z^{k-1}) - (K \cup L)$ was added to L in the respective step. Since $z^k \in N(z^{k-1})$ by (4.19), z^k was either already present in $K \cup L$, or newly added to L, in both cases $z^k \in Y_0$. Furthermore, since x is a feasible solution of $P(z^{k-1})$ and $T_{z^{k-1}}x = T_{z^k}x$ holds as $x_{j_k} = 0$, it is also a feasible solution of $P(z^k)$. This concludes the proof by induction; since $z^m = z$, we have $z \in Y_0$. □

As it can be seen, this detailed proof is a formalization of the following idea: if we take a path from x_c to $x \in C_0$, then the only change of signs occurs when the path passes through a point with one or more zero components; all the respective sign vectors are added to L in the course of the algorithm, hence they belong to Y_0.

Now we finally prove that the algorithm really performs the task for which it was designed:

Theorem 152. *For each $n \times n$ interval matrix $\mathbf{A}$ and each $b \in \mathbb{R}^n$, the algorithm in a finite number of steps checks regularity or singularity of $\mathbf{A}$.*

Proof: First, only elements of the finite set Z are being inserted into L and no element may be reinserted, hence the algorithm terminates in a finite number of steps. If some problem (4.18) is proved unbounded, then $\mathbf{A}$ is singular by Theorem 149. Hence, we only have to prove that if $\mathbf{A}$ has not been found singular and if the list L becomes empty, then $\mathbf{A}$ is regular.

The component C_0 may be written in the form

$$C_0 = \bigcup_{z \in Y} X_z(A^I, b),$$

where Y may be chosen so that $X_z(A^I, b) \neq \emptyset$ for each $z \in Y$. We shall prove that

$$Y \subseteq Y_0 \tag{4.20}$$

holds. Since $L = \emptyset$, this will imply that all $X_z(A^I, b)$, $z \in Y$ have been checked to be bounded, hence $\mathbf{A}$ is regular by Theorem 149.

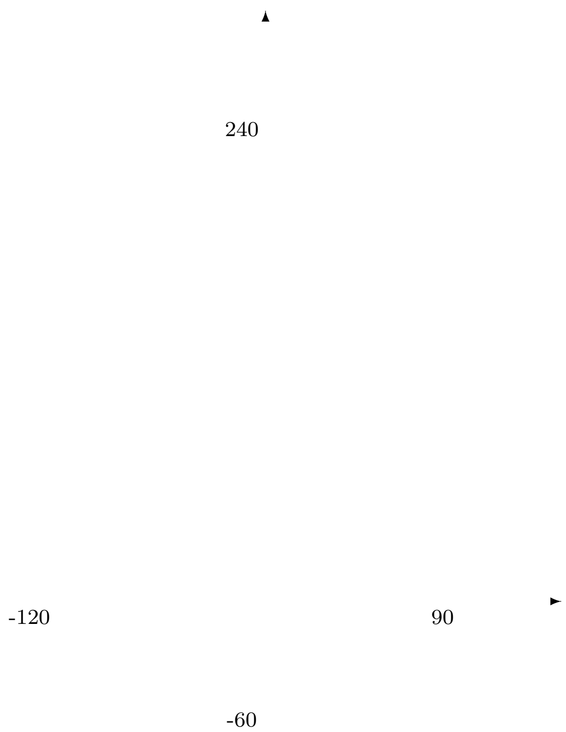
To prove (4.20), take a $\overline{z} \in Y$. Choose an $x \in X_{\overline{z}}(\mathbf{A}, b)$, so that $T_{\overline{z}}x \geq 0$. Since C_0 is connected and contains $x_c = A_c^{-1}b$, there exists a path from x_c to x, contained entirely in C_0. In view of convexity of the sets $X_z(A^I, b), z \in Z$, the path may be chosen in a piecewise linear form $x^0x^1 \dots x^m$, where $x^0 = x_c$, $x^m = x$ and the segment with endpoints x^i, x^{i+1} is always a part of a single orthant ($i = 0, \dots, m-1$). We shall prove by induction on i that for each $i = 0, \dots, m$, if $T_z x^i \geq 0$ for some $z \in Z$, then $z \in Y_0$. Since $z_c = \operatorname{sgn} x_c$ is inserted into L at the beginning of the main loop and $T_{z_c}x_c \geq 0$, the assertion for $x^0 = x_c$ follows from Lemma 151. Let the assertion be true for x^i, $i \geq 0$. Since the whole segment with endpoints x^i and x^{i+1} is a part of a single orthant, there exists a $\tilde{z} \in Z$ such that $T_{\tilde{z}}x^i \geq 0$ and $T_{\tilde{z}}x^{i+1} \geq 0$. Then $\tilde{z} \in Y_0$ by assumption concerning x^i, hence from $T_{\tilde{z}}x^{i+1} \geq 0$, $\tilde{z} \in Y_0$ we obtain from Lemma 151 that each $z \in Z$ with $T_z x^{i+1} \geq 0$ belongs to Y_0. This concludes the proof by induction. Hence, since $T_{\overline{z}}x^m = T_{\overline{z}}x \geq 0$, we have $\overline{z} \in Y_0$. This proves (4.20). □

4.2.5 Examples

Example. Consider the example by Hansen [19]: $\mathbf{A} = [\underline{A}, \overline{A}]$, $\mathbf{b} = [\underline{b}, \overline{b}]$, where

$$\underline{A} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}, \quad \overline{A} = \begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} 0 \\ 60 \end{pmatrix}, \quad \overline{b} = \begin{pmatrix} 120 \\ 240 \end{pmatrix}.$$

Since for each $z \in Y_2$ the intersection of the set of weak solutions X with the orthant $\{x \in \mathbb{R}^2;\, T_z x \geq 0\}$ is described by (4.5), considering separately all four orthants we arrive at this picture of the set X:



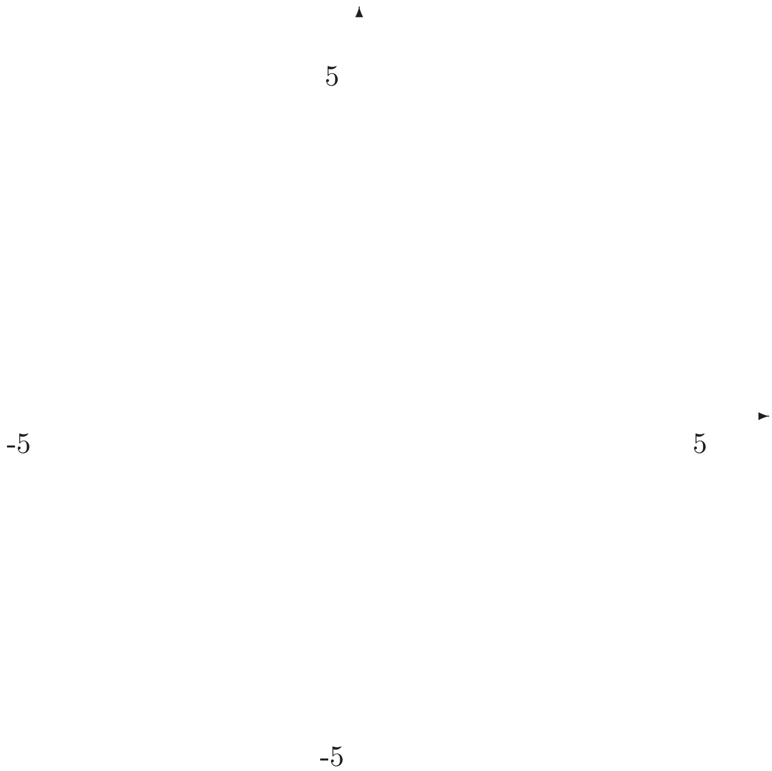


Figure 4.1: The example by Barth and Nuding

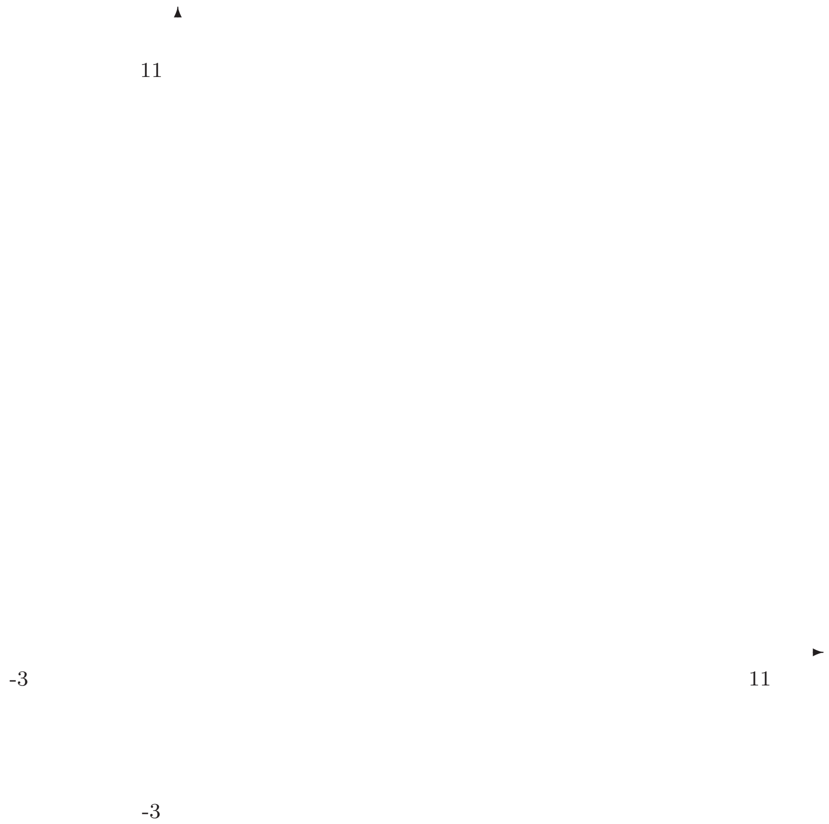


Figure 4.2: The example by Nickel

4.3 The points x_y

4.3.1 Existence and uniqueness

Lemma 153. *Let* $\mathbf{A}$ *be regular and let*

$$A_{yz'}x' = A_{yz''}x'' \tag{4.21}$$

hold for some $y, z', z'' \in Y_n$ *and* $x' \neq x''$*. Then there exists a* j *satisfying* $z'_j z''_j = -1$ *and* $x'_j x''_j > 0$.

Proof: Assume to the contrary that for each j, $z'_j z''_j = -1$ implies $x'_j x''_j \leq 0$ and hence also $|x'_j - x''_j| = |x'_j| + |x''_j|$. We shall prove that in this case

$$|T_{z'}x' - T_{z''}x''| \leq |x' - x''|, \tag{4.22}$$

i.e. that

$$|z'_j x'_j - z''_j x''_j| \leq |x'_j - x''_j|$$

holds for each j. In fact, this is obvious for $z'_j z''_j = 1$. If $z'_j z''_j = -1$, then

$$|z'_j x'_j - z''_j x''_j| = |x'_j + x''_j| \leq |x'_j| + |x''_j| = |x'_j - x''_j|,$$

which proves (4.22). Now, from (4.21) we have

$$(A_c - T_y \Delta T_{z'})x' = (A_c - T_y \Delta T_{z''})x''$$

which implies

$$|A_c(x' - x'')| = |T_y \Delta (T_{z'}x' - T_{z''}x'')| \leq \Delta |T_{z'}x' - T_{z''}x''| \leq \Delta |x' - x''|,$$

hence $\mathbf{A}$ is singular, a contradiction. □

Theorem 154. *Let* $\mathbf{A}$ *be regular. Then for each* $y \in Y_n$ *the equation*

$$A_c x - T_y \Delta |x| = b_y \tag{4.23}$$

has a unique solution $x_y \in X$ *and there holds*

$$\operatorname{Conv} X = \operatorname{Conv} \{x_y \,;\, y \in Y_n\}.$$

Proof: We shall first simplify the description of the algorithm by proving by induction that after each updating of C, the current values of z, x and C satisfy

$$x = A_{yz}^{-1} b_y, \tag{4.24}$$
$$C = A_{yz}^{-1} T_y \Delta. \tag{4.25}$$

This is obviously so for the initial values of z, x and C. Thus let (4.24), (4.25) hold true at some step and let $\tilde{z}$, $\tilde{x}$ and $\tilde{C}$ be the updated values, i.e.

$$\begin{aligned}
\tilde{z}_k &= -z_k, \\
\alpha &= 2\tilde{z}_k/(1-2\tilde{z}_k C_{kk}) = -2z_k/(1+2z_k C_{kk}), \\
\tilde{x} &= x + \alpha x_k C_{\cdot k}, \\
\tilde{C} &= C + \alpha C_{\cdot k} C_{k\cdot}
\end{aligned}$$

Since the matrix

$$A_{y\tilde{z}} = A_c - T_y \Delta (T_z - 2z_k e_k e_k^T) = A_{yz} + (2z_k T_y \Delta e_k) e_k^T \in \mathbf{A}$$

is nonsingular, it follows from the Sherman-Morrison theorem that

$$0 \neq 1 + e_k^T A_{yz}^{-1} 2z_k T_y \Delta e_k = 1 + 2z_k C_{kk},$$

hence α is well defined and

$$A_{y\tilde{z}}^{-1} = A_{yz}^{-1} - \frac{A_{yz}^{-1} 2z_k T_y \Delta e_k e_k^T A_{yz}^{-1}}{1+2z_k C_{kk}} = A_{yz}^{-1} + \alpha C_{\cdot k} e_k^T A_{yz}^{-1}.$$

Then we have

$$A_{y\tilde{z}}^{-1} b_y = A_{yz}^{-1} b_y + \alpha C_{\cdot k} e_k^T A_{yz}^{-1} b_y = x + \alpha x_k C_{\cdot k} = \tilde{x}$$

and

$$A_{y\tilde{z}}^{-1} T_y \Delta = A_{yz}^{-1} T_y \Delta + \alpha C_{\cdot k} e_k^T A_{yz}^{-1} T_y \Delta = C + \alpha C_{\cdot k} C_{k\cdot} = \tilde{C},$$

which proves (4.24), (4.25) by induction.

Hence we can see that the matrix C plays a purely auxiliary role, helping to avoid explicit computation of $x = A_{yz}^{-1} b_y$ at each step, and its elimination brings the algorithm to a more transparent (but computationally less efficient) form:

$z := \operatorname{sgn}(A_c^{-1} b_y)$;
$x := A_{yz}^{-1} b_y$;
while $z_j x_j < 0$ for some j **do**
  $k := \min\{j;\ z_j x_j < 0\}$;
  $z_k := -z_k$;
  $x := A_{yz}^{-1} b_y$;
$x_y := x$.

Using this form, we shall now prove finiteness of the sequence of k's generated in the loop of the algorithm by induction, showing that each k can occur there at most 2^{n-k} times ($k = n, \ldots, 1$).

Case $k = n$: Assume that n appears at least twice in the sequence, and let z', x' and z'', x'' correspond to any two nearest occurrences of it (i.e., there is no other occurrence of n between them). Then $z'_j x'_j \geq 0$, $z''_j x''_j \geq 0$ for $j = 1, \ldots, n-1$, and

$z'_n x'_n < 0$, $z''_n x''_n < 0$, $z'_n z''_n = -1$, which implies $z'_n x'_n z''_n x''_n > 0$ and $x'_n x''_n < 0$. Hence, $z'_j x'_j z''_j x''_j \geq 0$ for each j. But since

$$A_{yz'} x' = b_y = A_{yz''} x'' \tag{4.26}$$

holds and $x' \neq x''$ (since $x'_n x''_n < 0$), it follows from Lemma 153 that there exists a j with $z'_j z''_j = -1$ and $x'_j x''_j > 0$ implying $z'_j x'_j z''_j x''_j < 0$, a contradiction; hence n occurs at most once in the sequence.

Case $k < n$: Again, let z', x' and z'', x'' correspond to any two nearest occurrences of k, so that $z'_j x'_j z''_j x''_j \geq 0$ for $j = 1, \dots, k-1$, $z'_k x'_k < 0$, $z''_k x''_k < 0$ and $z'_k z''_k = -1$, which implies $z'_k x'_k z''_k x''_k > 0$ and $x'_k x''_k < 0$. Since (4.26) holds, and $x' \neq x''$ because of $x'_k x''_k < 0$, Lemma 153 implies existence of a j with $z'_j z''_j = -1$ and $x'_j x''_j > 0$, hence $z'_j x'_j z''_j x''_j < 0$, so that $j > k$; since $z'_j z''_j = -1$, j must have entered the sequence between the two occurrences of k. Hence between any two nearest occurrences of k there is an occurrence of some $j > k$ in the sequence; this means by induction that k cannot occur there more than $(2^{n-k-1} + \ldots + 2 + 1) + 1 = 2^{n-k}$ times.

Hence the algorithm terminates in a finite number of steps with an x satisfying $A_{yz} x = b_y$ and $T_z x \geq 0$ for some $z \in Y$. Then $x \in X$ and $T_z x = |x|$, so that $A_c x - T_y \Delta |x| = A_{yz} x = b_y$ and x is a solution to (4.23). To prove uniqueness, assume to the contrary that (4.23) has solutions x' and x'', $x' \neq x''$. Put $z' = \operatorname{sgn} x'$, $z'' = \operatorname{sgn} x''$, then $T_{z'} x' \geq 0$, $T_{z''} x'' \geq 0$ and (4.26) holds, hence by Lemma 153 there is a j with $z'_j z''_j = -1$ and $x'_j x''_j > 0$, implying $z'_j x'_j z''_j x''_j < 0$ contrary to $z'_j x'_j \geq 0$ and $z''_j x''_j \geq 0$, a contradiction.

Hence the equation (4.23) has a unique solution x_y which can be computed by the algorithm in a finite number of steps. □

```
function [x_y, flag] = xyvector (A, b, y)
[x_y, flag] = signaccord (A_c, -T_y Δ, b_y);
 if flag = 'singular', return
 else flag = 'x_y computed';
 end
```

Figure 4.3: An algorithm for computing x_y.

4.3.2 Properties of the sign-accord algorithm

dodat p

Theorem 155. *If* $\mathbf{A}$ *is singular and* $\delta > 0$*, then there exists a* $y \in Y_n$ *for which the sign accord algorithm detects singularity.*

Theorem 156. *If* $\delta > 0$ *and if for each* $y \in Y_n$ *the sign accord algorithm produces a vector* x_y *(i.e., it does not fail), then* $\mathbf{A}$ *is regular.*

Theorem 157. *If* $\mathbf{A}$ *is regular and* $\delta > 0$*, then all the points* x_y*,* $y \in Y_n$*, are mutually different.*

Proof: [69], p. 37. □

Proposition 158. *Let*

$$|A_c^{-1}|\Delta|x_y| < |x_y|$$

hold for some $y \in Y_n$*. Then the sign accord algorithm finds* x_y *in the first iteration (i.e., it circumvents the "while" loop).*

Proof: [75], p. 40. □

Theorem on explicit formulae for the bounds in case $\Delta = pq^T$ (including λ, μ etc.):
Proof: [76], pp. 4-6. □

4.3.3 An iterative method for computing the x_y's

Theorem 159. *Let $M \geq 0$ and R satisfy*

$$M(I - |I - RA_c| - |R|\Delta) \geq I. \tag{4.27}$$

Then for each Y_n the sequence $\{x^i\}_{i=0}^{\infty}$ given by $x^0 = Rb_y$ and

$$x^{i+1} = (I - RA_c)x^i + RT_y\Delta|x^i| + Rb_y \tag{4.28}$$

$(i = 0, 1, \dots)$ converges to x_y and for each $i \geq 1$ there holds

$$|x_y - x^i| \leq (M - I)|x^i - x^{i-1}|. \tag{4.29}$$

Proof: Denote $G = |I - RA_c| + |R|\Delta$. By Theorem 43, (4.27) implies that $\mathbf{A}$ is strongly regular, $\varrho(G) < 1$, $G^i \to 0$ for $i \to \infty$, $\sum_{i=0}^{\infty} G^i = (I - G)^{-1} \leq M$, and R is nonsingular. Let $i \geq 1$. Subtracting the equations

$$\begin{aligned} x^{i+1} &= (I - RA_c)x^i + RT_y\Delta|x^i| + Rb_y, \\ x^i &= (I - RA_c)x^{i-1} + RT_y\Delta|x^{i-1}| + Rb_y, \end{aligned}$$

we get

$$|x^{i+1} - x^i| \leq |I - RA_c| \cdot |x^i - x^{i-1}| + |R|\Delta||x^i| - |x^{i-1}|| \leq G|x^i - x^{i-1}|$$

and for each $m \geq 1$ by induction

$$\begin{aligned} |x^{i+m} - x^i| &= |\sum_{j=0}^{m-1}(x^{i+j+1} - x^{i+j})| \leq \sum_{j=0}^{m-1}|x^{i+j+1} - x^{i+j}| \leq \sum_{j=0}^{m-1} G^{j+1}|x^i - x^{i-1}| \\ &\leq (\sum_{j=0}^{\infty} G^{j+1})|x^i - x^{i-1}| \leq (M - I)|x^i - x^{i-1}| \leq (M - I)G^{i-1}|x^1 - x^0|. \end{aligned}$$

From the final inequality

$$|x^{i+m} - x^i| \leq (M - I)G^{i-1}|x^1 - x^0|,$$

in view of the fact that $G^{i-1} \to 0$ as $i \to \infty$, we can see that the sequence $\{x^i\}$ is Cauchian, thus convergent, $x^i \to x^*$. Taking the limit in (4.28) and employing nonsingularity of R, we obtain that x^* satisfies

$$A_c x^* - T_y\Delta|x^*| = b_y,$$

and from the uniqueness of the solution of this equation (Theorem 154) we conclude that $x^* = x_y$. The estimation (4.29) follows from the above-established inequality

$$|x^{i+m} - x^i| \leq (M - I)|x^i - x^{i-1}|$$

by taking $m \to \infty$. □

4.3.4 Computing x_y in the special case $\Delta = pq^T$

Theorem 160. *Let* $\mathbf{A} = [A_c - pq^T, A_c + pq^T]$ *be regular. Then for each* $y \in Y_n$ *there holds*

$$x_y = \alpha_y A_c^{-1} T_y p + A_c^{-1} b_y,$$

where α_y *is the unique solution of the scalar equation*

$$\alpha = q^T |\alpha A_c^{-1} T_y p + A_c^{-1} b_y|.$$

Furthermore, if $q^T|A_c^{-1}|p < 1$*, then* $\alpha_y = \lim_{i \to \infty} \alpha_i$*, where the sequence* $\{\alpha_i\}_{i=0}^{\infty}$ *is given by*

$$\begin{aligned} \alpha_0 &= 0, \\ \alpha_{i+1} &= q^T |\alpha_i A_c^{-1} T_y p + A_c^{-1} b_y| \qquad (i = 0, 1, \ldots). \end{aligned}$$

Proof: Unpublished, but may be inferred from the proof of Theorem 3.4 in [81], p. 54.
□

4.3.5 The convex hull theorem

Theorem 161. (Convex hull theorem) *Let* $\mathbf{A}$ *be regular. Then*

$$\operatorname{Conv} X = \operatorname{Conv}\{x_y\,;\, y \in Y_n\}.$$

Proof: By Theorem 154, $x_y \in X$ for each $y \in Y_m$, hence also $\operatorname{Conv}\{x_y;\, y \in Y_m\} \subseteq \operatorname{Conv} X$. To prove the converse inclusion, take $A \in \mathbf{A}$ and $b \in \mathbf{b}$. Let $y \in Y_m$. Since

$$|T_y((A - A_c)x_y + b_c - b)| \leq \Delta|x_y| + \delta,$$

we have

$$\begin{aligned} T_y(Ax_y - b) &= T_y(A_c x_y - b_c) + T_y((A - A_c)x_y + b_c - b) \\ &\geq T_y(A_c x_y - b_c) - \Delta|x_y| - \delta \\ &= T_y(A_c x_y - T_y\Delta|x_y| - b_y) = 0 \end{aligned}$$

(since x_y solves (4.23)), hence

$$T_y A x_y \geq T_y b.$$

Then existence lemma ... implies that the solution x of the equation $Ax = b$ (which is unique because of regularity of $\mathbf{A}$) belongs to $\operatorname{Conv}\{x_y;\, y \in Y_m\}$. Since A and b were arbitrary in $\mathbf{A}$, $\mathbf{b}$, this gives $X \subseteq \operatorname{Conv}\{x_y;\, y \in Y_m\}$, and thereby also $\operatorname{Conv} X \subseteq \operatorname{Conv}\{x_y;\, y \in Y_m\}$, which concludes the proof. □

4.3.6 (Non)convexity of the solution set

Theorem 162. *Let* $\mathbf{A}$ *be regular. Then the solution set* X *of the system* $\mathbf{A}x = \mathbf{b}$ *is nonconvex if and only if there exist* $y, z \in Y$ *and* $i, j \in \{1, \ldots, n\}$ *such that* $y_i = z_i$, $(x_y)_j(x_z)_j < 0$ *and* $\Delta_{ij} > 0$. *Moreover, if this is the case, then no point of the segment connecting* x_y *with* x_z, *except the endpoints, belongs to* X.

Proof: We shall first prove the "if" part. Assuming that y, z, i and j with the properties listed exist, take real numbers $\lambda > 0$ and $\mu > 0$ with $\lambda + \mu = 1$ and put $x = \lambda x_y + \mu x_z$. Then $|x|_j < \lambda|x_y|_j + \mu|x_z|_j$, whereas $|x|_k \leq \lambda|x_y|_k + \mu|x_z|_k$ for each $k \neq j$. Since x_y and x_z satisfy

$$\begin{aligned}(A_c x_y - b_c)_i &= y_i(\Delta|x_y| + \delta)_i, \\ (A_c x_z - b_c)_i &= z_i(\Delta|x_z| + \delta)_i = y_i(\Delta|x_z| + \delta)_i\end{aligned}$$

due to (4.23), we obtain, using the positivity of Δ_{ij}, that

$$|A_c x - b_c|_i = |\lambda(A_c x_y - b_c) + \mu(A_c x_z - b_c)|_i = (\Delta(\lambda|x_y| + \mu|x_z|) + \delta)_i > (\Delta|x| + \delta)_i,$$

which in view of the Oettli-Prager theorem means that $x \notin X$. Since x was an arbitrary interior point of the segment connecting x_y with x_z, we can see that no such point belongs to X, and X is nonconvex.

To prove the "only if" part of the theorem, assume on the contrary that for each $y, z \in Y$ and each $i, j \in \{1, \ldots, n\}$, $y_i = z_i$ and $\Delta_{ij} > 0$ imply $(x_y)_j(x_z)_j \geq 0$. We shall prove that in this case each convex combination of vectors x_y belongs to X. This, in the light of the Convex hull theorem 161, will imply that $\operatorname{Conv} X \subseteq X$, proving that X is convex. So let $x = \sum_{y \in Y_m} \lambda_y x_y$, where λ_y, $y \in Y_m$, are nonnegative real numbers satisfying $\sum_{y \in Y_m} \lambda_y = 1$. Then from (4.23) we have

$$\begin{aligned}(A_c x - b_c)_i &= \sum_{y \in Y_m} \lambda_y (A_c x_y - b_c)_i = \sum_{y \in Y_m} \lambda_y y_i (\Delta|x_y| + \delta)_i \\ &= \sum_{j=1}^{n} \Delta_{ij} \Big(\sum_{y \in Y_m} \lambda_y y_i |x_y|_j\Big) + \sum_{y \in Y_m} \lambda_y y_i \delta_i\end{aligned}$$

and using our assumption that $y_i = z_i$ and $\Delta_{ij} > 0$ imply $(x_y)_j(x_z)_j \geq 0$, we obtain

$$(A_c x - b_c)_i = \sum_{j=1}^{n} \Delta_{ij} \Big(\Big|\sum_{y \in Y_m,\, y_i = 1} \lambda_y x_y\Big|_j - \Big|\sum_{y \in Y_m,\, y_i = -1} \lambda_y x_y\Big|_j\Big) + \sum_{y \in Y_m} \lambda_y y_i \delta_i.$$

Taking absolute values we have

$$|A_c x - b_c|_i \leq \sum_{j=1}^{n} \Delta_{ij} \Big|\sum_{y \in Y_m} \lambda_y x_y\Big|_j + \delta_i = (\Delta|x| + \delta)_i$$

for each $i \in \{1, \ldots, n\}$ and hence $|A_c x - b_c| \leq \Delta|x| + \delta$. This implies $x \in X$ in view of the Oettli-Prager theorem, and hence $\operatorname{Conv} X \subseteq X$. □

Corollary 163. *Let* **A** *be regular and let* $\Delta > 0$*. The* X *is nonconvex if and only if there exist* $y, z \in Y_n$*,* $y \neq -z$*, such that* $(x_y)_j(x_z)_j < 0$ *for some* j*.*

The reason for the assumption $y \neq -z$ is explained in the next theorem:

Theorem 164. *Let* **A** *be regular. Then for each* $y \in Y$ *the whole segment connecting* x_y *with* x_{-y} *belongs to* X*.*

4.4 Interval hull

4.4.1 Definition and basic formulae

Theorem 165. *Let* **A** *be regular. Then we have*

$$\begin{aligned}\underline{x} &= \min_{y\in Y_n} x_y = \min_{y,z\in Y_n} A^{-1}_{yz} b_y,\\ \overline{x} &= \max_{y\in Y_n} x_y = \max_{y,z\in Y_n} A^{-1}_{yz} b_y.\end{aligned}$$

4.4.2 NP-hardness of computing the hull

Given a nonsingular matrix $A \in \mathbb{R}^{(n-1)\times(n-1)}$ and a real number $\varepsilon > 0$, consider a linear interval system

$$\mathbf{A}x = \mathbf{b}, \tag{4.30}$$

where

$$\mathbf{A} = \begin{pmatrix} \varepsilon^2 & [-\varepsilon e^T, \varepsilon e^T] \\ 0 & A \end{pmatrix} \tag{4.31}$$

and

$$\mathbf{b} = \begin{pmatrix} 0 \\ [-\varepsilon e, \varepsilon e] \end{pmatrix} \tag{4.32}$$

with $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^{n-1}$. This means that the centers and radii are given by

$$A_c = \begin{pmatrix} \varepsilon^2 & 0^T \\ 0 & A \end{pmatrix}, \qquad \Delta = \begin{pmatrix} 0 & \varepsilon e^T \\ 0 & 0 \end{pmatrix},$$

$$b_c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \delta = \begin{pmatrix} 0 \\ \varepsilon e \end{pmatrix},$$

which implies that

$$|A_c^{-1}|\Delta = \begin{pmatrix} \frac{1}{\varepsilon^2} & 0^T \\ 0 & |A^{-1}| \end{pmatrix} \begin{pmatrix} 0 & \varepsilon e^T \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\varepsilon} e^T \\ 0 & 0 \end{pmatrix},$$

so that

$$\varrho(|A_c^{-1}|\Delta) = 0,$$

hence not only the interval matrix $\mathbf{A}$ is strongly regular, but also the spectral radius of $|A_c^{-1}|\Delta$ attains the lowest possible value independently of ε. Next, $A_c^{-1}b_c = 0$ and

$$|A_c^{-1}|\delta = \begin{pmatrix} 0 \\ \varepsilon |A^{-1}| e \end{pmatrix}.$$

Now we can state the basic result concerning the system (4.30)-(4.32):

Theorem 166. *Let A be nonsingular and let $\varepsilon > 0$. Then for the interval hull $[\underline{x}, \overline{x}]$ of the system (4.30)-(4.32) we have*

$$\overline{x} = -\underline{x} = \begin{pmatrix} \|A^{-1}\|_{\infty,1} \\ \varepsilon |A^{-1}| e \end{pmatrix}. \tag{4.33}$$

Proof: (a) Since $\mathbf{b}$ is symmetric about 0, the same holds for the solution set X of (4.30)-(4.32) (because if $x \in X$, then $A'x = b'$ for some $A' \in \mathbf{A}$ and $b' \in \mathbf{b}$, hence $A'(-x) = -b' \in \mathbf{b}$ and $-x \in X$), which implies that $\underline{x} = -\overline{x}$. Thus we are confined to evaluate $\overline{x}$ only. According to Theorem 165, we have

$$\overline{x} = \max_{y \in Y_m} x_y,$$

where for each $y \in Y_m$, x_y is the unique solution of the equation (4.23). Let us write $y = (y_1, y'^T)^T$, where $y' = (y_2, \ldots, y_n)^T$, and let us decompose x_y accordingly as $x_y = (x_1, x'^T)^T$. Then the equation (4.23) for the system (4.30)-(4.32) has the form

$$\begin{pmatrix} \varepsilon^2 & 0^T \\ 0 & A \end{pmatrix} \begin{pmatrix} x_1 \\ x' \end{pmatrix} = T_{\begin{pmatrix} y_1 \\ y' \end{pmatrix}} \left(\begin{pmatrix} 0 & \varepsilon e^T \\ 0 & 0 \end{pmatrix} \left| \begin{matrix} x_1 \\ x' \end{matrix} \right| + \begin{pmatrix} 0 \\ \varepsilon e \end{pmatrix} \right)$$

or equivalently

$$\varepsilon^2 x_1 = y_1 \varepsilon e^T |x'|,$$

$$Ax' = T_{y'} \varepsilon e = \varepsilon y',$$

which gives

$$x' = \varepsilon A^{-1} y',$$

$$x_1 = y_1 e^T |A^{-1} y'| = y_1 \|A^{-1} y'\|_1,$$

hence

$$x_y = \begin{pmatrix} y_1 \|A^{-1} y'\|_1 \\ \varepsilon A^{-1} y' \end{pmatrix},$$

and from Theorem 165 in view of (4.56) we obtain

$$\overline{x} = \max_{y \in Y_m} x_y = \begin{pmatrix} \|A^{-1}\|_{\infty,1} \\ \varepsilon |A^{-1}| e \end{pmatrix} = \begin{pmatrix} \|A^{-1}\|_{\infty,1} \\ \varepsilon d \end{pmatrix}.$$

(b) Since the right-hand side of the preconditioned system (4.55) is again symmetric about 0, we again have $\underline{\underline{x}} = -\overline{\overline{x}}$. The equation (4.23) for the preconditioned system (4.55), (4.31), (4.32) has the form

$$\begin{pmatrix} x_1 \\ x' \end{pmatrix} = T_{\begin{pmatrix} y_1 \\ y' \end{pmatrix}} \left(\begin{pmatrix} 0 & \frac{1}{\varepsilon} e^T \\ 0 & 0 \end{pmatrix} \left| \begin{matrix} x_1 \\ x' \end{matrix} \right| + \begin{pmatrix} 0 \\ \varepsilon |A^{-1}| e \end{pmatrix} \right),$$

which gives

$$x' = T_{y'} \varepsilon |A^{-1}| e = \varepsilon T_{y'} d,$$

$$x_1 = y_1 \tfrac{1}{\varepsilon} e^T |x'| = y_1 e^T |A^{-1}| e = y_1 \|d\|_1,$$

hence

$$x_y = \begin{pmatrix} y_1 \|d\|_1 \\ \varepsilon T_{y'} d \end{pmatrix}$$

and

$$\overline{\overline{x}} = \max_{y \in Y_m} x_y = \begin{pmatrix} \|d\|_1 \\ \varepsilon d \end{pmatrix},$$

which concludes the proof. □

Theorem 167. *Computing the interval hull of the solution set X is NP-hard even for systems with rational interval matrices satisfying*

$$\varrho(|A_c^{-1}|\Delta) = 0.$$

Proof: Given a rational $n \times n$ matrix A, construct the $(n+1) \times (n+1)$ interval matrix $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ with

$$A_c = \begin{pmatrix} 1 & 0^T \\ 0 & A \end{pmatrix}, \quad \Delta = \begin{pmatrix} 0 & e^T \\ 0 & 0 \end{pmatrix},$$

and the $(n+1)$-dimensional interval vector $\mathbf{b} = [b_c - \delta, b_c + \delta]$ with

$$b_c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \delta = \begin{pmatrix} 0 \\ e \end{pmatrix}$$

($e \in \mathbb{R}^n$). We have

$$|A_c^{-1}|\Delta = \begin{pmatrix} 0 & e^T \\ 0 & 0 \end{pmatrix},$$

hence

$$\varrho(|A_c^{-1}|\Delta) = 0.$$

Then each system $Ax = b$ with $A \in \mathbf{A}$, $b \in \mathbf{b}$ has the form

$$\begin{aligned} x_1 + c^T x' &= 0, \\ Ax' &= d \end{aligned}$$

for some $c \in [-e, e]$ and $d \in [-e, e]$, where $x' = (x_2, \ldots, x_{n+1})^T$. If $[\underline{x}, \overline{x}]$ is the interval hull of the solution set of $\mathbf{A}x = \mathbf{b}$, then for $\overline{x}_1$ we have

$$\overline{x}_1 = \max\{c^T x';\ c \in [-e, e],\ -e \le Ax' \le e\} = \max\{e^T |x'|;\ -e \le Ax' \le e\},$$

hence

$$\overline{x}_1 \ge 1$$

holds if and only if the system

$$-e \le Ax' \le e,$$
$$e^T |x'| \ge 1$$

has a solution. Since the latter problem is NP-complete (Theorem 22), $\overline{x}_1$ is NP-hard to compute and the same holds for $[\underline{x}, \overline{x}]$. □

4.4.3 Special case: Rank one radius

Theorem 168. *Let* $\mathbf{A} = [A_c - pq^T, A_c + pq^T]$ *be regular. Then we have*

$$\begin{aligned} \underline{x} &= \min_{y,z \in Y_n} \Big(I + \frac{A_c^{-1} T_p y z^T T_q}{1 - z^T T_q A_c^{-1} T_p y} \Big) A_c^{-1} b_y, \\ \overline{x} &= \max_{y,z \in Y_n} \Big(I + \frac{A_c^{-1} T_p y z^T T_q}{1 - z^T T_q A_c^{-1} T_p y} \Big) A_c^{-1} b_y. \end{aligned}$$

Theorem 169. *Let*

$$|A_c^{-1}| pq^T + \alpha |A_c^{-1}| < |A_c^{-1}|$$

and

$$(1-\alpha)|A_c^{-1}|\delta + \alpha |x_c| + q^T(|x_c| + |A_c^{-1}|\delta)|A_c^{-1}|p < |x_c|$$

hold, where

$$\alpha = q^T |A_c^{-1}| p$$

and

$$x_c = A_c^{-1} b_c.$$

Then $[A_c - pq^T, A_c + pq^T]$ *is regular and for each* i *we have*

$$\underline{x}_i = (x_c - |A_c^{-1}|\delta)_i - \frac{q^T|x_c| - \mu_i}{1 + \lambda_i} (|A_c^{-1}|p)_i,$$

$$\overline{x}_i = (x_c + |A_c^{-1}|\delta)_i + \frac{q^T|x_c| + \mu_i}{1 - \lambda_i} (|A_c^{-1}|p)_i,$$

where

$$\begin{aligned} \lambda_i &= z^T T_q A_c^{-1} T_p y(i), \\ \mu_i &= z^T T_q A_c^{-1} T_\delta y(i), \\ z &= \operatorname{sgn} x_c, \\ y(i) &= \operatorname{sgn} (A_c^{-1})_{i\cdot}. \end{aligned}$$

4.4.4 Primal approach

function$[\underline{x}, \overline{x}, flag]$=**xyhull** (**A**,**b**)
% For illustrative purposes (computes all the x_y's).
% Recommended for small-size examples only.
$z := 0 \in \mathbb{R}^n$; $y = e \in \mathbb{R}^n$;
$[x_y, flag] =$ **signaccord** $(A_c, -T_y\Delta, b_y)$;
if $flag =$ *'singular'*, $\underline{x} = [\,]$; $\overline{x} = [\,]$; **return**, **end**
$\underline{x} = x_y$; $\overline{x} = x_y$;
while $z \neq e$
    $k := \min\{i\,;\ z_i = 0\}$;
    for $i := 1$ **to** $k-1$, $z_i := 0$; **end**
    $z_k := 1$; $y_k := -y_k$;
    $[x_y, flag] =$ **signaccord** $(A_c, -T_y\Delta, b_y)$;
    if $flag =$ *'singular'*, $\underline{x} = [\,]$; $\overline{x} = [\,]$; **return**, **end**
    $\underline{x} = \min\{\underline{x}, x_y\}$;
    $\overline{x} = \max\{\overline{x}, x_y\}$;
end
$flag =$ *'hull computed'*;

Figure 4.4: An algorithm for computing the hull using the points x_y.

Theorem 170. *Let* **A** *be regular and let* $Ax = b$ *hold for some* $A \in \mathbf{A}$, $b \in \mathbf{b}$. *Let* $i \in \{1, \ldots, n\}$. *Then for each* $y \in Y_n$ *satisfying*

$$(A^{-1})_{ij} y_j \geq 0 \qquad (j = 1, \ldots, n)$$

there holds

$$(x_{-y})_i \leq x_i \leq (x_y)_i.$$

Proof: Let $Ax = b$ for some $A \in \mathbf{A}$, $b \in \mathbf{b}$, and let $y \in Y_n$ be arbitrary. Put $h = T_y A(x_y - x)$. We shall prove that $h \geq 0$. To this end, first take into account that $|T_y(A - A_c)x_y| \leq \Delta|x_y|$ and $|T_y(b_c - b)| \leq \delta$, hence $T_y(A - A_c)x_y \geq -\Delta|x_y|$ and $T_y(b_c - b) \geq -\delta$. Then we have

$$\begin{aligned} h &= T_y A(x_y - x) = T_y(A_c x_y - b_c) + T_y(A - A_c)x_y + T_y(b_c - b) \\ &\geq T_y(A_c x_y - b_c) - \Delta|x_y| - \delta = T_y(A_c x_y - T_y\Delta|x_y| - b_y) = 0 \end{aligned}$$

because x_y solves the equation $A_c x - T_y\Delta|x| = b_y$. Hence we have $T_y A(x_y - x) = h \geq 0$, which implies

$$x_y - x = A^{-1} T_y h. \tag{4.34}$$

Now, if $i \in \{1, \dots, n\}$ and if y is chosen so that $(A^{-1})_{ij} y_j \geq 0$ for each j, then from (4.34) and from the nonnegativity of h we obtain

$$(x_y)_i - x_i = \sum_{j=1}^{n} (A^{-1})_{ij} y_j h_j \geq 0$$

and

$$(x_{-y})_i - x_i = -\sum_{j=1}^{n} (A^{-1})_{ij} y_j h_j \leq 0,$$

which gives

$$(x_{-y})_i \leq x_i \leq (x_y)_i.$$

□

Theorem 171. *Let $M \geq 0$ and R satisfy the condition*

$$M(I - |I - RA_c| - |R|\Delta) \geq I,$$

and let

$$\begin{aligned} \underset{\sim}{B} &= R - (M - I)|R|, \\ \widetilde{B} &= R + (M - I)|R|. \end{aligned}$$

Then for each $i \in \{1, \dots, n\}$ we have

$$\begin{aligned} \underline{x}_i &= \min_{y \in Y(i)} (x_{-y})_i, \\ \overline{x}_i &= \max_{y \in Y(i)} (x_y)_i, \end{aligned}$$

where

$$Y(i) = \{y \in Y_n \,;\, y_j = 1 \text{ if } \underset{\sim}{B}_{ij} > 0,\ y_j = -1 \text{ if } \widetilde{B}_{ij} < 0\}.$$

4.4.5 The hull under inverse stability

Let us remind that a regular interval matrix $\mathbf{A}$ is called inverse stable (Subsection 3.3.18) if for each $i, j \in \{1, \dots, n\}$, either $(A^{-1})_{ij} > 0$ for each $A \in \mathbf{A}$, or $(A^{-1})_{ij} < 0$ for each $A \in \mathbf{A}$.

Theorem 172. *If $\mathbf{A}$ is inverse stable, then for each i we have*

$$\begin{aligned} \underline{x}_i &= (x_{-y(i)})_i, \\ \overline{x}_i &= (x_{y(i)})_i, \end{aligned}$$

where $y(i) = \operatorname{sgn}(A_c^{-1})_{i\cdot}$.

Theorem 173. *Let*

$$M(I - |I - RA_c| - |R|\Delta) \geq I,$$
$$(M - I)|R| < |R|$$

hold for some $M \geq 0$ and R. Then then for each i we have

$$\begin{aligned} \underline{x}_i &= (x_{-y(i)})_i, \\ \overline{x}_i &= (x_{y(i)})_i, \end{aligned}$$

where $y(i) = \operatorname{sgn} R_{i\bullet}$.

Theorem 174. *Let $\mathbf{A}$ be inverse stable and let $Z = \operatorname{sgn}(A_c^{-1})^T$. Then the matrix equations*

$$\begin{aligned} A_cX + (Z \circ \Delta)|X| &= b_ce^T - Z \circ (\delta e^T), \\ A_cX - (Z \circ \Delta)|X| &= b_ce^T + Z \circ (\delta e^T) \end{aligned}$$

have unique matrix solutions $\underline{X}$ and $\overline{X}$, respectively, and there holds

$$\begin{aligned} \underline{x} &= \operatorname{diag} \underline{X}, \\ \overline{x} &= \operatorname{diag} \overline{X}. \end{aligned}$$

Proof: [82], p. 138. □

Theorem 175. *Let $\mathbf{A}$ be strongly regular and inverse stable and let the solution set X be a part of a single orthant $\mathbb{R}^n_z$. Then we have*

$$\begin{aligned} \underline{x} &= x_c - \operatorname{diag}\Big(A_c^{-1} \sum_{j=0}^{\infty} (-1)^j M_j\Big), \\ \overline{x} &= x_c + \operatorname{diag}\Big(A_c^{-1} \sum_{j=0}^{\infty} M_j\Big), \end{aligned}$$

where the matrices M_j are given by

$$\begin{aligned} M_0 &= Z \circ ((\Delta|x_c| + \delta)e^T), \\ M_j &= Z \circ (BM_{j-1}) \qquad (j = 1, 2, \dots), \end{aligned}$$

and $x_c = A_c^{-1} b_c$, $Z = \operatorname{sgn}(A_c^{-1})^T$, $B = \Delta T_z A_c^{-1}$.

Proof: [77], p. 373. □

Denote

$$\beta = \max\big\{ \max_{ij} \Delta_{ij}, \max_i \delta_i \big\}.$$

Then we can easily prove by induction that

$$A_c^{-1} M_j = O(\beta^{j+1})$$

for each $j \geq 0$. In particular,

$$\begin{aligned} \underline{x} &= x_c - |A_c^{-1}|(\Delta|x_c| + \delta) + \operatorname{diag}(A_c^{-1} M_1) + O(\beta^3), \\ \overline{x} &= x_c + |A_c^{-1}|(\Delta|x_c| + \delta) + \operatorname{diag}(A_c^{-1} M_1) + O(\beta^3), \end{aligned} \tag{4.35}$$

so that

$$\tfrac{1}{2}(\overline{x} - \underline{x}) = |A_c^{-1}|(\Delta|x_c| + \delta) + O(\beta^3).$$

This improves Miller's estimate in [44] where the error was given by $O(\beta^2)$.

4.4.6 Special case: Inverse sign pattern

Definition. $\mathbf{A}$ is said to be of inverse sign pattern (z, y) if there exist $z, y \in Y_n$ such that $T_z A^{-1} T_y \geq 0$ holds for each $A \in \mathbf{A}$. If $\mathbf{A}$ is of inverse sign pattern (e, e), then it is called inverse nonnegative.

Theorem 176. $\mathbf{A}$ *is of inverse sign pattern* (z, y) *if and only if*

$$T_z A_{yz}^{-1} T_y \geq 0,$$

$$T_z A_{-yz}^{-1} T_y \geq 0$$

hold.

Proof: If $\mathbf{A}$ is of inverse sign pattern (z, y), then $T_z A_{yz}^{-1} T_y \geq 0$ and $T_z A_{-yz}^{-1} T_y \geq 0$. To prove the converse statement, assume that (3.24), (3.25) hold and consider an auxiliary interval matrix

$$[\underset{\sim}{A}, \widetilde{A}] = [T_y A_c T_z - \Delta, T_y A_c T_z + \Delta].$$

Then $\underset{\sim}{A} = T_y A_{yz} T_z$ and $\widetilde{A} = T_y A_{-yz} T_z$, hence $\underset{\sim}{A}^{-1} = T_z A_{yz}^{-1} T_y \geq 0$ and $\widetilde{A}^{-1} = T_z A_{-yz}^{-1} T_y \geq 0$, so that the matrix $D_0 = \widetilde{A}^{-1}(\widetilde{A} - \underset{\sim}{A})$ is nonnegative and satisfies

$$(I - D_0)^{-1} = (\widetilde{A}^{-1} \underset{\sim}{A})^{-1} = \underset{\sim}{A}^{-1} \widetilde{A} = I + \underset{\sim}{A}^{-1}(\widetilde{A} - \underset{\sim}{A}) \geq I,$$

which gives that $\varrho(D_0) < 1$. Now let $A \in \mathbf{A}$; then we have

$$T_y A T_z = \widetilde{A}(I - \widetilde{A}^{-1}(\widetilde{A} - T_y A T_z)). \tag{4.36}$$

Since $|T_y A T_z - T_y A_c T_z| \leq \Delta$, we have that $T_y A T_z \in [\underset{\sim}{A}, \widetilde{A}]$ and $\varrho(\widetilde{A}^{-1}(\widetilde{A} - T_y A T_z)) \leq \varrho(\widetilde{A}^{-1}(\widetilde{A} - \underset{\sim}{A})) = \varrho(D_0) < 1$, hence the right-hand side in (4.36) is invertible and we obtain

$$T_z A^{-1} T_y = (T_y A T_z)^{-1} = \sum_{j=0}^{\infty} (\widetilde{A}^{-1}(\widetilde{A} - T_y A T_z))^j \widetilde{A}^{-1} \geq 0, \tag{4.37}$$

which shows that $\mathbf{A}$ is of inverse sign pattern (z, y). □

Theorem 177. *If* $\mathbf{A}$ *is of inverse sign pattern* (z, y)*, then*

$$\underline{x} = \min\{x_{-y}, x_y\},$$

$$\overline{x} = \max\{x_{-y}, x_y\}.$$

Proof: If $\mathbf{A}$ is of inverse sign pattern (z, y), then we have

$$z_i(A^{-1})_{ij}y_j \geq 0 \qquad (i, j = 1, \ldots, n).$$

Thus for $z_i = 1$ there holds

$$(A^{-1})_{ij}y_j \geq 0 \qquad (j = 1, \ldots, n)$$

and for $z_i = -1$ there holds

$$(A^{-1})_{ij}(-y)_j \geq 0. \qquad (j = 1, \ldots, n).$$

Hence by Theorem 170 we have

$$\min\{(x_{-y})_i, (x_y)_i\} \leq x_i \leq \max\{(x_{-y})_i, (x_y)_i\}$$

for each i, which means that

$$\underline{x} = \min\{x_{-y}, x_y\},$$
$$\overline{x} = \max\{x_{-y}, x_y\}.$$

□

Theorem 178. *Let* $\mathbf{A}$ *be inverse nonnegative and let* $\overline{A}^{-1}\underline{b} \geq 0$*. Then we have*

$$\underline{x} = \overline{A}^{-1}\underline{b},$$

$$\overline{x} = \underline{A}^{-1}\overline{b}.$$

4.4.7 Dual approach

Theorem 179. *Let* $\mathbf{A}$ *be regular. Then for each* $z \in Y_n$ *the equation*

$$QA_c - |Q|\Delta T_z = I$$

has a unique matrix solution Q_z *which can be computed by the following finite algorithm:*

for $i := 1$ **to** n **do**
  $y := \operatorname{sgn}(A_c^T)^{-1} e_i$;
  $x := (A_{yz}^T)^{-1} e_i$;
  while $y_j x_j < 0$ for some j **do**
    $k := \min\{j;\ y_j x_j < 0\}$;
    $y_k := -y_k$;
    $x := (A_{yz}^T)^{-1} e_i$;
  $(Q_z)_{i\cdot} := x^T$.

Proof: If $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ is regular, then it transpose $\mathbf{A}^T = [A_c^T - \Delta^T, A_c^T + \Delta^T]$ is also regular, hence the sign accord algorithm when applied to it is finite and the procedure described in the theorem yields for each $z \in Y_n$ a matrix Q_z satisfying

$$A_c^T (Q_z^T)_{\cdot i} - T_z \Delta^T |(Q_z^T)_{\cdot i}| = e_i \tag{4.38}$$

for each i, hence

$$A_c^T Q_z^T - T_z \Delta^T |Q_z^T| = I$$

and

$$Q_z A_c - |Q_z| \Delta T_z = I.$$

Uniqueness of Q_z follows from the uniqueness of solution of the equation (4.38) stated in Theorem 154. □

Notations For each $z \in Y_n$ define

$$\begin{aligned} \underline{x}_z &= Q_{-z} b_c - |Q_{-z}|\delta, \\ \overline{x}_z &= Q_z b_c + |Q_z|\delta. \end{aligned}$$

Theorem 180. *Let* $\mathbf{A}$ *be regular. Then for each* $z \in Y_n$ *there holds*

$$X \cap \mathbb{R}^n_z \subseteq [\underline{x}_z, \overline{x}_z],$$

and both the right-hand side bounds are componentwise attained over X *(although possibly not over* $X \cap \mathbb{R}^n_z$*).*

Proposition 181 *For each $z \in Y_n$ and each $i \in \{1, \ldots, n\}$ we have*

$$(\underline{x}_z)_i = (A_{yz}^{-1} b_y)_i,$$

where

$$y^T = -\mathrm{sgn}\,(Q_{-z})_{i\cdot},$$

and

$$(\overline{x}_z)_i = (A_{yz}^{-1} b_y)_i,$$

where

$$y^T = \mathrm{sgn}\,(Q_z)_{i\cdot}.$$

Theorem 182. *Let* $\mathbf{A}$ *be regular and let Z be any subset of Y_n such that*

$$X \subseteq \bigcup_{z \in Z} \mathbb{R}^n_z.$$

Then we have

$$\underline{x} = \min_{z \in Z} \underline{x}_z,$$

$$\overline{x} = \max_{z \in Z} \overline{x}_z.$$

4.4.8 A general algorithm for computing the hull

function $[\underline{x}, \overline{x}, flag] =$ **hull** $(\mathbf{A}, \mathbf{b})$
% Recommended for the general case.
if A_c is singular
  $\underline{x} = [\,]$; $\overline{x} = [\,]$; $flag =$ *'singular'*; **return**
end
$\underline{x} = A_c^{-1} b_c$; $\overline{x} = \underline{x}$;
$z = \operatorname{sgn} \underline{x}$; $Z = \{z\}$; $D = \emptyset$;
while $Z \neq \emptyset$
  select $z \in Z$; $Z = Z - \{z\}$; $D = D \cup \{z\}$;
  $[Q_{-z}, flag] =$ **qzmatrix**$(\mathbf{A}, -z)$;
  if $flag =$ *'singular'*, $\underline{x} = [\,]$; $\overline{x} = [\,]$; **return**, **end**
  $\underset{\sim}{x} = Q_{-z} b_c - |Q_{-z}|\delta$;
  $[Q_z, flag] =$ **qzmatrix**$(\mathbf{A}, z)$;
  if $flag =$ *'singular'*, $\underline{x} = [\,]$; $\overline{x} = [\,]$; **return**, **end**
  $\tilde{x} = Q_z b_c + |Q_z|\delta$;
  if $\underset{\sim}{x} \leq \tilde{x}$
    $\underline{x} = \min\{\underline{x}, \underset{\sim}{x}\}$;
    $\overline{x} = \max\{\overline{x}, \tilde{x}\}$;
    for $j = 1 : n$
      $z' = z$; $z'_j = -z'_j$;
      if $(\underset{\sim}{x}_j \tilde{x}_j \leq 0$ **and** $z' \notin Z \cup D)$, $Z = Z \cup \{z'\}$; **end**
    end
  end
end
$flag =$ *'hull computed'*;

Figure 4.5: A general algorithm for computing the hull.

4.4.9 An iterative algorithm for computing the hull

Theorem 183. *Let $M \geq 0$ and R satisfy the strong regularity condition. Then for each $z \in Y_n$ the sequence generated by $Q^0 = R$ and*

$$Q^{i+1} = Q^i(I - A_c R) + |Q^i|\Delta T_z R + R$$

$(i = 0, 1, 2, \ldots)$ converges to Q_z and for each $i \geq 0$ we have

$$|\overline{x}_z - (Q^i b_c + |Q^i|\delta)| \leq |Q^{i+1} - Q^i|q,$$

where

$$q = M(|b_c| + \delta).$$

Iterative algorithm (as implemented in the 'optimal' part of intlinst.m)

At the start: R, $M \geq 0$ satisfying

$$(I - G)M \geq I, \tag{4.39}$$

where

$$G = |I - A_c R| + \Delta|R|, \tag{4.40}$$

and $\varepsilon > 0$.

if A_c is singular **then** terminate: **A** is singular
else
  $q := M(|b_c| + \delta)$;
  $\underset{\sim}{x} := A_c^{-1} b_c$; $\tilde{x} := \underset{\sim}{x}$; $z := \operatorname{sgn} \tilde{x}$;
  $Z := \{z\}$; $Z_d := \emptyset$;
  while $Z \neq \emptyset$ **do**
    select $z \in Z$; $Z := Z - \{z\}$; $Z_d := Z_d \cup \{z\}$;
    $Q := R$; $Q' := Q(I - A_c R) - |Q|\Delta T_z R + R$;
    while *not* $|Q' - Q|q < \varepsilon e$ **do**
      $Q := Q'$;
      $Q' := Q(I - A_c R) - |Q|\Delta T_z R + R$;
    $\underset{\sim}{x}_z := Q b_c - |Q|\delta - |Q' - Q|q$;
    $Q := R$; $Q' := Q(I - A_c R) + |Q|\Delta T_z R + R$;
    while *not* $|Q' - Q|q < \varepsilon e$ **do**
      $Q := Q'$;
      $Q' := Q(I - A_c R) + |Q|\Delta T_z R + R$;
    $\tilde{x}_z := Q b_c + |Q|\delta + |Q' - Q|q$;
    $\underset{\sim}{x} := \min\{\underset{\sim}{x}, \underset{\sim}{x}_z\}$;
    $\tilde{x} := \max\{\tilde{x}, \tilde{x}_z\}$;
    for $j := 1$ **to** n **do**
      if $(\underset{\sim}{x}_z)_j (\tilde{x}_z)_j \leq 0$ and $z - 2z_j e_j \notin Z \cup Z_d$
      then $Z := Z \cup \{z - 2z_j e_j\}$.

Theorem 184. *If R and $M \geq 0$ satisfy (4.39), (4.40), then for each $\varepsilon > 0$ the algorithm in a finite number of steps computes an enclosure $[\underset{\sim}{x}, \tilde{x}]$ of X satisfying*

$$[\underset{\sim}{x} + 2\varepsilon e, \tilde{x} - 2\varepsilon e] \subset [\underline{x}, \overline{x}] \subset [\underset{\sim}{x}, \tilde{x}].$$

4.4.10 Special case: Solution set lying in a single orthant

Theorem 185. *Let $X \cap (\mathbb{R}^n_z)^\circ \neq \emptyset$ for some $z \in Y_n$. Then the solution set satisfies*

$$X \subseteq (\mathbb{R}^n_z)^\circ$$

if and only if

$$T_z \underline{x}_z > 0$$

and

$$T_z \overline{x}_z > 0$$

hold.

4.5 Enclosures

4.5.1 Definition

4.5.2 Theoretical characterization of enclosures

Theorem 186. *Let* $\mathbf{A}$ *be regular and* $\Delta \neq 0$*. Then for each* $y \in Y_n$*, the set*

$$\begin{aligned} X_y &= \{x;\, T_y A_c x - \Delta|x| \geq T_y b_c + \delta\} \\ &= \{x;\, T_y A_c x - \Delta t \geq T_y b_c + \delta,\, -t \leq x \leq t\} \end{aligned}$$

is an unbounded convex polyhedron and x_y *is a vertex of it.*

Theorem 187. *Let* $\mathbf{A}$ *be regular. Then an interval vector* $\mathbf{x}$ *is an enclosure of the solution set of* $\mathbf{A}x = \mathbf{b}$ *if and only if it intersects all the sets* X_y*,* $y \in Y_n$*.*

Theorem 188. *Let* $\mathbf{A}$ *be regular. Then an interval vector* $[\underline{x}, \overline{x}]$ *is an enclosure of the solution set of* $\mathbf{A}x = \mathbf{b}$ *if and only if for each* $y \in Y_n$ *the inequality*

$$|A_c x - b_c| \geq \Delta|x| + \delta$$

has a solution $x_y \in [\underline{x}, \overline{x}]$ *satisfying* $T_y(A_c x_y - b_c) \geq 0$*.*

4.5.3 The Hansen-Bliek-Rohn enclosure: General form

Theorem 189. *Let $M \geq 0$ and R satisfy the strong regularity condition*

$$M(I - |I - RA_c| - |R|\Delta) \geq I. \tag{4.41}$$

Then we have

$$X \subseteq [\min\{T_\alpha^{-1}\underset{\sim}{x}, T_\beta^{-1}\underset{\sim}{x}\}, \max\{T_\alpha^{-1}\tilde{x}, T_\beta^{-1}\tilde{x}\}], \tag{4.42}$$

where

$$\begin{aligned}
\mu &= \operatorname{diag}(M), \\
r &= \operatorname{diag}(I - RA_c), \\
h &= \operatorname{diag}(M(I - |I - RA_c| - |R|\Delta) - I), \\
\alpha &= e + T_\mu(|r| - r) + h, \\
\beta &= 2\mu - e - T_\mu(|r| + r) - h, \\
x_c &= Rb_c, \\
x^* &= M(|x_c| + |R\delta|), \\
\underset{\sim}{x} &= -x^* + T_\mu(x_c + |x_c|), \\
\tilde{x} &= x^* + T_\mu(x_c - |x_c|).
\end{aligned}$$

Proof: 1) First we prove that each matrix A with $|A - A_c| \leq \Delta$ is nonsingular. Premultiplying the inequality (4.41) by the nonnegative matrix G yields $MG^2 + G + I \leq MG + I \leq M$ and by induction $\sum_{j=0}^{k} G^j \leq MG^{k+1} + \sum_{j=0}^{k} G^j \leq M$ for each $k \geq 0$, hence $\sum_0^\infty G^j$ is convergent which, as well known, implies that $\varrho(G) < 1$. Since $|I-RA| = |I-RA_c+R(A_c-A)| \leq |I-RA_c|+|R|\Delta = G$, we have $\varrho(I-RA) \leq \varrho(G) < 1$ which means that the matrix $RA = I-(I-RA)$ is nonsingular, hence A is nonsingular.

2) Next we prove that $\beta_i \geq \alpha_i \geq 1$ for each i. From the definition of h_i we have $m_i = (MG)_{ii} + 1 + h_i \geq m_i|r_i| + 1 + h_i$ which can be easily rearranged to $2m_i - 1 - (|r_i| + r_i)m_i - h_i \geq 1 + (|r_i| - r_i)m_i + h_i$, giving $\beta_i \geq \alpha_i$; the inequality $\alpha_i \geq 1$ follows from nonnegativity of m_i and h_i.

3) Let x solve $Ax = b$ for some A, b with $|A - A_c| \leq \Delta$ and $|b - b_c| \leq \delta$. Then we have

$$x = (I - RA)x + Rb = (I - RA_c)x + R(A_c - A)x + Rb_c + R(b - b_c) \tag{4.43}$$

and taking absolute values we get

$$|x| \leq G|x| + |Rb_c| + |R|\delta. \tag{4.44}$$

Let $i \in \{1, \ldots, n\}$. Then from the ith equation in (4.43) we have

$$\begin{aligned}
x_i &\leq ((I - RA_c)x)_i + (|R|\Delta|x|)_i + (Rb_c)_i + (|R|\delta)_i \\
&= (G|x| + |Rb_c| + |R|\delta)_i + ((I - RA_c)x - |I - RA_c||x| + Rb_c - |Rb_c|)_i.
\end{aligned} \tag{4.45}$$

Put $x' = (|x_1|, \dots, |x_{i-1}|, x_i, |x_{i+1}|, \dots, |x_n|)^T$. Then from (4.44) and (4.45) we have $x' \le G|x| + |Rb_c| + |R|\delta + ((I - RA_c)x - |I - RA_c||x| + Rb_c - |Rb_c|)_i e_i$, where e_i is the ith column of I. Premultiplying this inequality by the nonnegative vector $e_i^T M$ and using the matrix $H := M - MG - I \ge 0$, we obtain $(Mx')_i \le ((M - H - I)|x|)_i + ((I - RA_c)x - |I - RA_c||x|)_i m_i + \tilde{x}_i$ and consequently

$$(M(x' - |x|))_i + (H|x|)_i + |x_i| + (|I - RA_c||x| - (I - RA_c)x)_i m_i \le \tilde{x}_i. \tag{4.46}$$

Since $(M(x' - |x|))_i = m_i(x_i - |x_i|)$, $(H|x|)_i \ge h_i|x_i|$ and $(|I - RA_c||x| - (I - RA_c)x)_i \ge |r_i||x_i| - r_i x_i$, from (4.46) we finally obtain an inequality containing variable x_i only:

$$m_i(x_i - |x_i|) + h_i|x_i| + |x_i| + (|r_i||x_i| - r_i x_i)m_i \le \tilde{x}_i. \tag{4.47}$$

If $x_i \ge 0$, then this inequality becomes $\alpha_i x_i \le \tilde{x}_i$, implying $x_i \le \tilde{x}_i/\alpha_i$, and if $x_i < 0$, then (4.47) turns into $\beta_i x_i \le \tilde{x}_i$, giving $x_i \le \tilde{x}_i/\beta_i$, which together yields

$$x_i \le \max\{\tilde{x}_i/\alpha_i,\ \tilde{x}_i/\beta_i\}. \tag{4.48}$$

In this way we have obtained the upper bound in (4.42). To prove the lower one, notice that $-x$ satisfies $A(-x) = -b$, where $|A - A_c| \le \Delta$ and $|(-b) - (-b_c)| \le \delta$. Hence we can use the previously obtained result if we set $b_c := -b_c$, which affects $\tilde{x}_i$ only. Then from (4.48) we get $-x_i \le \max\{-\underset{\sim}{x}_i/\alpha_i,\ -\underset{\sim}{x}_i/\beta_i\}$ which, after premultiplying by -1, gives the lower bound in (4.42).

4) Finally, to prove the optimality result for the case $A_c = I$ and $\varrho(\Delta) < 1$, take $R = I$ and $M = (I - \Delta)^{-1}$, then $M \ge 0$, $G = \Delta$ and (4.41) is satisfied as an equation; moreover, for each i we have $r_i = h_i = 0$, $\alpha_i = 1$, $\beta_i = 2m_i - 1$, hence (4.42) has the form

$$\min\{\underset{\sim}{x}_i,\ \underset{\sim}{x}_i/\beta_i\} \le x_i \le \max\{\tilde{x}_i,\ \tilde{x}_i/\beta_i\}. \tag{4.49}$$

To prove that the upper bound is really attained, let us fix an $i \in \{1, \dots, n\}$ and define a diagonal matrix D by $D_{jj} = 1$ if $j \ne i$ and $(b_c)_j \ge 0$, $D_{jj} = -1$ if $j \ne i$ and $(b_c)_j < 0$, and $D_{jj} = 1$ if $j = i$, and let $\tilde{b} = Db_c + \delta$. Then it can be easily verified that $\tilde{x}_i = (M\tilde{b})_i$ holds. First, define $x' = DM\tilde{b}$. Since $M = (I - \Delta)^{-1}$ implies $\Delta M = M\Delta = M - I$, we have $(I - D\Delta D)x' = DM\tilde{b} - D(M - I)\tilde{b} = D\tilde{b} = b_c + D\delta$, which means that x' solves the system $(I - D\Delta D)x' = b_c + D\delta$ where $|(I - D\Delta D) - I| = \Delta$, $|(b_c + D\delta) - b_c| = \delta$ and $x'_i = e_i^T DM\tilde{b} = e_i^T M\tilde{b} = (M\tilde{b})_i = \tilde{x}_i$, which shows that $\tilde{x}_i$ is attained. Second, let $x'' = DM(\tilde{b} - 2(\tilde{x}_i/\beta_i)\Delta e_i)$ and define a diagonal matrix D' by $D'_{ii} = -1$ and $D'_{jj} = D_{jj}$ otherwise. Then $(I - D\Delta D')DM = DM - D\Delta(I - 2e_i e_i^T)M = DM - D(M - I) + 2D\Delta e_i e_i^T M = D + 2D\Delta e_i e_i^T M$, hence $(I - D\Delta D')x'' = (D + 2D\Delta e_i e_i^T M)(\tilde{b} - 2(\tilde{x}_i/\beta_i)\Delta e_i) = D\tilde{b} + 2\tilde{x}_i D\Delta e_i(-(1/\beta_i) + 1 - (2/\beta_i)(m_i - 1)) = D\tilde{b} = b_c + D\delta$, which shows that x'' is a solution to the system $(I - D\Delta D')x'' = b_c + D\delta$ where $|(I - D\Delta D') - I| = \Delta$, $|(b_c + D\delta) - b_c| = \delta$ and $x''_i = e_i^T DM(\tilde{b} - 2(\tilde{x}_i/\beta_i)\Delta e_i) = \tilde{x}_i - 2(\tilde{x}_i/\beta_i)(m_i - 1) = \tilde{x}_i/\beta_i$. This proves that $\tilde{x}_i/\beta_i$ is attained, hence also the upper bound $\max\{\tilde{x}_i,\ \tilde{x}_i/\beta_i\}$ in (4.49) is attained. The proof for the lower bound follows from the result just obtained by applying it to the case $b_c := -b_c$ as in the part 3). □

4.5.4 HBR enclosure: Original form using exact inverses

Theorem 190. *Let* $\mathbf{A}$ *be strongly regular. Then we have*

$$X \subseteq [\underline{\underline{x}}, \overline{\overline{x}}], \tag{4.50}$$

where

$$\begin{aligned} M &= (I - |A_c^{-1}|\Delta)^{-1}, \\ \mu &= \operatorname{diag}(M), \\ T_\nu &= (2T_\mu - I)^{-1}, \\ x_c &= A_c^{-1} b_c, \\ x^* &= M(|x_c| + |A_c^{-1}\delta|), \\ \underset{\sim}{x} &= -x^* + T_\mu(x_c + |x_c|), \\ \tilde{x} &= x^* + T_\mu(x_c - |x_c|), \\ \underline{\underline{x}} &= \min\{\underset{\sim}{x}, T_\nu \underset{\sim}{x}\}, \\ \overline{\overline{x}} &= \max\{\tilde{x}, T_\nu \tilde{x}\}. \end{aligned}$$

Proof: First we note that because of strong regularity we have

$$M = \sum_{j=0}^{\infty} (|A_c^{-1}|\Delta)^j \geq I \geq 0,$$

thus also $2T_\mu - I \geq I$, so that the diagonal matrix $T_\nu = (2T_\mu - I)^{-1}$ exists and $\nu_i = 1/(2M_{ii} - 1)$ for each i.

To prove (4.50), take an $x \in X$, then by the Oettli-Prager theorem it satisfies

$$|A_c x - b_c| \leq \Delta|x| + \delta,$$

hence

$$x - x_c \leq |x - x_c| = |A_c^{-1}(A_c x - b_c)| \leq |A_c^{-1}||A_c x - b_c| \leq |A_c^{-1}|(\Delta|x| + \delta) \tag{4.51}$$

and similarly

$$|x| - |x_c| \leq ||x| - |x_c|| \leq |x - x_c| \leq |A_c^{-1}|(\Delta|x| + \delta). \tag{4.52}$$

Now, let us fix an $i \in \{1, \ldots, n\}$. Then from (4.51) we have

$$x_i \leq (x_c)_i + (|A_c^{-1}|(\Delta|x| + \delta))_i \tag{4.53}$$

and (4.52) gives

$$|x_j| \leq |x_c|_j + (|A_c^{-1}|(\Delta|x| + \delta))_j \tag{4.54}$$

for each $j \neq i$. Since $x_i = |x_i| + (x_i - |x_i|)$ and the same holds for $(x_c)_i$, we can put (4.53) and (4.54) together as

$$|x| + (x_i - |x_i|)e_i \leq |x_c| + ((x_c)_i - |x_c|_i)e_i + |A_c^{-1}|(\Delta|x| + \delta),$$

which implies

$$(I - |A_c^{-1}|\Delta)|x| + (x_i - |x_i|)e_i \leq |x_c| + |A_c^{-1}|\delta + ((x_c)_i - |x_c|_i)e_i.$$

Premultiplying this inequality by the nonnegative vector $e_i^T M$, we finally obtain an inequality containing variable x_i only:

$$|x_i| + (x_i - |x_i|)M_{ii} \leq x_i^* + ((x_c)_i - |x_c|_i)M_{ii} = \tilde{x}_i.$$

If $x_i \geq 0$, then this inequality becomes

$$x_i \leq \tilde{x}_i,$$

and if $x_i < 0$, then it turns into

$$x_i \leq \tilde{x}_i/(2M_{ii} - 1) = \nu_i \tilde{x}_i,$$

in both cases

$$x_i \leq \max\{\tilde{x}_i, \nu_i \tilde{x}_i\}.$$

Since i was arbitrary, we conclude that

$$x \leq \max\{\tilde{x}, T_\nu \tilde{x}\},$$

which is the upper bound in (4.50). To prove the lower bound, notice that if $Ax = b$ for some $A \in \mathbf{A}$ and $b \in \mathbf{b}$, then $A(-x) = -b$, hence $-x$ belongs to the solution set of the system $\mathbf{A}x = [-b_c - \delta, -b_c + \delta]$, and we can apply the previous result to this system by setting $b_c := -b_c$. In this way we obtain

$$-x \leq \max\{x^* + T_\mu(-x_c - |x_c|), T_\nu(x^* + T_\mu(-x_c - |x_c|))\},$$

hence

$$x \geq \min\{-x^* + T_\mu(x_c + |x_c|), T_\nu(-x^* + T_\mu(x_c + |x_c|))\} = \min\{\underset{\sim}{x}, T_\nu \underset{\sim}{x}\},$$

which is the lower bound in (4.50). The theorem is proved. □

4.5.5 Overestimation of the HBR enclosure

Theorem 191. *Let* $\mathbf{A}$ *be strongly regular and let* $M = (I - |A_c^{-1}|\Delta)^{-1}$. *Then for each* $i \in \{1, \ldots, n\}$ *we have*

$$\begin{aligned} \underline{\underline{x}}_i &\le \underline{x}_i \le \underline{\underline{x}}_i + \underline{d}_i, \\ \overline{\overline{x}}_i - \overline{d}_i &\le \overline{x}_i \le \overline{\overline{x}}_i, \end{aligned}$$

where

$$\begin{aligned} \underline{d}_i &= e_i^T M|(T_{\underline{z}} A_c^{-1} T_{\underline{z}} - |A_c^{-1}|)(\underline{\xi}_i \Delta M e_i + \Delta x^* + \delta)|, \\ \overline{d}_i &= e_i^T M|(T_{\overline{z}} A_c^{-1} T_{\overline{z}} - |A_c^{-1}|)(\overline{\xi}_i \Delta M e_i + \Delta x^* + \delta)|, \\ \underline{\xi}_i &= (|\underline{\underline{x}}| + \underline{\underline{x}} - x_c - |x_c|)_i, \\ \overline{\xi}_i &= (|\overline{\overline{x}}| - \overline{\overline{x}} + x_c - |x_c|)_i \end{aligned}$$

and $\underline{z}$, $\overline{z}$ *are given by*

$$\underline{z}_j = \begin{cases} \operatorname{sgn}(x_c)_j & \text{if } j \ne i, \\ -1 & \text{if } j = i, \end{cases}$$

$$\overline{z}_j = \begin{cases} \operatorname{sgn}(x_c)_j & \text{if } j \ne i, \\ 1 & \text{if } j = i \end{cases}$$

$(j = 1, \ldots, n)$.

Theorem 192. *Let* A *be strongly regular. Then we have:*

(i) $[\underline{x}, \overline{x}] = [\underline{\underline{x}}, \overline{\overline{x}}]$ *if* A_c *is a diagonal matrix with positive diagonal entries,*
(ii) $\underline{x} = \underline{\underline{x}}$ *if* $A_c^{-1} \ge 0$ *and* $A_c^{-1} b_c \le 0$,
(iii) $\overline{x} = \overline{\overline{x}}$ *if* $A_c^{-1} \ge 0$ *and* $A_c^{-1} b_c \ge 0$.

4.5.6 The HBR algorithm

function $[\underline{\underline{x}}, \overline{\overline{x}}, \underline{d}, \overline{d}, flag] = \mathbf{hbr}\,(\mathbf{A}, \mathbf{b})$
if (A_c is singular **or** $I - |A_c^{-1}|\Delta$ is singular **or** $(I - |A_c^{-1}|\Delta)^{-1} \not\geq I$)
    $\underline{\underline{x}} = [\,];\ \overline{\overline{x}} = [\,];\ \underline{d} = [\,];\ \overline{d} = [\,];\ flag = \textit{'enclosure not computed'};$
    return
end
$M = (I - |A_c^{-1}|\Delta)^{-1};$
$\mu = \mathrm{diag}\,(M);$
$T_\nu = (2T_\mu - I)^{-1};$
$x_c = A_c^{-1} b_c;$
$x^* = M(|x_c| + |A_c^{-1}\delta|);$
$\underset{\sim}{x} = -x^* + T_\mu(x_c + |x_c|);$
$\tilde{x} = x^* + T_\mu(x_c - |x_c|);$
$\underline{\underline{x}} = \min\{\underset{\sim}{x}, T_\nu \underset{\sim}{x}\};$
$\overline{\overline{x}} = \max\{\tilde{x}, T_\nu \tilde{x}\};$
$flag = \textit{'enclosure computed'};$
$z = \mathrm{sgn}\, x_c;$
$\underline{\xi} = |\underline{\underline{x}}| + \underline{\underline{x}} - x_c - |x_c|;$
$\overline{\xi} = |\overline{\overline{x}}| - \overline{\overline{x}} + x_c - |x_c|;$
for $i = 1 : n$
    $z' = z;\ z'_i = -1;$
    $\underline{d}_i = (M|(T_{z'} A_c^{-1} T_{z'} - |A_c^{-1}|)(\underline{\xi}_i \Delta M e_i + \Delta x^* + \delta)|)_i;$
    $z'_i = 1;$
    $\overline{d}_i = (M|(T_{z'} A_c^{-1} T_{z'} - |A_c^{-1}|)(\overline{\xi}_i \Delta M e_i + \Delta x^* + \delta)|)_i;$
end

Figure 4.6: The Hansen-Bliek-Rohn enclosure algorithm.

4.5.7 Preconditioning

$$[I - |A_c^{-1}|\Delta,\, I + |A_c^{-1}|\Delta]x = [A_c^{-1}b_c - |A_c^{-1}|\delta,\, A_c^{-1}b_c + |A_c^{-1}|\delta], \tag{4.55}$$

4.5.8 Example: A 100% overestimation in case $n = 4$

$$[I - |A_c^{-1}|\Delta,\ I + |A_c^{-1}|\Delta]x = [A_c^{-1}b_c - |A_c^{-1}|\delta,\ A_c^{-1}b_c + |A_c^{-1}|\delta],$$
$$\|A\|_{\infty,1} = \max_{\|x\|_\infty = 1} \|Ax\|_1 = \max_{y \in Y_m} \|Ay\|_1 \tag{4.56}$$

Given a nonsingular matrix $A \in \mathbb{R}^{(n-1)\times(n-1)}$ and a real number $\varepsilon > 0$, consider a linear interval system

$$\mathbf{A}x = \mathbf{b},$$

where

$$\mathbf{A} = \begin{pmatrix} \varepsilon^2 & [-\varepsilon e^T, \varepsilon e^T] \\ 0 & A \end{pmatrix}$$

and

$$\mathbf{b} = \begin{pmatrix} 0 \\ [-\varepsilon e, \varepsilon e] \end{pmatrix}$$

with $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^{n-1}$. This means that the centers and radii are given by

$$A_c = \begin{pmatrix} \varepsilon^2 & 0^T \\ 0 & A \end{pmatrix}, \qquad \Delta = \begin{pmatrix} 0 & \varepsilon e^T \\ 0 & 0 \end{pmatrix},$$

$$b_c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \delta = \begin{pmatrix} 0 \\ \varepsilon e \end{pmatrix},$$

which implies that

$$|A_c^{-1}|\Delta = \begin{pmatrix} \frac{1}{\varepsilon^2} & 0^T \\ 0 & |A^{-1}| \end{pmatrix} \begin{pmatrix} 0 & \varepsilon e^T \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\varepsilon} e^T \\ 0 & 0 \end{pmatrix},$$

so that

$$\varrho(|A_c^{-1}|\Delta) = 0,$$

hence not only the interval matrix $\mathbf{A}$ is strongly regular, but also the spectral radius of $|A_c^{-1}|\Delta$ attains the lowest possible value independently of ε. Next, $A_c^{-1}b_c = 0$ and

$$|A_c^{-1}|\delta = \begin{pmatrix} 0 \\ \varepsilon |A^{-1}| e \end{pmatrix}.$$

Now we can state the basic result concerning the system (4.30)-(4.32):

Theorem 193. *Let A be nonsingular and let $\varepsilon > 0$. Then for the interval hull $[\underline{x}, \overline{x}]$ and for the preconditioned interval hull $[\underline{\underline{x}}, \overline{\overline{x}}]$ of the system (4.30)-(4.32) we have*

$$\overline{x} = -\underline{x} = \begin{pmatrix} \|A^{-1}\|_{\infty,1} \\ \varepsilon d \end{pmatrix},$$

$$\overline{\overline{x}} = -\underline{\underline{x}} = \begin{pmatrix} \|d\|_1 \\ \varepsilon d \end{pmatrix},$$

where $d = |A^{-1}|e$.

Proof: (a) Since $\mathbf{b}$ is symmetric about 0, the same holds for the solution set X of (4.30)-(4.32) (because if $x \in X$, then $A'x = b'$ for some $A' \in \mathbf{A}$ and $b' \in \mathbf{b}$, hence $A'(-x) = -b' \in \mathbf{b}$ and $-x \in X$), which implies that $\underline{x} = -\overline{x}$. Thus we are confined to evaluate $\overline{x}$ only. According to Theorem 165, we have

$$\overline{x} = \max_{y \in Y_m} x_y,$$

where for each $y \in Y_m$, x_y is the unique solution of the equation (4.23). Let us write $y = (y_1, y'^T)^T$, where $y' = (y_2, \ldots, y_n)^T$, and let us decompose x_y accordingly as $x_y = (x_1, x'^T)^T$. Then the equation (4.23) for the system (4.30)-(4.32) has the form

$$\begin{pmatrix} \varepsilon^2 & 0^T \\ 0 & A \end{pmatrix} \begin{pmatrix} x_1 \\ x' \end{pmatrix} = T_{\begin{pmatrix} y_1 \\ y' \end{pmatrix}} \left(\begin{pmatrix} 0 & \varepsilon e^T \\ 0 & 0 \end{pmatrix} \left| \begin{matrix} x_1 \\ x' \end{matrix} \right| + \begin{pmatrix} 0 \\ \varepsilon e \end{pmatrix} \right)$$

or equivalently

$$\varepsilon^2 x_1 = y_1 \varepsilon e^T |x'|,$$

$$Ax' = T_{y'} \varepsilon e = \varepsilon y',$$

which gives

$$x' = \varepsilon A^{-1} y',$$

$$x_1 = y_1 e^T |A^{-1} y'| = y_1 \|A^{-1} y'\|_1,$$

hence

$$x_y = \begin{pmatrix} y_1 \|A^{-1} y'\|_1 \\ \varepsilon A^{-1} y' \end{pmatrix},$$

and from Theorem 165 in view of (4.56) we obtain

$$\overline{x} = \max_{y \in Y_m} x_y = \begin{pmatrix} \|A^{-1}\|_{\infty,1} \\ \varepsilon |A^{-1}| e \end{pmatrix} = \begin{pmatrix} \|A^{-1}\|_{\infty,1} \\ \varepsilon d \end{pmatrix}.$$

(b) Since the right-hand side of the preconditioned system (4.55) is again symmetric about 0, we again have $\underline{\underline{x}} = -\overline{\overline{x}}$. The equation (4.23) for the preconditioned system (4.55), (4.31), (4.32) has the form

$$\begin{pmatrix} x_1 \\ x' \end{pmatrix} = T_{\begin{pmatrix} y_1 \\ y' \end{pmatrix}} \left(\begin{pmatrix} 0 & \frac{1}{\varepsilon} e^T \\ 0 & 0 \end{pmatrix} \left| \begin{matrix} x_1 \\ x' \end{matrix} \right| + \begin{pmatrix} 0 \\ \varepsilon |A^{-1}| e \end{pmatrix} \right),$$

which gives

$$x' = T_{y'} \varepsilon |A^{-1}| e = \varepsilon T_{y'} d,$$

$$x_1 = y_1 \tfrac{1}{\varepsilon} e^T |x'| = y_1 e^T |A^{-1}| e = y_1 \|d\|_1,$$

hence

$$x_y = \begin{pmatrix} y_1 \|d\|_1 \\ \varepsilon T_{y'} d \end{pmatrix}$$

and

$$\overline{\overline{x}} = \max_{y \in Y_m} x_y = \begin{pmatrix} \|d\|_1 \\ \varepsilon d \end{pmatrix},$$

which concludes the proof. □

Now we can see the main point: the values of $\overline{x}_1 = \|A^{-1}\|_{\infty,1}$ and $\overline{\overline{x}}_1 = e^T|A^{-1}|e$ are *independent of* ε. To achieve the result wanted, it remains to choose an appropriate matrix A. But before doing so, we note that the formula (4.33) yields another proof of the NP-hardness of computing the interval hull (proved originally in [100]): since computing the norm $\|\cdot\|_{\infty,1}$ is NP-hard (as proved in [97]), by (4.33) computing $\overline{x}_1$, and thus also $[\underline{x}, \overline{x}]$, is NP-hard as well.

Consider the example (4.30)-(4.32) with

$$A = \begin{pmatrix} 1 & -3 & -3 \\ -3 & 1 & -3 \\ -3 & -3 & 1 \end{pmatrix}, \tag{4.57}$$

or, explicitly written,

$$\begin{pmatrix} \varepsilon^2 & [-\varepsilon, \varepsilon] & [-\varepsilon, \varepsilon] & [-\varepsilon, \varepsilon] \\ 0 & 1 & -3 & -3 \\ 0 & -3 & 1 & -3 \\ 0 & -3 & -3 & 1 \end{pmatrix} x = \begin{pmatrix} 0 \\ [-\varepsilon, \varepsilon] \\ [-\varepsilon, \varepsilon] \\ [-\varepsilon, \varepsilon] \end{pmatrix}. \tag{4.58}$$

As a direct application of Theorem 166 we obtain:

Theorem 194. *For each $\varepsilon > 0$, for the interval hull $[\underline{x}, \overline{x}]$ and for the preconditioned interval hull $[\underline{\underline{x}}, \overline{\overline{x}}]$ of the linear interval system (4.58) we have*

$$\overline{x} = -\underline{x} = \begin{pmatrix} 0.6 \\ 0.4\varepsilon \\ 0.4\varepsilon \\ 0.4\varepsilon \end{pmatrix}, \tag{4.59}$$

$$\overline{\overline{x}} = -\underline{\underline{x}} = \begin{pmatrix} 1.2 \\ 0.4\varepsilon \\ 0.4\varepsilon \\ 0.4\varepsilon \end{pmatrix}. \tag{4.60}$$

Proof: We are left with substituting

$$A^{-1} = \begin{pmatrix} 0.10 & -0.15 & -0.15 \\ -0.15 & 0.10 & -0.15 \\ -0.15 & -0.15 & 0.10 \end{pmatrix}$$

into (4.33) and (**??**), using (4.56) for evaluation of $\|A^{-1}\|_{\infty,1}$, which yields (4.59) and (4.60). □

We have proved that for the system (4.58) there holds $\overline{\overline{x}}_1 = 2\overline{x}_1 = 1.2$ independently of ε. The matrix A in (4.57), although being of quite regular structure at a glance, was in fact found through extensive experiencing in MATLAB involving computation of several tens of thousands of randomly generated examples (of size 3×3) aimed at maximizing the value of

$$\frac{e^T|A^{-1}|e}{\|A^{-1}\|_{\infty,1}}. \tag{4.61}$$

For the best result found the ratio was slightly less than 2 and the coefficients of A were close to integers; then rounding to nearest integers produced the matrix (4.57) for which the value of (4.61) is 2. However, notice from (4.59), (4.60) that $\underline{\underline{x}}_i = \underline{x}_i$, $\overline{\overline{x}}_i = \overline{x}_i$ for $i \geq 2$. Theorem 166 may yield another related results, but we have not pursued the matter any further.

4.5.9 Comparison with the Bauer-Skeel bounds

Theorem 195. (Bauer 1966, Skeel 1979) *If*

$$\varrho(|A_c^{-1}|\Delta) < 1,$$

then for each A, b such that $|A - A_c| \le \Delta$ and $|b - b_c| \le \delta$, A is nonsingular and the solution of

$$Ax = b$$

satisfies

$$-x^* + x_c + |x_c| \le x \le x^* + x_c - |x_c|,$$

where

$$\begin{aligned} M &= (I - |A_c^{-1}|\Delta)^{-1}, \\ x^* &= M(|x_c| + |A_c^{-1}|\delta). \end{aligned}$$

Note. Usually presented as $|x - x_c| \le x^* - |x_c|$, with $\delta = 0$ or in normwise setting. Two inversions needed.

For comparison, denote the Bauer-Skeel bounds by

$$\underline{x} \le x \le \overline{x}$$

and the HBR bounds by

$$\underline{\underline{x}} \le x \le \overline{\overline{x}},$$

i.e.

$$\begin{aligned} \underline{x} &= -x^* + x_c + |x_c|, \\ \overline{x} &= x^* + x_c - |x_c|, \\ \underline{\underline{x}} &= \min\{\underset{\sim}{x}, T\underset{\sim}{x}\}, \\ \overline{\overline{x}} &= \max\{\tilde{x}, T\tilde{x}\}. \end{aligned}$$

It turns out that crucial for the comparison is the fact that

$$M_{ii} \ge 1 \text{ for each } i.$$

Theorem 196. *Under the common assumption $\varrho(|A_c^{-1}|\Delta) < 1$, for each i we have*

$$\overline{x}_i - \overline{\overline{x}}_i \ge \min\left\{(M_{ii} - 1)(|x_c|_i - (x_c)_i),\ \tfrac{2(M_{ii}-1)}{2M_{ii}-1}(x_i^* - |x_c|_i)\right\} \ge 0,$$

$$\underline{\underline{x}}_i - \underline{x}_i \ge \min\left\{(M_{ii} - 1)(|x_c|_i + (x_c)_i),\ \tfrac{2(M_{ii}-1)}{2M_{ii}-1}(x_i^* - |x_c|_i)\right\} \ge 0.$$

In particular,

$$\underline{x} \le \underline{\underline{x}} \le \overline{\overline{x}} \le \overline{x},$$

i.e. the HBR bounds are never worse than the Bauer-Skeel bounds.

Remark. Nonnegativity follows from the facts that $M \geq I$ and
$x^* = M(|x_c| + |A_c^{-1}|\delta) \geq |x_c|$.

Theorem 197. *Let the spectral condition hold. Then for each i such that $M_{ii} > 1$ and $(x_c)_i \neq 0$ we have*

$$(\overline{x}_i - \underline{x}_i) - (\overline{\overline{x}}_i - \underline{\underline{x}}_i) \geq \frac{2(M_{ii}-1)^2}{2M_{ii}-1}|x_c|_i > 0,$$

hence

$$\overline{\overline{x}}_i - \underline{\underline{x}}_i < \overline{x}_i - \underline{x}_i,$$

i.e., the ith HBR bound is better than the Bauer-Skeel bound.

Remark. Recall that $M = (I - |A_c^{-1}|\Delta)^{-1} = \sum_{j=0}^{\infty}(|A_c^{-1}|\Delta)^j \geq I$.
Hence $M_{ii} > 1$ e.g. if $(|A_c^{-1}|\Delta)_{ii} > 0$.

We can conclude that the HBR bounds are "almost always" better than the Bauer-Skeel bounds.

Still, how good are the HBR bounds themselves?

4.5.10 Many other enclosures

4.5.11 Improvement of an enclosure

Conditions:

$$0 \notin [\underset{\sim}{B}_{ki}, \tilde{B}_{ki}],\ 0 \notin [\underset{\sim}{x}_j, \tilde{x}_j],\ \underset{\sim}{B}_{ki}\tilde{x}_j < 0 \tag{4.62}$$

$$0 \notin [\underset{\sim}{B}_{ki}, \tilde{B}_{ki}],\ 0 \notin [\underset{\sim}{x}_j, \tilde{x}_j],\ \underset{\sim}{B}_{ki}\tilde{x}_j > 0 \tag{4.63}$$

$$\underset{\sim}{B}_{ki} > 0 \tag{4.64}$$

$$\tilde{B}_{ki} < 0 \tag{4.65}$$

Theorem 198. *Let* $[\underline{x}, \overline{x}]$ *be the interval hull of the solution set of a system*

$$\mathbf{A}x = \mathbf{b}$$

and let $[\underline{x}, \overline{x}] \subseteq [\underset{\sim}{x}, \tilde{x}]$, $\mathbf{A}^{-1} \subseteq [\underset{\sim}{B}, \tilde{B}]$. *Then for each* $k \in \{1, \dots, n\}$ *we have:*

(i) the interval hull $[\underline{x}_{(k)}, \overline{x}_{(k)}]$ *of the solution set of the system*

$$[\underline{A}_{(k)}, \overline{A}_{(k)}]x = [\underline{b}_{(k)}, \overline{b}_{(k)}]$$

given by

$$(\underline{A}_{(k)})_{ij} = \begin{cases} \overline{A}_{ij} & \text{if (4.63) holds,} \\ \underline{A}_{ij} & \text{otherwise} \end{cases}, \quad (\overline{A}_{(k)})_{ij} = \begin{cases} \underline{A}_{ij} & \text{if (4.62) holds,} \\ \overline{A}_{ij} & \text{otherwise} \end{cases},$$

$$(\underline{b}_{(k)})_i = \begin{cases} \overline{b}_i & \text{if (4.65) holds,} \\ \underline{b}_i & \text{otherwise} \end{cases}, \quad (\overline{b}_{(k)})_i = \begin{cases} \underline{b}_i & \text{if (4.64) holds,} \\ \overline{b}_i & \text{otherwise} \end{cases}$$

$(i, j = 1, \dots, n)$ *satisfies* $(\underline{x}_{(k)})_k = \underline{x}_k$,

(ii) the interval hull $[\underline{x}^{(k)}, \overline{x}^{(k)}]$ *of the solution set of the system*

$$[\underline{A}^{(k)}, \overline{A}^{(k)}]x = [\underline{b}^{(k)}, \overline{b}^{(k)}]$$

given by

$$(\underline{A}^{(k)})_{ij} = \begin{cases} \overline{A}_{ij} & \text{if (4.62) holds,} \\ \underline{A}_{ij} & \text{otherwise} \end{cases}, \quad (\overline{A}^{(k)})_{ij} = \begin{cases} \underline{A}_{ij} & \text{if (4.63) holds,} \\ \overline{A}_{ij} & \text{otherwise} \end{cases},$$

$$(\underline{b}^{(k)})_i = \begin{cases} \overline{b}_i & \text{if (4.64) holds,} \\ \underline{b}_i & \text{otherwise} \end{cases}, \quad (\overline{b}^{(k)})_i = \begin{cases} \underline{b}_i & \text{if (4.65) holds,} \\ \overline{b}_i & \text{otherwise} \end{cases}$$

$(i, j = 1, \dots, n)$ *satisfies* $(\overline{x}^{(k)})_k = \overline{x}_k$.

4.6 Dependent data

4.6.1 The basic problem

4.6.2 The approach

4.6.3 The symmetric case

Chapter 5

Systems of interval linear equations and inequalities (rectangular case)

5.1 Introduction

Unless said otherwise, all the proofs of theorems contained in this chapter can be found in [98].

5.2 Solvability and feasibility

5.2.1 Introduction and notations

This chapter deals with solvability and feasibility (i.e., nonnegative solvability) of systems of interval linear equations and inequalities. After a few preliminary sections, we delineate in Section 5.2.3 eight decision problems (weak solvability of equations through strong feasibility of inequalities) which are then solved in eight successive Sections 5.2.4 to 5.2.12. It turns out that four problems are solvable in polynomial time and four are NP-hard. Some of the results are easy (Theorem 208), some difficult to prove (Theorem 209), some are surprising (Theorem 222). Although solutions of several of them have been already known, the complete classification of the eight problems given here is new. Some special cases (tolerance, control, and algebraic solutions, systems with square matrices) are treated in Sections 5.4.2 to 5.4.5. The last Section 5.5 contains additional notes and references to the material of this chapter. Some of the results find later applications in interval linear programming (Chapter 6).

We shall use the following notations. The ith row of a matrix A is denoted by $A_{i\cdot}$, the jth column by $A_{\cdot j}$. For two matrices A, B of the same size, inequalities like $A \leq B$ or $A < B$ are understood componentwise. A is called nonnegative if $0 \leq A$; A^T is the transpose of A. The absolute value of a matrix $A = (a_{ij})$ is defined by $|A| = (|a_{ij}|)$. We shall use the following easy-to-prove properties valid whenever the respective operations and inequalities are defined:

(i) $A \leq B$ and $0 \leq C$ imply $AC \leq BC$,
(ii) $A \leq |A|$,
(iii) $|A| \leq B$ if and only if $-B \leq A \leq B$,
(iv) $|A + B| \leq |A| + |B|$,
(v) $||A| - |B|| \leq |A - B|$,
(vi) $|AB| \leq |A||B|$.

The same notations and results also apply to vectors which are always considered one-column matrices. Hence, for $a = (a_i)$ and $b = (b_i)$, $a^T b = \sum_i a_i b_i$ is the scalar product whereas ab^T is the matrix $(a_i b_j)$. Maximum (or minimum) of two vectors a, b is understood componentwise, i.e., $(\max\{a, b\})_i = \max\{a_i, b_i\}$ for each i. In particular, for vectors a^+, a^- defined by $a^+ = \max\{a, 0\}$, $a^- = \max\{-a, 0\}$ we have $a = a^+ - a^-$, $|a| = a^+ + a^-$, $a^+ \geq 0$, $a^- \geq 0$ and $(a^+)^T a^- = 0$. I denotes the unit matrix, e_j is the jth column of I and $e = (1, \ldots, 1)^T$ is the vector of all ones (in these cases we do not designate explicitly the dimension which can always be inferred from the context). In our descriptions to follow, important role will be played by the set Y_m of all ± 1 vectors in $\mathbb{R}^m$, i.e.,

$$Y_m = \{y \in \mathbb{R}^m;\ |y| = e\}.$$

Obviously, the cardinality of Y_m is 2^m. For each $x \in \mathbb{R}^m$ we define its sign vector $\operatorname{sgn} x$by

$$(\operatorname{sgn} x)_i = \begin{cases} 1 & \text{if } x_i \geq 0, \\ -1 & \text{if } x_i < 0 \end{cases} \qquad (i = 1, \ldots, m),$$

so that $\operatorname{sgn} x \in Y_m$. For a given vector $y \in \mathbb{R}^m$ we denote

$$T_y = \operatorname{diag}(y_1, \dots, y_m) = \begin{pmatrix} y_1 & 0 & \dots & 0 \\ 0 & y_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y_m \end{pmatrix}. \tag{5.1}$$

With a few exceptions (mainly in the proof of Theorem 204), we shall use the notation T_y for vectors $y \in Y_m$ only, in which case we have $T_{-y} = -T_y$, $T_y^{-1} = T_y$ and $|T_y| = I$. For each $x \in \mathbb{R}^m$ we can write $|x| = T_z x$, where $z = \operatorname{sgn} x$; we shall often use this trick to remove the absolute value of a vector. Notice that $T_z x = (z_i x_i)_{i=1}^m$.

5.2.2 Solvability and feasibility

From this section on we consider systems of interval linear equations $\mathbf{A}x = \mathbf{b}$ or systems of interval linear inequalities $\mathbf{A}x \le \mathbf{b}$. Unless said otherwise, it is always assumed that $\mathbf{A}$ is $m \times n$ and $\mathbf{b}$ is an m-dimensional interval vector, where m and n are arbitrary positive integers.

Let us recall (Section 1.3.10) that a system of linear equations $Ax = b$ is called *solvable* if it has a solution, and *feasible* if it has a nonnegative solution. Throughout this chapter the reader is kindly asked to bear in mind that *feasibility means nonnegative solvability*.

The basic result concerning feasibility of linear equations was proved by Farkas [9] in 1902. As it will be used at some crucial points in the sequel, we give here an elementary, but somewhat lengthy proof of it. The ideas of the proof will not be exploited later, so that the reader may skip the proof without loss of continuity.

Theorem 199. *[Farkas] A system*

$$Ax = b \tag{5.2}$$

is feasible if and only if each p with $A^T p \ge 0$ satisfies $b^T p \ge 0$.

Proof: (a) If the system (5.2) has a solution $x \ge 0$ and if $A^T p \ge 0$ holds for some $p \in \mathbb{R}^m$, then $b^T p = (Ax)^T p = x^T (A^T p) \ge 0$. This proves the "only if" part of the theorem.

(b) We shall prove the "if" part by contradiction, proving that if the system (5.2) does not possess a nonnegative solution, then there exists a $p \in \mathbb{R}^m$ satisfying $A^T p \ge 0$ and $b^T p < 0$; for the purposes of the proof it is advantageous to write down this system in the column form

$$p^T A_{.j} \ \ge \ 0 \qquad (j = 1, \ldots, n), \tag{5.3}$$

$$p^T b \ < \ 0. \tag{5.4}$$

We shall prove this assertion by induction on n.

(b1) If $n = 1$, then A consists of a single column a. Let $W = \{\alpha a;\ \alpha \in \mathbb{R}\}$ be the subspace spanned by a. According to the orthogonal decomposition theorem (Meyer [43], p. 405), b can be written in the form

$$b = b_W + b_{W^\perp},$$

where $b_W \in W$ and $b_{W^\perp} \in W^\perp$, $W^\perp$ being the orthogonal complement of W. We shall consider two cases. If $b_{W^\perp} = 0$, then $b \in W$, so that $b = \alpha a$ for some $\alpha \in \mathbb{R}$. Since $Ax = b$ does not possess a nonnegative solution due to the assumption, it must be $\alpha < 0$ and $a \ne 0$, so that if we put $p = a$, then $p^T a = \|a\|_2^2 \ge 0$ and

$p^T b = \alpha \|a\|_2^2 < 0$, hence p satisfies (5.3), (5.4). If $b_{W^\perp} \neq 0$, put $p = -b_{W^\perp}$, then $p^T a = 0$ and $p^T b = -\|b_{W^\perp}\|_2^2 < 0$, so that p again satisfies (5.3), (5.4).

(b2) Let the induction hypothesis hold for $n-1 \geq 1$ and let a system (5.2), where $A \in \mathbb{R}^{m\times n}$, does not possess a nonnegative solution. Then neither does the system

$$\sum_{j=1}^{n-1} A_{\cdot j} x_j = b$$

(otherwise for $x_n = 0$ we would get a nonnegative solution of (5.2)), hence according to the induction hypothesis there exists a $\overline{p} \in \mathbb{R}^m$ satisfying

$$\overline{p}^T A_{\cdot j} \geq 0 \qquad (j = 1, \ldots, n-1), \tag{5.5}$$

$$\overline{p}^T b < 0. \tag{5.6}$$

If $\overline{p}^T A_{\cdot n} \geq 0$, then p satisfies (5.3), (5.4) and we are done. Thus assume that

$$\overline{p}^T A_{\cdot n} < 0. \tag{5.7}$$

Put

$$\begin{aligned} \alpha_j &= \overline{p}^T A_{\cdot j} \qquad (j = 1, \ldots, n), \\ \beta &= \overline{p}^T b, \end{aligned}$$

then $\alpha_1 \geq 0, \ldots, \alpha_{n-1} \geq 0$, $\alpha_n < 0$ and $\beta < 0$. Consider the system

$$\sum_{j=1}^{n-1} (\alpha_n A_{\cdot j} - \alpha_j A_{\cdot n}) x_j = \alpha_n b - \beta A_{\cdot n}. \tag{5.8}$$

If it had a nonnegative solution $x_1, \ldots, x_{n-1}$, then we could rearrange it to the form

$$\sum_{j=1}^{n-1} A_{\cdot j} x_j + A_{\cdot n} x_n = b, \tag{5.9}$$

where

$$x_n = \frac{\beta - \sum_{j=1}^{n-1} \alpha_j x_j}{\alpha_n} > 0$$

due to (5.5), (5.6), (5.7), so that the system (5.9), and thus also (5.2), would have a nonnegative solution $x_1, \ldots, x_n$ contrary to the assumption. Therefore the system (5.8) does not possess a nonnegative solution and thus according to the induction hypothesis there exists a $\tilde{p}$ such that

$$\tilde{p}^T (\alpha_n A_{\cdot j} - \alpha_j A_{\cdot n}) \geq 0 \qquad (j = 1, \ldots, n-1), \tag{5.10}$$

$$\tilde{p}^T (\alpha_n b - \beta A_{\cdot n}) < 0. \tag{5.11}$$

Now we set

$$p = \alpha_n \tilde{p} - (\tilde{p}^T A_{\cdot n})\overline{p}$$

and we shall show that p satisfies (5.3), (5.4). For $j = 1, \ldots, n-1$ we have according to (5.10)

$$p^T A_{\cdot j} = \alpha_n \tilde{p}^T A_{\cdot j} - (\tilde{p}^T A_{\cdot n})\overline{p}^T A_{\cdot j} \geq \alpha_j \tilde{p}^T A_{\cdot n} - (\tilde{p}^T A_{\cdot n})\alpha_j = 0, \tag{5.12}$$

for $j = n$ we get

$$p^T A_{\cdot n} = \alpha_n \tilde{p}^T A_{\cdot n} - (\tilde{p}^T A_{\cdot n})\overline{p}^T A_{\cdot n} = \alpha_n \tilde{p}^T A_{\cdot n} - (\tilde{p}^T A_{\cdot n})\alpha_n = 0, \tag{5.13}$$

and finally from (5.11)

$$p^T b = \alpha_n \tilde{p}^T b - (\tilde{p}^T A_{\cdot n})\overline{p}^T b < \beta \tilde{p}^T A_{\cdot n} - (\tilde{p}^T A_{\cdot n})\beta = 0, \tag{5.14}$$

so that (5.12), (5.13), (5.14) imply (5.3) and (5.4), hence p is a vector having the asserted properties, which completes the proof by induction. □

With the help of Farkas theorem we can now characterize solvability of systems of linear equations:

Theorem 200. *A system $Ax = b$ is solvable if and only if each p with $A^T p = 0$ satisfies $b^T p = 0$.*

Proof: If x solves $Ax = b$ and $A^T p = 0$ holds for some p, then $b^T p = p^T b = p^T Ax = (A^T p)^T x = 0$. Conversely, let the condition hold. Then for each p such that $A^T p \geq 0$ and $A^T p \leq 0$ we have $b^T p \geq 0$. But this, according to the Farkas theorem, is just the sufficient condition for the system

$$Ax_1 - Ax_2 = b \tag{5.15}$$

to be feasible. Hence (5.15) has a solution $x_1 \geq 0$, $x_2 \geq 0$, thus $A(x_1 - x_2) = b$ and $Ax = b$ is solvable. □

For systems of linear inequalities we introduce the notions of solvability and feasibility in the same way: a system $Ax \leq b$ is called *solvable* if it has a solution, and *feasible* if it has a nonnegative solution. Again, we can use Farkas theorem for characterizing solvability and feasibility:

Theorem 201. *A system $Ax \leq b$ is solvable if and only if each $p \geq 0$ with $A^T p = 0$ satisfies $b^T p \geq 0$.*

Proof: If x solves $Ax \leq b$ and $A^T p = 0$ holds for some $p \geq 0$, then $b^T p = p^T b \geq p^T Ax = 0$. Conversely, let the condition hold, so that each $p \geq 0$ with $A^T p \geq 0$,

$A^T p \le 0$ satisfies $b^T p \ge 0$. This, however, in view of the Farkas theorem means that the system

$$Ax_1 - Ax_2 + x_3 = b$$

is feasible. Hence due to the nonnegativity of x_3 we have $A(x_1 - x_2) \le b$, and the system $Ax \le b$ is solvable. □

Theorem 202. *A system $Ax \le b$ is feasible if and only if each $p \ge 0$ with $A^T p \ge 0$ satisfies $b^T p \ge 0$.*

Proof: If $x \ge 0$ solves $Ax \le b$ and $A^T p \ge 0$ holds for some $p \ge 0$, then $b^T p = p^T b = p^T Ax = (A^T p)^T x \ge 0$. Conversely, let the condition hold; then it is exactly the Farkas condition for the system

$$Ax_1 + x_2 = b \tag{5.16}$$

to be feasible. Hence (5.16) has a solution $x_1 \ge 0$, $x_2 \ge 0$, which implies $Ax_1 \le b$, so that the system $Ax \le b$ is feasible. □

Finally, we sum up the results achieved in this section in the form of a table which reveals similarities and differences among the four necessary and sufficient conditions:

Problem	Condition
solvability of $Ax = b$	$(\forall p)(A^T p = 0 \Rightarrow b^T p = 0)$
feasibility of $Ax = b$	$(\forall p)(A^T p \ge 0 \Rightarrow b^T p \ge 0)$
solvability of $Ax \le b$	$(\forall p \ge 0)(A^T p = 0 \Rightarrow b^T p \ge 0)$
feasibility of $Ax \le b$	$(\forall p \ge 0)(A^T p \ge 0 \Rightarrow b^T p \ge 0)$

An important result published by Khachiyan [34] in 1979 says that feasibility of a system of linear equations can be checked (and a solution to it, if it exists, found) in polynomial time. Since all three other problems, as shown in the proofs, can be reduced to this one, it follows that all four problems can be solved in polynomial time.

Lemma 203 (existence lemma) *Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and let for each $y \in Y$ the inequality*

$$T_y Ax \ge T_y b \tag{5.17}$$

have a solution x_y. Then the equation

$$Ax = b$$

has a solution in the set

$$\mathrm{Conv}\{x_y;\ y \in Y\}. \tag{5.18}$$

Proof: We shall prove that the system of linear equations

$$\sum_{y \in Y} \lambda_y A x_y = b, \tag{5.19}$$

$$\sum_{y \in Y} \lambda_y = 1 \tag{5.20}$$

has a solution $\lambda_y \geq 0$, $y \in Y$. In view of the Farkas theorem, it suffices to show that for each $p \in \mathbb{R}^m$ and for each $p_0 \in \mathbb{R}$,

$$p^T A x_y + p_0 \geq 0 \text{ for each } y \in Y \tag{5.21}$$

implies

$$p^T b + p_0 \geq 0. \tag{5.22}$$

Thus let p, p_0 satisfy (5.21). Put $y = -\operatorname{sgn} p$, then $p = -T_y|p|$ and from (5.17), (5.21) we have

$$p^T b + p_0 = -|p|^T T_y b + p_0 \geq -|p|^T T_y A x_y + p_0 = p^T A x_y + p_0 \geq 0,$$

which proves (5.22). Hence the system (5.19), (5.20) has a solution $\lambda_y \geq 0$, $y \in Y$. Put $x = \sum_{y \in Y} \lambda_y x_y$, then $Ax = b$ by (5.19) and x belongs to the set (5.18) by (5.20). $\square$

5.2.3 Weak and strong solvability/feasibility

Let $\mathbf{A}$ be an $m \times n$ interval matrix and $\mathbf{b}$ an m-dimensional interval vector. Under a system of interval linear equations

$$\mathbf{A}x = \mathbf{b} \tag{5.23}$$

we understand the *family* of all systems of linear equations

$$Ax = b \tag{5.24}$$

with data satisfying

$$A \in \mathbf{A},\ b \in \mathbf{b}, \tag{5.25}$$

and similarly a system of interval linear inequalities

$$\mathbf{A}x \leq \mathbf{b} \tag{5.26}$$

is the *family* of all systems

$$Ax \leq b$$

whose data satisfy

$$A \in \mathbf{A},\ b \in \mathbf{b}.$$

We introduce the following definitions.

Definition. A system (5.23) is said to be *weakly* solvable (feasible) if *some* system (5.24) with data (5.25) is solvable (feasible), and it is called *strongly* solvable (feasible) if *each* system (5.24) with data (5.25) is solvable (feasible). In the same way we define weak and strong solvability (feasibility) of a system of interval linear inequalities (5.26).

Hence, the word "weakly" refers to validity of the respective property for some system in the family whereas the word "strongly" refers to its validity for all systems in the family.

Introduction of weak and strong properties has an obvious motivation. Assume we are to decide whether some system $A_0x = b_0$ is solvable, but the exact data of this system are not directly available to us (they come from some measurements, are afflicted with rounding errors, etc.); instead, we only know that they satisfy $A_0 \in \mathbf{A}$, $b_0 \in \mathbf{b}$. Then we can be sure that our system $A_0x = b_0$ is solvable only if we know that the system (5.23) is strongly solvable, and in a similar way we can be sure that the system $A_0x = b_0$ is not solvable only if we know that the system (5.23) is not weakly solvable. A similar reasoning also holds for feasibility and for interval linear inequalities.

In this way, combining weak and strong solvability or feasibility of systems of interval linear equations or inequalities, we arrive at eight decision problems:

- weak solvability of equations,

- weak feasibility of equations,
- strong solvability of equations,
- strong feasibility of equations,
- weak solvability of inequalities,
- weak feasibility of inequalities,
- strong solvability of inequalities,
- strong feasibility of inequalities.

We shall study these problems separately in the next eight sections. It will be shown that all of them can be solved by finite means, however in half of the cases the number of steps is exponential in matrix size and the respective problems will be proved to be NP-hard.

5.2.4 Weak solvability of equations

In this section we shall study the first of the eight decision problems delineated in Section 5.2.3, namely weak solvability of systems of interval linear equations. As before, we shall assume that $\mathbf{A}$ is an $m \times n$ interval matrix and $\mathbf{b}$ is an m-dimensional interval vector, where m and n are arbitrary positive integers.

First we shall introduce a useful auxiliary term: a vector $x \in \mathbb{R}^n$ is called a *weak solution* of $\mathbf{A}x = \mathbf{b}$ if it satisfies $Ax = b$ for some $A \in \mathbf{A}$, $b \in \mathbf{b}$. Oettli and Prager [57] proved in 1964 the following nice and far-reaching characterization of weak solutions:

Theorem 204. (Oettli-Prager) *A vector $x \in \mathbb{R}^n$ is a weak solution of $\mathbf{A}x = \mathbf{b}$ if and only if it satisfies*

$$|A_c x - b_c| \le \Delta|x| + \delta. \tag{5.27}$$

Proof: If x is a weak solution, then $Ax = b$ for some $A \in \mathbf{A}$, $b \in \mathbf{b}$, which gives $|A_c x - b_c| = |(A_c - A)x + b - b_c| \le \Delta|x| + \delta$. Conversely, let $|A_c x - b_c| \le \Delta|x| + \delta$ hold for some x. Define $y \in \mathbb{R}^m$ by

$$y_i = \begin{cases} \frac{(A_c x - b_c)_i}{(\Delta|x|+\delta)_i} & \text{if } (\Delta|x| + \delta)_i > 0, \\ 1 & \text{if } (\Delta|x| + \delta)_i = 0 \end{cases} \qquad (i = 1, \ldots, m), \tag{5.28}$$

then $|y| \le e$ and

$$A_c x - b_c = T_y(\Delta|x| + \delta). \tag{5.29}$$

Put $z = \operatorname{sgn} x$, then $|x| = T_z x$ and from (5.29) we get

$$(A_c - T_y \Delta T_z)x = b_c + T_y \delta. \tag{5.30}$$

Since $|y| \le e$ and $z \in Y_n$, we have $|T_y \Delta T_z| \le \Delta$ and $|T_y \delta| \le \delta$, so that $A_c - T_y \Delta T_z \in \mathbf{A}$ and $b_c + T_y \delta \in \mathbf{b}$, which implies that x is a weak solution of $\mathbf{A}x = \mathbf{b}$. □

The main merit of the Oettli-Prager theorem consists in the fact that it describes the set of all weak solutions by means of a single, but nonlinear, inequality (5.27). In the proof we have also established a constructive result which is worth stating independently:

Proposition 205 *If x solves (5.27), then it satisfies*

$$(A_c - T_y \Delta T_z)x = b_c + T_y \delta, \tag{5.31}$$

where y is given by

$$y_i = \begin{cases} (A_c x - b_c)_i / (\Delta|x| + \delta)_i & \text{if } (\Delta|x| + \delta)_i > 0, \\ 1 & \text{if } (\Delta|x| + \delta)_i = 0 \end{cases} \qquad (i = 1, \ldots, m), \tag{5.32}$$

and $z = \operatorname{sgn} x$.

Weak solvability of a system $\mathbf{A}x = \mathbf{b}$, as it was defined in Section 5.2.3, is equivalent to existence of a weak solution to it. Hence we can employ the Oettli-Prager theorem to characterize weak solvability of interval linear equations. Let us remind that in accordance with the general definition (3.7) we have $A_{ez} = A_c - \Delta T_z$ and $A_{-ez} = A_c + \Delta T_z$.

Theorem 206. *A system $\mathbf{A}x = \mathbf{b}$ is weakly solvable if and only if the system*

$$A_{ez}x \leq \overline{b}, \tag{5.33}$$
$$-A_{-ez}x \leq -\underline{b} \tag{5.34}$$

is solvable for some $z \in Y_n$.

Proof: If $\mathbf{A}x = \mathbf{b}$ is weakly solvable, then it has a weak solution x which according to Theorem 204 satisfies (5.27) and thus also

$$-\Delta|x| - \delta \leq A_c x - b_c \leq \Delta|x| + \delta. \tag{5.35}$$

If we put $z = \operatorname{sgn} x$, then $|x| = T_z x$ and (5.35) turns into $A_{ez}x = (A_c - \Delta T_z)x \leq b_c + \delta = \overline{b}$ and $A_{-ez}x = (A_c + \Delta T_z)x \geq b_c - \delta = \underline{b}$ which shows that x satisfies (5.33), (5.34). Conversely, let (5.33), (5.34) hold for some x and $z \in Y_n$. Then we have

$$-\Delta T_z x - \delta \leq A_c x - b_c \leq \Delta T_z x + \delta$$

and consequently

$$|A_c x - b_c| \leq \Delta T_z x + \delta \leq \Delta|x| + \delta,$$

hence x satisfies (5.27) and therefore it is a weak solution of $\mathbf{A}x = \mathbf{b}$. □

This result shows that checking weak solvability of interval linear equations can be in principle performed by finite means by checking solvability of systems (5.33), (5.34), $z \in Y_n$ by some finite procedure (e.g., a linear programming technique). However, to verify that $\mathbf{A}x = \mathbf{b}$ is not weakly solvable, we have to check all the systems (5.33), (5.34), $z \in Y_n$, whose number in the worst case is 2^n. Clearly, this is nearly impossible even for relatively small values of n (say, $n = 30$). It turns out that the source of these difficulties does not lie with inadequateness of our description, but that it is inherently present in the problem itself which is NP-hard. In the proof of this statement we shall see an approach that will also be used several times later, namely a polynomial-time reduction of our standard NP-complete problem from Theorem 22 to the current problem, which will prove its NP-hardness.

Theorem 207. *Checking weak solvability of interval linear equations is NP-hard.*

Proof: Let A be a square matrix. We shall first prove that the system

$$-e \leq Ax \leq e, \tag{5.36}$$

$$e^T|x| \geq 1 \tag{5.37}$$

has a solution if and only if the system of interval linear equations

$$\begin{aligned} [A, A]x &= [-e, e], & (5.38) \\ [-e^T, e^T]x &= [1, 1] & (5.39) \end{aligned}$$

is weakly solvable. If x solves (5.36), (5.37) and if we set $x' = \frac{x}{e^T|x|}$, then $|Ax'| = \frac{1}{e^T|x|}|Ax| \leq |Ax| \leq e$ and $e^T|x'| = 1$, hence x' satisfies $Ax' = b$, $z^T x' = 1$ for some $b \in [-e, e]$ and $z^T = (\operatorname{sgn} x')^T \in [-e^T, e^T]$, which means that (5.38), (5.39) is weakly solvable. Conversely, let (5.38), (5.39) have a weak solution x; then $Ax = b$ and $c^T x = 1$ for some $b \in [-e, e]$ and $c^T \in [-e^T, e^T]$, hence $|Ax| \leq e$ and $1 = c^T x \leq |c|^T |x| \leq e^T |x|$, so that x solves (5.36), (5.37). We have shown that the problem of checking solvability of (5.36), (5.37) can be reduced in polynomial time to that of checking weak solvability of (5.38), (5.39). Since the former problem is NP-complete by Theorem 22, the latter one is NP-hard. □

5.2.5 Weak feasibility of equations

Using the notion of a weak solution introduced in Section 5.2.4, we can say that a system $\mathbf{A}x = \mathbf{b}$ is weakly feasible (in the sense of the definition made in Section 5.2.3) if and only if it has a nonnegative weak solution. Hence we can again use the Oettli-Prager theorem to obtain a characterization of weak feasibility:

Theorem 208. *A system* $\mathbf{A}x = \mathbf{b}$ *is weakly feasible if and only if the system*

$$\begin{aligned} \underline{A}x &\le \overline{b}, \qquad (5.40) \\ -\overline{A}x &\le -\underline{b} \qquad (5.41) \end{aligned}$$

is feasible.

Proof: If $\mathbf{A}x = \mathbf{b}$ is weakly feasible, then it possesses a nonnegative weak solution x which by Theorem 204 satisfies

$$|A_c x - b_c| \le \Delta x + \delta \qquad (5.42)$$

and thus also

$$-\Delta x - \delta \le A_c x - b_c \le \Delta x + \delta, \qquad (5.43)$$

which is (5.40), (5.41). Conversely, if (5.40), (5.41) has a nonnegative solution x, then it satisfies (5.43) and (5.42) and by the same Theorem 204 it is a nonnegative weak solution to $\mathbf{A}x = \mathbf{b}$ which means that this system is weakly feasible. □

Hence, only one system of linear inequalities (5.40), (5.41) is to be checked in this case. Referring to the last paragraph of Section 5.2.2, we can conclude that checking weak feasibility of interval linear equations can be performed in polynomial time whereas checking weak solvability, as we have seen in Theorem 207, is NP-hard.

5.2.6 Strong solvability of equations

By definition (Section 5.2.3), $\mathbf{A}x = \mathbf{b}$ is strongly solvable if each system $Ax = b$ with $A \in \mathbf{A}$, $b \in \mathbf{b}$ is solvable. If $\underline{A}_{ij} < \overline{A}_{ij}$ for some i, j or $\underline{b}_i < \overline{b}_i$ for some i, then the family $\mathbf{A}x = \mathbf{b}$ consists of infinitely many linear systems. Therefore the fact that solvability of these infinitely many systems can be characterized in terms of feasibility of finitely many systems is nontrivial, and so is the proof of the following theorem which also establishes a useful additional property. $\mathrm{Conv}\, X$ denotes the convex hull of X, i.e., the intersection of all convex subsets of $\mathbb{R}^n$ containing X.

Theorem 209. *A system $\mathbf{A}x = \mathbf{b}$ is strongly solvable if and only if for each $y \in Y_m$ the system*

$$A_{ye}x^1 - A_{-ye}x^2 = b_y, \tag{5.44}$$

$$x^1 \geq 0,\ x^2 \geq 0 \tag{5.45}$$

has a solution x_y^1, x_y^2. Moreover, if this is the case, then for each $A \in \mathbf{A}$, $b \in \mathbf{b}$ the system $Ax = b$ has a solution in the set

$$\mathrm{Conv}\{x_y^1 - x_y^2;\ y \in Y_m\}.$$

Proof: "Only if": Let $\mathbf{A}x = \mathbf{b}$ be strongly solvable. Assume to the contrary that (5.44), (5.45) does not have a solution for some $y \in Y_m$. Then Farkas theorem implies existence of a $p \in \mathbb{R}^m$ satisfying

$$(A_c - T_y\Delta)^T p \;\geq\; 0, \tag{5.46}$$

$$(A_c + T_y\Delta)^T p \;\leq\; 0, \tag{5.47}$$

$$b_y^T p \;<\; 0. \tag{5.48}$$

Now (5.46) and (5.47) together give

$$\Delta^T T_y p \leq A_c^T p \leq -\Delta^T T_y p,$$

hence

$$|A_c^T p| \leq -\Delta^T T_y p = |-\Delta^T T_y p| \leq \Delta^T |p|,$$

and the Oettli-Prager theorem as applied to the system $[A_c^T - \Delta^T, A_c^T + \Delta^T]x = [0, 0]$ shows that there exists a matrix $A \in \mathbf{A}$ such that

$$A^T p = 0. \tag{5.49}$$

In the light of Theorem 200, (5.49) and (5.48) mean that the system

$$Ax = b_y$$

has no solution, which contradicts our assumption of strong solvability since $A \in \mathbf{A}$ and $b_y \in \mathbf{b}$.

"If": Conversely, let for each $y \in Y_m$ the system (5.44), (5.45) have a solution x_y^1, x_y^2. Let $A \in \mathbf{A}$, $b \in \mathbf{b}$. To prove that the system $Ax = b$ has a solution, take an arbitrary $y \in Y_m$ and put $x_y = x_y^1 - x_y^2$. Then we have

$$\begin{aligned} T_y(Ax_y - b) &= T_y(A_c x_y - b_c) + T_y(A - A_c)x_y + T_y(b_c - b) \\ &\geq T_y(A_c x_y - b_c) - \Delta|x_y| - \delta \end{aligned}$$

since $|T_y(A - A_c)x_y| \leq \Delta|x_y|$, which implies $T_y(A - A_c)x_y \geq -\Delta|x_y|$, and similarly $|T_y(b_c - b)| \leq \delta$ implies $T_y(b_c - b) \geq -\delta$, thus

$$\begin{aligned} T_y(Ax_y - b) &\geq T_y(A_c(x_y^1 - x_y^2) - b_c) - \Delta|x_y^1 - x_y^2| - \delta \\ &\geq T_y(A_c(x_y^1 - x_y^2) - b_c) - \Delta(x_y^1 + x_y^2) - \delta \\ &= T_y((A_c - T_y\Delta)x_y^1 - (A_c + T_y\Delta)x_y^2 - (b_c + T_y\delta)) \\ &= T_y(A_{ye}x_y^1 - A_{-ye}x_y^2 - b_y) \\ &= 0 \end{aligned}$$

since x_y^1, x_y^2 solve (5.44), (5.45). In this way we have proved that for each $y \in Y_m$, x_y satisfies

$$T_y A x_y \geq T_y b. \tag{5.50}$$

Using (5.50), we shall next prove that the system of linear equations

$$\sum_{y \in Y_m} \lambda_y A x_y = b, \tag{5.51}$$

$$\sum_{y \in Y_m} \lambda_y = 1 \tag{5.52}$$

has a solution $\lambda_y \geq 0$, $y \in Y_m$. In view of Farkas theorem, it suffices to show that for each $p \in \mathbb{R}^m$ and each $p_0 \in \mathbb{R}$,

$$p^T A x_y + p_0 \geq 0 \text{ for each } y \in Y_m \tag{5.53}$$

implies

$$p^T b + p_0 \geq 0. \tag{5.54}$$

Thus let p and p_0 satisfy (5.53). Put $y = -\operatorname{sgn} p$, then $p = -T_y|p|$ and from (5.50), (5.53) we have

$$p^T b + p_0 = -|p|^T T_y b + p_0 \geq -|p|^T T_y A x_y + p_0 = p^T A x_y + p_0 \geq 0,$$

which proves (5.54). Hence the system (5.51), (5.52) has a solution $\lambda_y \geq 0$, $y \in Y_m$. Put $x = \sum_{y \in Y_m} \lambda_y x_y$, then $Ax = b$ by (5.51) and x belongs to the set $\operatorname{Conv}\{x_y;\, y \in Y_m\} = \operatorname{Conv}\{x_y^1 - x_y^2;\, y \in Y_m\}$ by (5.52). This proves the "if" part, and also the additional assertion. □

Let us have a closer look at the form of the systems (5.44). If $y_k = 1$, then the kth rows of A_{ye} and A_{-ye} are equal to the kth rows of $\underline{A}$ and $\overline{A}$, respectively, and $(b_y)_k = \overline{b}_k$. This means that in this case the kth equation of (5.44) has the form

$$(\underline{A}x^1 - \overline{A}x^2)_k = \overline{b}_k, \tag{5.55}$$

and similarly in case $y_k = -1$ it is of the form

$$(\overline{A}x^1 - \underline{A}x^2)_k = \underline{b}_k. \tag{5.56}$$

Hence we can see that the family of systems (5.44) for all $y \in Y_m$ is just the family of all systems whose kth equations are either of the form (5.55), or of the form (5.56) for $k = 1, \ldots, m$. Now we can use the algorithm of Section 1.3.1 to generate the systems $A_{ye}x^1 - A_{-ye}x^2 = b_y$ in such a way that any pair of successive systems differs in exactly one equation. In this way, a feasible solution x^1, x^2 of the preceding system satisfies all but at most one of the equations of the next generated system, so that this solution x^1, x^2 can be used as the initial iteration for the procedure for checking feasibility of the next system (the procedure is not specified in the algorithm; e.g., phase I of the simplex method may be used for this purpose). The complete description of the algorithm is as follows:

$z := 0$; $y := e$; $strosolv := true$;
$A := \underline{A}$; $B := \overline{A}$; $b := \overline{b}$;
if $Ax^1 - Bx^2 = b$ is not feasible **then** $strosolv := false$; **end**
while $z \neq e$ & $strosolv$
    $k := \min\{i;\ z_i = 0\}$;
    for $i := 1$ **to** $k-1$, $z_i := 0$; **end**
    $z_k := 1$; $y_k := -y_k$;
    if $y_k = 1$ **then** $A_{k\cdot} := \underline{A}_{k\cdot}$; $B_{k\cdot} := \overline{A}_{k\cdot}$; $b_k := \overline{b}_k$;
        else $A_{k\cdot} := \overline{A}_{k\cdot}$; $B_{k\cdot} := \underline{A}_{k\cdot}$; $b_k := \underline{b}_k$;
    end
    if $Ax^1 - Bx^2 = b$ is not feasible **then** $strosolv := false$; **end**
end
% $\mathbf{A}x = \mathbf{b}$ is strongly solvable if and only if $strosolv = true$.

A small change can greatly improve the performance of the algorithm. Observe that if

$$\underline{A}_{k\cdot} = \overline{A}_{k\cdot} \quad \text{and} \quad \underline{b}_k = \overline{b}_k \tag{5.57}$$

holds for some k, then the equations (5.55) and (5.56) are the same and there is no need to solve the same system anew. Hence only rows satisfying

$$\underline{A}_{k\cdot} \neq \overline{A}_{k\cdot} \quad \text{or} \quad \underline{b}_k < \overline{b}_k \tag{5.58}$$

play any role. Let us reorder the equations of $\mathbf{A}x = \mathbf{b}$ so that those satisfying (5.58) go first, followed by those with (5.57). Hence, for the reordered system the matrix (Δ, δ) has first q nonzero rows, followed by $m-q$ zero rows $(0 \le q \le m)$. Now we can employ the algorithm in literally the same formulation, but started with $z := 0 \in \mathbb{R}^q$, $y := e \in \mathbb{R}^q$ (instead of $z, y \in \mathbb{R}^m$ in the original version). In this way, in case of strong solvability 2^q systems $A_{ye}x^1 - A_{-ye}x^2 = b_y$ are to be checked for feasibility. Clearly, the whole procedure can be considered acceptable for moderate values of q only.

Since the number of systems to be checked is in the worst case exponential in the matrix size, we may suspect the problem to be NP-hard. It turns out to be indeed the case, and the NP-complete problem of Theorem 22 can again be used for the purpose of the proof of this result.

Theorem 210. *Checking strong solvability of interval linear equations is NP-hard.*

Proof: Let A be square $n \times n$. We shall prove that the system

$$-e \le Ax \le e, \tag{5.59}$$

$$e^T|x| \ge 1 \tag{5.60}$$

has a solution if and only if the system of interval linear equations

$$[A - ee^T, A + ee^T]x = [0, e] \tag{5.61}$$

is *not* strongly solvable. "If": Assume that (5.61) is not strongly solvable, so that $A'x = b'$ does not have a solution for some $A' \in [A - ee^T, A + ee^T]$ and $b' \in [0, e]$. Then A' must be singular, hence $A'x' = 0$ for some $x' \ne 0$. Then x' is a weak solution of the system $[A - ee^T, A + ee^T]x = [0, 0]$, hence $|Ax'| \le ee^T|x'|$ by the Oettli-Prager theorem. Now if we set $x = \frac{x'}{e^T|x'|}$, then $|Ax| \le e$ and $e^T|x| = 1$, so that x solves (5.59), (5.60). "Only if" by contradiction: Assume that (5.61) is strongly solvable, and let A' be an arbitrary matrix in $[A - ee^T, A + ee^T]$. Then for each $j = 1, \dots, n$ the system $A'x = e_j$ (where $e_j \in [0, e]$ is the jth column of the unit matrix I) has a solution x^j, hence the matrix X consisting of columns $x^1, \dots, x^n$ satisfies $A'X = I$, so that A' is nonsingular. Hence, strong solvability of (5.61) implies nonsingularity of each $A' \in [A - ee^T, A + ee^T]$. Assume now that (5.59), (5.60) has a solution x. Then $|Ax| \le e \le ee^T|x|$, and the Oettli-Prager theorem implies that x solves $A'x = 0$ for some $A' \in [A - ee^T, A + ee^T]$, hence A' is singular which contradicts the above fact that each $A' \in [A - ee^T, A + ee^T]$ is nonsingular. This contradiction shows that strong solvability of (5.61) precludes existence of a solution to (5.59), (5.60), which proves the "only if" part of the assertion. In view of the established equivalence, we can see that the problem of checking solvability of (5.59), (5.60) can be reduced in polynomial time to that of checking strong solvability of (5.61). By Theorem 22, the former problem is NP-complete; hence the latter one is NP-hard. □

In an analogy with weak solutions, we may also introduce strong solutions of systems of interval linear equations. A vector x is said to be a *strong solution* of $\mathbf{A}x = \mathbf{b}$ if it satisfies $Ax = b$ for each $A \in \mathbf{A}$, $b \in \mathbf{b}$. We have this characterization of strong solutions:

Theorem 211. *A vector $x \in \mathbb{R}^n$ is a strong solution of $\mathbf{A}x = \mathbf{b}$ if and only if it satisfies*

$$A_c x = b_c, \tag{5.62}$$

$$\Delta|x| = \delta = 0. \tag{5.63}$$

Proof: Let x be a strong solution of $\mathbf{A}x = \mathbf{b}$. Put $z = \operatorname{sgn} x$, then $|x| = T_z x$, and x satisfies both

$$A_c x = b_c \tag{5.64}$$

and

$$(A_c + \Delta T_z)x = b_c - \delta. \tag{5.65}$$

Subtracting (5.64) from (5.65), we obtain

$$\Delta|x| = \Delta T_z x = -\delta,$$

where $\Delta|x| \geq 0$ and $-\delta \leq 0$, hence $\Delta|x| = \delta = 0$. Conversely, if (5.62) and (5.63) hold, then for each $A \in \mathbf{A}$, $b \in \mathbf{b}$ we have

$$|Ax - b| = |A_c x - b_c + (A - A_c)x + b_c - b| \leq \Delta|x| + \delta = 0,$$

so that $Ax = b$, hence x is a strong solution of $\mathbf{A}x = \mathbf{b}$. □

The condition $\Delta|x| = 0$ in (5.63) says that it must be $x_j = 0$ for each j with $\Delta_{.j} \neq 0$. Hence, putting $J = \{j;\, \Delta_{.j} \neq 0\}$, we may reformulate (5.62), (5.63) in the form

$$\sum_{j \notin J} (A_c)_{.j} x_j = b_c, \tag{5.66}$$

$$x_j = 0 \qquad (j \in J), \tag{5.67}$$

$$\delta = 0, \tag{5.68}$$

which shows that checking existence of a strong solution (and, in the positive case, also computation of it) may be performed by solving a single system of linear equations (5.66). But on the whole the system (5.66)-(5.68) shows that strong solutions exist on rare occasions only, as it could have been expected already from the definition.

5.2.7 Strong feasibility of equations

By definition in Section 5.2.3, a system $\mathbf{A}x = \mathbf{b}$ is strongly feasible if each system $Ax = b$ with $A \in \mathbf{A}$, $b \in \mathbf{b}$ is feasible. It turns out that characterization of strong feasibility can be easily derived from that of strong solvability:

Theorem 212. *A system $\mathbf{A}x = \mathbf{b}$ is strongly feasible if and only if for each $y \in Y_m$ the system*

$$A_{ye}x = b_y \tag{5.69}$$

has a nonnegative solution x_y. Moreover, if this is the case, then for each $A \in \mathbf{A}$, $b \in \mathbf{b}$ the system $Ax = b$ has a solution in the set

$$\mathrm{Conv}\{x_y;\, y \in Y_m\}.$$

Proof: If $\mathbf{A}x = \mathbf{b}$ is strongly feasible, then each system (5.69) has a nonnegative solution since $A_{ye} \in \mathbf{A}$ and $b_y \in \mathbf{b}$ for each $y \in Y_m$. Conversely, if for each $y \in Y_m$ the system (5.69) has a nonnegative solution x_y, then setting $x_y^1 = x_y$, $x_y^2 = 0$ for each $y \in Y_m$, we can see that x_y^1, x_y^2 solve (5.44), (5.45). This according to Theorem 209 means that each system $Ax = b$, $A \in \mathbf{A}$, $b \in \mathbf{b}$ has a solution in the set $\mathrm{Conv}\{x_y^1 - x_y^2;\, y \in Y_m\} = \mathrm{Conv}\{x_y;\, y \in Y_m\}$ which is a part of the nonnegative orthant, hence $\mathbf{A}x = \mathbf{b}$ is strongly feasible. □

Repeating the argument following the proof of Theorem 209, we can say that the kth row of (5.69) is of the form

$$(\underline{A}x)_k = \overline{b}_k$$

if $y_k = 1$ and of the form

$$(\overline{A}x)_k = \underline{b}_k$$

if $y_k = -1$. Hence, the algorithm for checking strong solvability can be easily adapted for the present purpose:

$z := 0$; $y := e$; $strofeas := true$;
$A := \underline{A}$; $b := \overline{b}$;
if $Ax = b$ is not feasible **then** $strofeas := false$; **end**
while $z \neq e$ & $strofeas$
  $k := \min\{i;\, z_i = 0\}$;
  for $i := 1$ **to** $k-1$, $z_i := 0$; **end**
  $z_k := 1$; $y_k := -y_k$;
  if $y_k = 1$ **then** $A_{k\cdot} := \underline{A}_{k\cdot}$; $b_k := \overline{b}_k$; **else** $A_{k\cdot} := \overline{A}_{k\cdot}$; $b_k := \underline{b}_k$; **end**
  if $Ax = b$ is not feasible **then** $strofeas := false$; **end**
end
% $\mathbf{A}x = \mathbf{b}$ is strongly feasible if and only if $strofeas = true$.

As in Section 5.2.6, the equations of $\mathbf{A}x = \mathbf{b}$ should be first reordered so that the first q of them satisfy (5.58) and the last $m - q$ of them are of the form (5.57). Then the algorithm remains in force if it is initialized with $z := 0 \in \mathbb{R}^q$, $y := e \in \mathbb{R}^q$.

In contrast to checking weak feasibility which is polynomial-time (Section 5.2.5), checking strong feasibility remains NP-hard. The proof, going along similar lines as before, is a little bit different since $n \times 2n$ matrices are needed here.

Theorem 213. *Checking strong feasibility of interval linear equations is NP-hard.*

Proof: Let A be square $n \times n$. We shall prove that the system

$$-e \leq Ax \leq e, \tag{5.70}$$

$$e^T|x| \geq 1 \tag{5.71}$$

has a solution if and only if the system of interval linear equations

$$[(A^T - ee^T, -A^T - ee^T), (A^T + ee^T, -A^T + ee^T)]x = [-e, e] \tag{5.72}$$

(with an $n \times 2n$ interval matrix) is *not* strongly feasible. "If": Let (5.72) be not strongly feasible; then according to Theorem 212 there exists a $y \in Y_m$ such that the system $A_{ye}x = b_y$ is not feasible. In our case this system has the form

$$(A^T - ye^T)x^1 + (-A^T - ye^T)x^2 = y.$$

Since it is not feasible, Farkas theorem assures existence of a vector x' satisfying

$$\begin{aligned} (A - ey^T)x' &\geq 0, && (5.73)\\ (-A - ey^T)x' &\geq 0, && (5.74)\\ y^T x' &< 0, && (5.75) \end{aligned}$$

then (5.73), (5.74) imply

$$|Ax'| \leq -ey^T x' = |-ey^T x'| \leq ee^T|x'|,$$

where $x' \neq 0$ by (5.75), hence the vector $x = \frac{x'}{e^T|x'|}$ satisfies $|Ax| \leq e$ and $e^T|x| = 1$, so that it is a solution to (5.70), (5.71). "Only if" by contradiction: Assume that (5.72) is strongly feasible. Let $A' \in [A - ee^T, A + ee^T]$, then $A'^T \in [A^T - ee^T, A^T + ee^T]$ and $-A'^T \in [-A^T - ee^T, -A^T + ee^T]$, so that strong feasibility of (5.72) implies that for each $j = 1, \ldots, n$ the equation

$$A'^T x^1 - A'^T x^2 = e_j$$

is feasible, i.e., the equation $A'^T x = e_j$ has a solution x^j. Then the matrix X consisting of columns $x^1, \ldots, x^n$ satisfies $A'^T X = I$, which proves that A'^T, and thus also A', is nonsingular. We have proved that strong feasibility of (5.72) implies nonsingularity of each $A' \in [A - ee^T, A + ee^T]$. As we have seen in the proof of Theorem 210, solvability of (5.70), (5.71) would mean existence of a singular matrix $A' \in [A - ee^T, A + ee^T]$, a contradiction. Hence (5.70), (5.71) is not solvable, which concludes the proof of the "only if" part. In view of Theorem 22, the established equivalence shows that checking strong feasibility is NP-hard. □

Theorem 214. *A system* $\mathbf{A}x = \mathbf{b}$ *is strongly feasible if and only if for each* $p \in \mathbb{R}^m$, $A_c^T p + \Delta^T |p| \geq 0$ *implies* $b_c^T p - \delta^T |p| \geq 0$.

Proof: First we prove that each system

$$Ax = b,\ x \geq 0 \tag{5.76}$$

with data satisfying

$$A \in \mathbf{A},\ b \in \mathbf{b} \tag{5.77}$$

has a solution if and only if

$$(\forall y)(A_c^T y + \Delta^T |y| \geq 0 \Rightarrow b_c^T y - \delta^T |y| \geq 0) \tag{5.78}$$

holds. "Only if": Let each system (5.76) with data (5.77) have a solution, and let $A_c^T y + \Delta^T |y| \geq 0$ for some $y \in \mathbb{R}^m$. Define a diagonal matrix T by $T_{ii} = 1$ if $y_i \geq 0$, $T_{ii} = -1$ if $y_i < 0$, and $T_{ij} = 0$ if $i \neq j$ $(i, j = 1, \ldots, m)$, then $|y| = Ty$. Consider now the system

$$(A_c + T\Delta)x = b_c - T\delta,\ x \geq 0. \tag{5.79}$$

Since $A_c + T\Delta \in \mathbf{A}$ and $b_c - T\delta \in \mathbf{b}$, the system (5.79) has a solution according to the assumption, and $(A_c + T\Delta)^T y = A_c^T y + \Delta^T |y| \geq 0$, hence FARKAS lemma applied to (5.79) gives that $b_c^T y - \delta^T |y| = (b_c - T\delta)^T y \geq 0$, which proves (5.78). "If": Assuming that (5.78) holds, consider a system (5.76) with data satisfying (5.77). Let $A^T y \geq 0$ for some y; then $A_c^T y + \Delta^T |y| \geq (A_c + A - A_c)^T y = A^T y \geq 0$, hence (5.78) gives that $b^T y = (b_c + b - b_c)^T y \geq b_c^T y - \delta^T |y| \geq 0$. Thus we have proved that for each y, $A^T y \geq 0$ implies $b^T y \geq 0$, and FARKAS lemma proves the existence of a solution to (5.76). □

Of the four decision problems related to interval linear equations we have investigated so far, three were found to be NP-hard and only one to be solvable in polynomial time. In the next four sections we shall see that this ratio becomes exactly reciprocal for interval linear inequalities: only one problem will be NP-hard, and three will be solvable in polynomial time.

5.2.8 Farkas-type theorems for equations

Theorem 215. *A system of interval linear equations* $\mathbf{A}x = \mathbf{b}$ *is*

- *weakly solvable if and only if*

$$(\exists z \in Y_n)(\forall p)(T_z A_c^T p \geq \Delta^T |p| \Rightarrow b_c^T p + \delta^T |p| \geq 0),$$

- *weakly feasible if and only if*

$$(\forall p)(A_c^T p \geq \Delta^T |p| \Rightarrow b_c^T p + \delta^T |p| \geq 0),$$

- *strongly solvable if and only if*

$$(\forall p)(|A_c^T p| \leq \Delta^T |p| \Rightarrow b_c^T p - \delta^T |p| \geq 0),$$

- *strongly feasible if and only if*

$$(\forall p)(A_c^T p + \Delta^T |p| \geq 0 \Rightarrow b_c^T p - \delta^T |p| \geq 0).$$

Proof: Weak feasibility [67], pp. 524-525, strong feasibility [96], pp. S1051-S1052; the other unpublished. □

Theorem 216. *If a system* $\mathbf{A}x = \mathbf{b}$ *is not weakly feasible, then there exists a fixed linear combination of rows which, when applied to any system* $Ax = b$ *with* $A \in \mathbf{A}$ *and* $b \in \mathbf{b}$*, always produces an equation which does not possess a nonnegative solution.*

Proof: [78], p. 94. □

5.2.9 Weak solvability of inequalities

As in Section 5.2.4, we first define $x \in \mathbb{R}^n$ to be a *weak solution* of a system of interval linear inequalities $\mathbf{A}x \le \mathbf{b}$ if it satisfies $Ax \le b$ for some $A \in \mathbf{A}$, $b \in \mathbf{b}$. Gerlach [14] proved in 1981 an analogue of the Oettli-Prager theorem for the case of interval linear inequalities:

Theorem 217. (Gerlach) *A vector x is a weak solution of $\mathbf{A}x \le \mathbf{b}$ if and only if it satisfies*

$$A_c x - \Delta|x| \le \overline{b}. \tag{5.80}$$

Proof: If x solves $Ax \le b$ for some $A \in \mathbf{A}$ and $b \in \mathbf{b}$, then

$$A_c x - b_c \le (A_c - A)x + b - b_c \le |(A_c - A)x + b - b_c| \le \Delta|x| + \delta,$$

which is (5.80). Conversely, let (5.80) hold for some x. Put $z = \operatorname{sgn} x$, then substituting $|x| = T_z x$ into (5.80) leads to

$$A_{ez} x \le \overline{b},$$

where $A_{ez} \in \mathbf{A}$ and $\overline{b} \in \mathbf{b}$, hence x is a weak solution of $\mathbf{A}x \le \mathbf{b}$. □

A system $\mathbf{A}x \le \mathbf{b}$ is weakly solvable (Section 5.2.3) if some system $Ax \le b$, $A \in \mathbf{A}$, $b \in \mathbf{b}$ is solvable; in other words, weak solvability is equivalent to existence of a weak solution. Hence, the Gerlach theorem provides us with the following characterization:

Theorem 218. *A system $\mathbf{A}x \le \mathbf{b}$ is weakly solvable if and only if the system*

$$A_{ez} x \le \overline{b} \tag{5.81}$$

is solvable for some $z \in Y_n$.

Proof: If x is a weak solution of $\mathbf{A}x \le \mathbf{b}$, then, as we have seen in the proof of the Gerlach theorem, it satisfies (5.81) for $z = \operatorname{sgn} x$. Conversely, if x satisfies (5.81) for some $z \in Y_n$, then it is a weak solution of the system $\mathbf{A}x \le \mathbf{b}$ which is then weakly solvable. □

The description suggests that the problem might be NP-hard, and it turns out to be again the case:

Theorem 219. *Checking weak solvability of interval linear inequalities is NP-hard.*

Proof: Given a square matrix A, the system

$$-e \le Ax \le e, \tag{5.82}$$

$$e^T|x| \geq 1 \tag{5.83}$$

can be rewritten equivalently as

$$\begin{pmatrix} A \\ -A \\ 0^T \end{pmatrix} x - \begin{pmatrix} 0 \\ 0 \\ e^T \end{pmatrix} |x| \leq \begin{pmatrix} e \\ e \\ -1 \end{pmatrix},$$

which is just the Gerlach inequality (5.80) for the system

$$\mathbf{A}x \leq \mathbf{b}, \tag{5.84}$$

where

$$A_c = \begin{pmatrix} A \\ -A \\ 0^T \end{pmatrix}, \quad \Delta = \begin{pmatrix} 0 \\ 0 \\ e^T \end{pmatrix}, \quad \underline{b} = \overline{b} = \begin{pmatrix} e \\ e \\ -1 \end{pmatrix}. \tag{5.85}$$

Hence the system (5.82), (5.83) has a solution if and only if the system of interval linear inequalities (5.84), (5.85) is weakly solvable. Thus the NP-complete problem of checking solvability of (5.82), (5.83) (Theorem 22) can be reduced in polynomial time to the problem of checking weak solvability of interval linear inequalities, which is then NP-hard. □

5.2.10 Weak feasibility of inequalities

Weak feasibility of inequalities was defined in Section 5.2.3 as existence of a nonnegative weak solution. For nonnegative x we can replace the term $|x|$ in the Gerlach inequality simply by x, thereby obtaining this simple characterization:

Theorem 220. *A system* $\mathbf{A}x \leq \mathbf{b}$ *is weakly feasible if and only if the system*

$$\underline{A}x \leq \overline{b} \tag{5.86}$$

is feasible.

Proof: If $x \geq 0$ satisfies $Ax \leq b$ for some $A \in \mathbf{A}$ and $b \in \mathbf{b}$, then

$$\underline{A}x \leq Ax \leq b \leq \overline{b}$$

and x is a feasible solution to (5.86). Conversely, feasibility of (5.86) obviously implies weak feasibility of $\mathbf{A}x \leq \mathbf{b}$. □

Since feasibility of only one system of linear inequalities is to be checked, the problem is solvable in polynomial time (see the last paragraph of Section 5.2.2).

5.2.11 Strong solvability of inequalities

By definition, a system $\mathbf{A}x \leq \mathbf{b}$ is strongly solvable if each system $Ax \leq b$ with $A \in \mathbf{A}$, $b \in \mathbf{b}$ is solvable. Since the problem of checking strong solvability of interval linear equations is NP-hard (Theorem 210), one might expect the same to be the case for interval linear inequalities. But this analogy is no more true, and we have this rather surprising result:

Theorem 221. *A system $\mathbf{A}x \leq \mathbf{b}$ is strongly solvable if and only if the system*

$$\overline{A}x^1 - \underline{A}x^2 \leq \underline{b} \tag{5.87}$$

is feasible.

Proof: "Only if": Assume to the contrary that the system (5.87) is not feasible; then neither is the system

$$\overline{A}x^1 - \underline{A}x^2 + x^3 = \underline{b},$$

and Farkas theorem implies existence of a vector $p \in \mathbb{R}^m$ satisfying

$$\overline{A}^T p \geq 0, \tag{5.88}$$
$$\underline{A}^T p \leq 0, \tag{5.89}$$
$$p \geq 0, \tag{5.90}$$
$$\underline{b}^T p < 0. \tag{5.91}$$

Then (5.88) and (5.89) give

$$-\Delta^T p \leq -A_c^T p \leq \Delta^T p,$$

hence

$$|A_c^T p| \leq \Delta^T p = \Delta^T |p|$$

because of (5.90), and the Oettli-Prager theorem as applied to the system

$$[A_c^T - \Delta^T, A_c^T + \Delta^T]x = [0, 0]$$

implies existence of a matrix $A \in \mathbf{A}$ satisfying

$$A^T p = 0,$$

which together with (5.90) and (5.91) shows in the light of Theorem 201 that the system

$$Ax \leq \underline{b}$$

does not have a solution, a contradiction.

"If": Let $x^1 \geq 0$, $x^2 \geq 0$ solve (5.87). Then for each $A \in \mathbf{A}$ and each $b \in \mathbf{b}$ we have

$$A(x^1 - x^2) \leq \overline{A}x^1 - \underline{A}x^2 \leq \underline{b} \leq b,$$

so that $x^1 - x^2$ solves $Ax \leq b$. Hence $\mathbf{A}x \leq \mathbf{b}$ is strongly solvable; even more, all the systems $Ax \leq b$, $A \in \mathbf{A}$, $b \in \mathbf{b}$ share a common solution $x^1 - x^2$. □

Hence checking strong solvability of inequalities can be performed in polynomial time. Let us call a vector x satisfying $Ax \leq b$ for each $A \in \mathbf{A}$, $b \in \mathbf{b}$ a *strong solution* of $\mathbf{A}x \leq \mathbf{b}$. We have simultaneously proved the following result:

Theorem 222. *If a system $\mathbf{A}x \leq \mathbf{b}$ is strongly solvable, then it has a strong solution.*

In other words, if each system $Ax \leq b$ with data satisfying $A \in \mathbf{A}$, $b \in \mathbf{b}$ has a solution of its own (depending on A and b, say $x(A, b)$), then all these systems share a common solution. This fact is certainly not obvious.

We have this characterization of strong solutions:

Theorem 223. *The following assertions are equivalent:*

(i) x is a strong solution of $\mathbf{A}x \leq \mathbf{b}$,

(ii) x satisfies

$$A_c x - b_c \leq -\Delta|x| - \delta, \tag{5.92}$$

(iii) $x = x^1 - x^2$, where x^1, x^2 satisfy

$$\overline{A}x^1 - \underline{A}x^2 \leq \underline{b}, \tag{5.93}$$

$$x^1 \geq 0,\ x^2 \geq 0. \tag{5.94}$$

Proof: We shall prove (i)⇒(ii)⇒(iii)⇒(i).

(i)⇒(ii): If $Ax \leq b$ for each $A \in \mathbf{A}$, $b \in \mathbf{b}$, then also $A_{-ez}x \leq \underline{b}$, where $z = \operatorname{sgn} x$, hence

$$A_c x + \Delta|x| = (A_c + \Delta T_z)x = A_{-ez}x \leq \underline{b} = b_c - \delta,$$

which implies (5.92).

(ii)⇒(iii): If x satisfies (5.92), then for $x^1 = x^+ = \max\{x, 0\}$, $x^2 = x^- = \max\{-x, 0\}$ we have $x^1 \geq 0$, $x^2 \geq 0$ and

$$\overline{A}x^1 - \underline{A}x^2 = A_c(x^1 - x^2) + \Delta(x^1 + x^2) = A_c x + \Delta|x| \leq b_c - \delta = \underline{b},$$

hence x^1, x^2 solve (5.93), (5.94) and $x = x^1 - x^2$.

(iii)⇒(i) was proved in the "if" part of the proof of Theorem 221. □

We can sum up these results in the form of a simple algorithm:

```
function [x, flag] = strongsol(A, b)
solve the system Āx¹ − A̲x² ≤ b̲, x¹ ≥ 0, x² ≥ 0;
if it has a solution x¹, x²
    x = x¹ − x²; flag = 'strong solution found'; return
else
    x = []; flag = 'not strongly solvable';
end
```

Figure 5.1: An algorithm for checking strong solvability of $\mathbf{A}x \leq \mathbf{b}$.

5.2.12 Strong feasibility of inequalities

Finally, checking strong feasibility of inequalities is easy to characterize and can be done in polynomial time:

Theorem 224. *A system $\mathbf{A}x \leq \mathbf{b}$ is strongly feasible if and only if the system*

$$\overline{A}x \leq \underline{b} \tag{5.95}$$

is feasible.

Proof: If $\mathbf{A}x \leq \mathbf{b}$ is strongly feasible, then (5.95) is feasible. Conversely, if (5.95) has a solution $x \geq 0$, then for each $A \in \mathbf{A}$, $b \in \mathbf{b}$ we have

$$Ax \leq \overline{A}x \leq \underline{b} \leq b,$$

hence $\mathbf{A}x \leq \mathbf{b}$ is strongly feasible. □

5.2.13 Farkas-type theorems for inequalities

Theorem 225. *A system of interval linear inequalities* $\mathbf{A}x \leq \mathbf{b}$ *is*

- *weakly solvable if and only if*

$$(\exists z \in Y_n)(\forall p \geq 0)(T_z A_c^T p \geq \Delta^T p \Rightarrow (b_c + \delta)^T p \geq 0),$$

- *weakly feasible if and only if*

$$(\forall p \geq 0)((A_c - \Delta)^T p \geq 0 \Rightarrow (b_c + \delta)^T p \geq 0),$$

- *strongly solvable if and only if*

$$(\forall p \geq 0)(|A_c^T p| \leq \Delta^T p \Rightarrow (b_c - \delta)^T p \geq 0),$$

- *strongly feasible if and only if*

$$(\forall p \geq 0)((A_c + \Delta)^T p \geq 0 \Rightarrow (b_c - \delta)^T p \geq 0).$$

Proof: Unpublished. □

5.2.14 Summary: Complexity results

We can now summarize the results of the previous eight sections in the form of a table.

system of	equations	weakly	solvable	NP-hard
			feasible	polynomial-time
		strongly	solvable	NP-hard
			feasible	NP-hard
	inequalities	weakly	solvable	NP-hard
			feasible	polynomial-time
		strongly	solvable	polynomial-time
			feasible	polynomial-time

We can draw several conclusions from it. For interval problems, on the average:

(i) properties of equations are more difficult to check than those of inequalities,
(ii) checking solvability is more difficult than checking feasibility,
(iii) there is no such distinction between weak and strong properties.

5.3 Radii of solvability and feasibility

5.3.1 Definitions

Definition. For a system $Ax = b$ we introduce:

$$\begin{aligned}
r_{se}(A,b) &= \inf\{\, \|(A,b)-(A',b')\|_{1,\infty};\ A'x=b' \text{ is unsolvable}\,\},\\
r_{ue}(A,b) &= \inf\{\, \|(A,b)-(A',b')\|_{1,\infty};\ A'x=b' \text{ is solvable}\,\},\\
r_{fe}(A,b) &= \inf\{\, \|(A,b)-(A',b')\|_{1,\infty};\ A'x=b' \text{ is infeasible}\,\},\\
r_{ie}(A,b) &= \inf\{\, \|(A,b)-(A',b')\|_{1,\infty};\ A'x=b' \text{ is feasible}\,\}.
\end{aligned}$$

Definition. For a system $Ax \le b$ we introduce:

$$\begin{aligned}
r_{si}(A,b) &= \inf\{\, \|(A,b)-(A',b')\|_{1,\infty};\ A'x\le b' \text{ is unsolvable}\,\},\\
r_{ui}(A,b) &= \inf\{\, \|(A,b)-(A',b')\|_{1,\infty};\ A'x\le b' \text{ is solvable}\,\},\\
r_{fi}(A,b) &= \inf\{\, \|(A,b)-(A',b')\|_{1,\infty};\ A'x\le b' \text{ is infeasible}\,\},\\
r_{ii}(A,b) &= \inf\{\, \|(A,b)-(A',b')\|_{1,\infty};\ A'x\le b' \text{ is feasible}\,\}.
\end{aligned}$$

Note. $\|A\|_{1,\infty} = \max_{ij} |a_{ij}|$.

5.3.2 Radii of (un)solvability and (in)feasibility for equations

Theorem 226. *For each $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ there holds:*

$$\begin{aligned}
r_{se}(A,b) &= \min_{\|p\|_1=1} \max\{\, \|A^T p\|_\infty,\, |b^T p|\,\},\\
r_{ue}(A,b) &= \min_{x} \frac{\|Ax-b\|_\infty}{\|x\|_1+1},\\
r_{fe}(A,b) &= \min_{\|p\|_1=1} \max\{\,0,\, \max_i(-A^T p)_i,\, b^T p\,\},\\
r_{ie}(A,b) &= \min_{x\ge 0} \frac{\|Ax-b\|_\infty}{\|x\|_1+1}.
\end{aligned}$$

Proof: Unpublished. □

5.3.3 Radii of (un)solvability and (in)feasibility for inequalities

Theorem 227. *For each $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ there holds:*

$$
\begin{aligned}
r_{si}(A,b) &= \min_{\substack{\|p\|_1=1\\ p\geq 0}} \max\{\|A^T p\|_\infty, b^T p\},\\
r_{ui}(A,b) &= \min_{x} \frac{\max\{0, \max_i (Ax-b)_i\}}{\|x\|_1+1},\\
r_{fi}(A,b) &= \min_{\substack{\|p\|_1=1\\ p\geq 0}} \max\{0, \max_i (-A^T p)_i, b^T p\},\\
r_{ii}(A,b) &= \min_{x\geq 0} \frac{\max\{0, \max_i (Ax-b)_i\}}{\|x\|_1+1}.
\end{aligned}
$$

Proof: Unpublished. □

5.4 Special types of solutions

5.4.1 (Z,z)-solutions: A generalization of the Oettli-Prager theorem

In 1995 S. P. Shary proposed a new unifying view of different concepts of solutions of interval linear equations by introducing quantifications over interval coefficients. His definition is reformulated here in order to make it, as well as the following result, as simple as possible.

Definition. Let $|Z| = E \in \mathbb{R}^{m\times n}$ and $|z| = e \in \mathbb{R}^m$. A vector $x \in \mathbb{R}^n$ is said to be a (Z,z)-solution of a system $\mathbf{A}x = \mathbf{b}$ if for each $A_{ij} \in [\underline{A}_{ij}, \overline{A}_{ij}]$ with $Z_{ij} = -1$ and for each $b_i \in [\underline{b}_i, \overline{b}_i]$ with $z_i = -1$ there exist $A_{ij} \in [\underline{A}_{ij}, \overline{A}_{ij}]$ with $Z_{ij} = 1$ and $b_i \in [\underline{b}_i, \overline{b}_i]$ with $z_i = 1$ such that $Ax = b$ holds[1].

Despite the complexity of this definition, it turns out that description of (Z,z)-solutions becomes wonderfully simple as soon as the Hadamard product is employed. The following theorem constitutes a generalization of the Oettli-Prager theorem as well as of several our previous results.

Theorem 228. (Shary-Lakeyev-Rohn) *A vector $x \in \mathbb{R}^n$ is a (Z,z)-solution of $\mathbf{A}x = \mathbf{b}$ if and only if it satisfies*

$$|A_c x - b_c| \le (Z \circ \Delta)|x| + z \circ \delta. \tag{5.96}$$

Proof: Given Z and z, first define interval matrices $\mathbf{A}_1$, $\mathbf{A}_2$ and interval vectors $\mathbf{b}_1$, $\mathbf{b}_2$ by

$$\begin{aligned}
\mathbf{A}_1 &= \{\tfrac{1}{2}(E-Z)\circ A;\ A \in \mathbf{A}\} = [A'_c - \Delta', A'_c + \Delta'],\\
\mathbf{A}_2 &= \{\tfrac{1}{2}(E+Z)\circ A;\ A \in \mathbf{A}\} = [A''_c - \Delta'', A''_c + \Delta''],\\
\mathbf{b}_1 &= \{\tfrac{1}{2}(e-z)\circ b;\ b \in \mathbf{b}\} = [b'_c - \delta', b'_c + \delta'],\\
\mathbf{b}_2 &= \{\tfrac{1}{2}(e+z)\circ b;\ b \in \mathbf{b}\} = [b''_c - \delta'', b''_c + \delta''],
\end{aligned}$$

where "$\circ$" denotes the Hadamard product. As we can see, $\mathbf{A}_1$ is obtained from $\mathbf{A}$ by zeroing the ijth interval coefficients with $Z_{ij} = 1$, $\mathbf{A}_2$ by zeroing those with $Z_{ij} = -1$, and an analogue holds for $\mathbf{b}_1$, $\mathbf{b}_2$. Then x is a (Z,z)-solution if and only if for each $A_1 \in \mathbf{A}_1, b_1 \in \mathbf{b}_1$ the equation

$$(A_1 + A_2)b = b_1 + b_2,$$

i.e., the equation

$$A_1 x - b_1 = b_2 - A_2 x,$$

is satisfied for some $A_2 \in \mathbf{A}_2$, $b_2 \in \mathbf{b}_2$, which is equivalent to

$$\{A_1 x - b_1;\ A_1 \in \mathbf{A}_1, b_1 \in \mathbf{b}_1\} \subseteq \{b_2 - A_2 x;\ A_2 \in \mathbf{A}_2, , b_2 \in \mathbf{b}_2\}. \tag{5.97}$$

[1] Thus "-1" corresponds to "$\forall$" and "1" to "$\exists$". It could be argued that the reverse order would be more natural, but we would have to pay for it by introducing minus signs into the main formula (5.96).

But according to Proposition 229,

$$\{A_1x - b_1;\ A_1 \in \mathbf{A}_1, b_1 \in \mathbf{b}_1\} = [A'_c x - \Delta'|x| - b'_c - \delta', A'_c x + \Delta'|x| - b'_c + \delta']$$

and

$$\{b_2 - A_2x;\ b_2 \in \mathbf{b}_2, A_2 \in \mathbf{A}_2\} = [-A''_c x - \Delta''|x| + b''_c - \delta'', -A''_c x + \Delta''|x| + b''_c + \delta''],$$

hence the inclusion (5.97) is equivalent to

$$-(\Delta'' - \Delta')|x| - (\delta'' - \delta') \leq (A'_c + A''_c)x - (b'_c + b''_c) \leq (\Delta'' - \Delta')|x| + (\delta'' - \delta'),$$

which gives

$$|(A'_c + A''_c)x - (b'_c + b''_c)| \leq (\Delta'' - \Delta')|x| + (\delta'' - \delta'). \tag{5.98}$$

Now, taking into account that $A'_c + A''_c = A_c$, $b'_c + b''_c = b_c$, $\Delta'' - \Delta' = Z \circ \Delta$, and $\delta'' - \delta' = z \circ \delta$, we obtain (5.96). □

In this way, the previously defined types of solutions become special cases of (Z, z)-solutions, and their descriptions turn out to be special cases of Theorem 228. So we obtain

- weak solutions for $Z = E$, $z = e$ (Theorem 204),
- strong solutions for $Z = -E$, $z = -e$ (Theorem 211),
- tolerance solutions for $Z = -E$, $z = e$ (Theorem 230, (ii)),
- control solutions for $Z = E$, $z = -e$ (Theorem 231, (ii)).

This shows that Theorem 228 (though little known so far) indeed offers a unified view of different types of solutions of interval linear equations. It could also be easily reformulated for interval linear inequalities, but we refrain from it here.

5.4.2 Tolerance solutions

So far we have investigated mainly decision problems and in frame of it four types of solutions (weak and strong solutions of both equations and inequalities) were introduced as auxiliary tools only. In this and in the next two sections we shall define additional three types of solutions motivated by some practical considerations.

In the present section we shall study tolerance solutions. A vector $x \in \mathbb{R}^n$ is said to be a *tolerance solution* of $\mathbf{A}x = \mathbf{b}$ if it satisfies $Ax \in \mathbf{b}$ for each $A \in \mathbf{A}$. The name of this type of solution reflects the fact that vector Ax stays within the prescribed tolerance $[\underline{b}, \overline{b}]$ independently of the choice of $A \in \mathbf{A}$. Original motivations for introducing and studying tolerance solutions came from the problem of crane construction (Nuding and Wilhelm [56]) and from the problem of input-output planning with inexact data [66].

The definition can also be recast by saying that x shall satisfy

$$\{Ax;\ A \in \mathbf{A}\} \subseteq \mathbf{b}. \tag{5.99}$$

We start therefore with a description of the left-hand-side set in (5.99).

Proposition 229 *Let $\mathbf{A}$ be an $m \times n$ interval matrix and let $x \in \mathbb{R}^n$. Then there holds*

$$\{Ax;\ A \in \mathbf{A}\} = [A_c x - \Delta|x|, A_c x + \Delta|x|]. \tag{5.100}$$

Proof: If $b \in \{Ax;\ A \in \mathbf{A}\}$, then $Ax = b$ for some $A \in \mathbf{A}$, hence x is a weak solution of

$$\mathbf{A}x = [b, b] \tag{5.101}$$

and by the Oettli-Prager theorem it satisfies

$$|A_c x - b| \le \Delta|x|, \tag{5.102}$$

hence

$$-\Delta|x| \le A_c x - b \le \Delta|x| \tag{5.103}$$

and

$$A_c x - \Delta|x| \le b \le A_c x + \Delta|x|. \tag{5.104}$$

We have proved that $\{Ax;\ A \in \mathbf{A}\} \subseteq [A_c x - \Delta|x|, A_c x + \Delta|x|]$. Conversely, if $b \in [A_c x - \Delta|x|, A_c x + \Delta|x|]$, then b satisfies (5.104), (5.103) and (5.102), hence x is a weak solution of (5.101) which gives that $b \in \{Ax;\ A \in \mathbf{A}\}$. This proves the converse inclusion, hence (5.100) holds. □

With the help of this auxiliary result we can give two equivalent descriptions of tolerance solutions:

Theorem 230. *The following assertions are equivalent:*

(i) x is a tolerance solution of $\mathbf{A}x = \mathbf{b}$,

(ii) x satisfies

$$|A_c x - b_c| \le -\Delta|x| + \delta, \tag{5.105}$$

(iii) $x = x_1 - x_2$, where x_1, x_2 satisfy

$$\overline{A}x_1 - \underline{A}x_2 \le \overline{b}, \tag{5.106}$$

$$\underline{A}x_1 - \overline{A}x_2 \ge \underline{b}, \tag{5.107}$$

$$x_1 \ge 0,\ x_2 \ge 0. \tag{5.108}$$

Proof: We prove (i)⇒(ii)⇒(iii)⇒(i).

(i)⇒(ii): According to Proposition 229,

$$\{Ax;\ A \in \mathbf{A}\} = [A_c x - \Delta|x|, A_c x + \Delta|x|].$$

Hence, if x is a tolerance solution, then

$$[A_c x - \Delta|x|, A_c x + \Delta|x|] \subseteq [b_c - \delta, b_c + \delta],$$

which implies

$$b_c - \delta \le A_c x - \Delta|x| \le A_c x + \Delta|x| \le b_c + \delta$$

and thus also

$$-(-\Delta|x| + \delta) \le A_c x - b_c \le -\Delta|x| + \delta, \tag{5.109}$$

which is (5.105).

(ii)⇒(iii): If x satisfies (5.105), then for $x_1 = x^+$, $x_2 = x^-$ we have $x = x_1 - x_2$, $|x| = x_1 + x_2$ and the inequalities (5.109) turn into

$$\Delta(x_1 + x_2) - \delta \le A_c(x_1 - x_2) - b_c \le -\Delta(x_1 + x_2) + \delta,$$

which gives (5.106), (5.107), and (5.108) is satisfied because $x^+ \ge 0$, $x^- \ge 0$.

(iii)⇒(i): If $x_1 \ge 0$, $x_2 \ge 0$ solve (5.106), (5.107), then for $x = x_1 - x_2$ and for each $A \in \mathbf{A}$ we have

$$Ax = A(x_1 - x_2) \le \overline{A}x_1 - \underline{A}x_2 \le \overline{b}$$

and

$$Ax = A(x_1 - x_2) \ge \underline{A}x_1 - \overline{A}x_2 \ge \underline{b}$$

which shows that $Ax \in \mathbf{b}$ for each $A \in \mathbf{A}$, hence x is a tolerance solution. □

There is a remarkable similarity between the inequality (5.105) and the Oettli-Prager inequality (5.27): both descriptions differ in the sign preceding the matrix Δ only. Yet this seemingly small difference has an astounding impact: while checking existence of solution of the Oettli-Prager inequality is NP-hard (Theorem 207), checking existence of a tolerance solution can be performed in polynomial time simply by checking solvability of the system (5.106)-(5.108). The description (iii) also shows that the set of tolerance solutions is a convex polyhedron, it allows to compute the range of components of tolerance solutions by solving the respective linear programming problems [74], etc.

5.4.3 Control solutions

A vector $x \in \mathbb{R}^n$ is called a *control solution* of $\mathbf{A}x = \mathbf{b}$ if for each $b \in \mathbf{b}$ there exists an $A \in \mathbf{A}$ such that $Ax = b$ holds; in other words, if

$$\mathbf{b} \subseteq \{Ax;\ A \in \mathbf{A}\}.$$

Control solutions were introduced by Shary [106] in 1992. The choice of the word "control" was probably motivated by the fact that each vector $b \in \mathbf{b}$ can be reached by Ax when properly controlling the coefficients of A within $\mathbf{A}$. We have this characterization:

Theorem 231. *The following assertions are equivalent:*

(i) x *is a control solution of* $\mathbf{A}x = \mathbf{b}$*,*

(ii) x *satisfies*

$$|A_c x - b_c| \leq \Delta|x| - \delta, \tag{5.110}$$

(iii) x *solves*

$$A_{ez}x \leq \underline{b}, \tag{5.111}$$

$$-A_{-ez}x \leq -\overline{b} \tag{5.112}$$

for some $z \in Y_n$*.*

Proof: We shall prove (i)⇒(ii)⇒(iii)⇒(i).

(i)⇒(ii): If x is a control solution, then by Proposition 229 it satisfies $[b_c - \delta, b_c + \delta] \subseteq \{Ax;\ A \in \mathbf{A}\} = [A_c x - \Delta|x|, A_c x + \Delta|x|]$, which implies

$$A_c x - \Delta|x| \leq b_c - \delta \leq b_c + \delta \leq A_c x + \Delta|x|$$

and

$$-(\Delta|x| - \delta) \leq A_c x - b_c \leq \Delta|x| - \delta, \tag{5.113}$$

hence

$$|A_c x - b_c| \leq \Delta|x| - \delta.$$

(ii)⇒(iii): If x satisfies (5.110), then (5.113) holds and with $z = \operatorname{sgn} x$ we can substitute $|x| = T_z x$ into (5.113) which leads to (5.111), (5.112).

(iii)⇒(i): If x solves (5.111), (5.112) for some $z \in Y_n$, then $|\Delta T_z x| \leq \Delta|x|$, hence

$$\begin{aligned} A_c x - \Delta|x| &\leq (A_c - \Delta T_z)x = A_{ez}x \leq \underline{b} \leq \overline{b} \leq A_{-ez}x \\ &= (A_c + \Delta T_z)x \leq A_c x + \Delta|x|, \end{aligned}$$

which implies

$$[\underline{b}, \overline{b}] \subseteq [A_c x - \Delta|x|, A_c x + \Delta|x|] = \{Ax;\ A \in \mathbf{A}\}$$

by Proposition 229, hence x is a control solution. □

Again, the inequality (5.110) differs from the Oettli-Prager inequality (5.27) in the sign preceding δ only. But this time the difference does not affect complexity of the problem:

Theorem 232. *Checking existence of control solutions is NP-hard.*

Proof: For a square matrix A, consider the system

$$-e \le Ax \le e, \tag{5.114}$$

$$e^T|x| \ge 1, \tag{5.115}$$

and the inequality

$$\left| \begin{pmatrix} A \\ 0^T \end{pmatrix} x - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right| \le \begin{pmatrix} ee^T \\ e^T \end{pmatrix} |x| - \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.116}$$

If x solves (5.114), (5.115), then it also solves (5.116). Conversely, if x solves (5.116), then $x \ne 0$ and $x' = \frac{x}{e^T|x|}$ solves (5.114), (5.115). Hence, the system (5.114), (5.115) has a solution if and only if the inequality (5.116) has a solution. But (5.116) is exactly the inequality (5.110) for the system of interval linear equations

$$\begin{aligned} [A - ee^T, A + ee^T]x &= [0, 0], & (5.117)\\ [-e^T, e^T]x &= [1, 1], & (5.118) \end{aligned}$$

which gives that (5.114), (5.115) has a solution if and only if (5.117), (5.118) has a control solution. Now an application of Theorem 22 concludes the proof. □

5.4.4 (Strictly) formal solutions

Definition. An interval vector $\mathbf{x}$ is called a formal solution of an interval linear system $\mathbf{A}x = \mathbf{b}$ if it satisfies $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$, where the matrix multiplication is performed in interval arithmetic.

Definition. An interval vector $\mathbf{x}$ is called a strictly formal solution of $\mathbf{A}x = \mathbf{b}$ if it is a formal solution of it and there exist $A', A'' \in \mathbf{A}$ and $x', x'' \in \mathbf{x}$ such that $A'x' = \underline{b}$ and $A''x'' = \overline{b}$ hold.

Theorem 233. *Let $\mathbf{A}x = \mathbf{b}$ have a strictly formal solution. Then the equations*

$$A_c x - \Delta|x| = \underline{b}, \tag{5.119}$$

$$A_c x + \Delta|x| = \overline{b} \tag{5.120}$$

have solutions and for each pair x^1, x^2 of solutions of (5.119), (5.120) the interval vector

$$\mathbf{x} = [\min\{x^1, x^2\}, \max\{x^1, x^2\}]$$

is a strictly formal solution of $\mathbf{A}x = \mathbf{b}$.

Proof: [84], p. 222. □

Theorem 234. *Let $\mathbf{A}$ be regular and let*

$$\mathbf{x}^* = [\min\{x^1, x^2\}, \max\{x^1, x^2\}],$$

where x^1 and x^2 are the unique solutions of the equations (5.119), (5.120) respectively (see Subsection ...). Then there holds: if $\mathbf{A}x = \mathbf{b}$ has a strictly formal solution, then $\mathbf{x}^*$ is a strictly formal solution.

Proof: [84], p. 222. □

5.4.5 Algebraic solutions

A vector $x \in \mathbb{R}^n$ is called an *algebraic solution* of $\mathbf{A}x = \mathbf{b}$ if it satisfies

$$\{Ax;\ A \in \mathbf{A}\} = \mathbf{b}. \tag{5.121}$$

Algebraic solutions were first introduced by Ratschek and Sauer in [61]. This type of solution is easy to characterize:

Theorem 235. *x is an algebraic solution of $\mathbf{A}x = \mathbf{b}$ if and only if it satisfies*

$$A_c x = b_c, \tag{5.122}$$
$$\Delta|x| = \delta. \tag{5.123}$$

Proof: By Proposition 229, (5.121) is equivalent to

$$[A_c x - \Delta|x|, A_c x + \Delta|x|] = [b_c - \delta, b_c + \delta], \tag{5.124}$$

which implies (5.122), (5.123). On the other hand, (5.122) and (5.123) imply (5.124) and thus also (5.121). □

It follows from Theorems 230 and 231, inequalities (5.105) and (5.110), that x is an algebraic solution of $\mathbf{A}x = \mathbf{b}$ if and only if it is both tolerance and control solution of it. If $m = n$ and A_c is nonsingular, then $\mathbf{A}x = \mathbf{b}$ has an algebraic solution if and only if the data satisfy

$$\Delta|A_c^{-1}b_c| = \delta, \tag{5.125}$$

in which case $x = A_c^{-1}b_c$ is the unique algebraic solution of it.

5.4.6 Summary: Solution types

We have introduced altogether eight types of solutions. We summarize the results in the following table which clearly illustrates the tiny differences in their descriptions.

Solution	Description	Reference
weak solution of $\mathbf{A}x = \mathbf{b}$	$\lvert A_c x - b_c\rvert \leq \Delta\lvert x\rvert + \delta$	(5.27)
strong solution of $\mathbf{A}x = \mathbf{b}$	$A_c x - b_c = \Delta\lvert x\rvert = \delta = 0$	(5.62), (5.63)
weak solution of $\mathbf{A}x \leq \mathbf{b}$	$A_c x - b_c \leq \Delta\lvert x\rvert + \delta$	(5.80)
strong solution of $\mathbf{A}x \leq \mathbf{b}$	$A_c x - b_c \leq -\Delta\lvert x\rvert - \delta$	(5.92)
tolerance solution	$\lvert A_c x - b_c\rvert \leq -\Delta\lvert x\rvert + \delta$	(5.105)
control solution	$\lvert A_c x - b_c\rvert \leq \Delta\lvert x\rvert - \delta$	(5.110)
algebraic solution	$A_c x - b_c = \Delta\lvert x\rvert - \delta = 0$	(5.122), (5.123)
(Z, z)-solution	$\lvert A_c x - b_c\rvert \leq (Z \circ \Delta)\lvert x\rvert + z \circ \delta$	(5.96)

5.5 Notes and references

In this section we give some additional notes and references to the material contained in this chapter.

Section 5.2.1. We use standard linear algebraic notations except for Y_m, T_y and $\operatorname{sgn} x$ (introduced in [81]).

Section 5.2.2. The word "feasibility", which is a one-word substitute for nonnegative solvability, was inspired by linear programming terminology. Theorem 199, also known as Farkas lemma, was proved by Farkas [9] in 1902. It is an important theoretical result (as evidenced throughout this chapter), but it does not give a constructive way of checking feasibility which must be done by another means (usually by a linear programming technique).

Section 3.2.2. Matrices A_{yz} and vectors b_y were introduced in [81]. Importance of the finite set of matrices A_{yz} becomes more apparent with problems involving square interval matrices only (as regularity, positive definiteness etc.). For example, an interval matrix $\mathbf{A}$ is regular (see Section 3.3.2) if and only if $\det A_{yz}$ is of the same sign for each $z, y \in Y_n$ (Baumann [4]); for further results of this type see the monograph by Kreinovich, Lakeyev, Rohn and Kahl [38], Chapters 21 and 22. As we have seen, in context of interval systems typically only matrices of the form A_{ye} or A_{ez} arise.

Section 5.2.3. The definition of an interval linear system $\mathbf{A}x = \mathbf{b}$ as a family of systems $Ax = b$, $A \in \mathbf{A}$, $b \in \mathbf{b}$ makes it possible to define various types of solutions. The notion of strong feasibility of interval linear equations was introduced in [68], and weak solvability as a counterpart of strong solvability was first studied by Rohn and Kreslová in [101]. Formulation and study of the complete set of the eight decision problems is new and forms the bulk of this chapter.

Section 5.2.4. The Oettli-Prager theorem is formulated here in the form (5.27) which has become standard, although not explicitly present in the original paper [57] where the authors preferred an entrywise formulation. The theorem is now considered a basic tool for both backward error analysis (Golub and van Loan [15], Higham [20]) and interval analysis (Neumaier [55]) of systems of linear equations. Another form of Proposition 205 (perhaps more attractive, but less useful) is described in [72], Theorem 1.2. NP-hardness of checking weak solvability of equations was proved by Lakeyev and Noskov [41] (preliminary announcement without proof in [40]) by another means. The proof given here employs polynomial reduction of our standard problem of Theorem 22 to the current problem, an approach adhered to throughout the chapter.

Section 5.2.5. Theorem 208 is a simple consequence of the Oettli-Prager theorem. It was rediscovered independently in [65].

Section 5.2.6. The proof of Theorem 209 is not straightforward and so was the history of it. The "if" part was formulated and proved in technical reports [71], [70] in 1984, but the author refrained from further journal publication because he considered the sufficient condition too strong. Only in 1996 he discovered by chance that it was also necessary (paradoxically, it was the easier part of the proof), which gave rise to Theorem 209 published in [99]. The second part of the proof of the "if" part relies in fact on a new existence theorem for systems of linear equations which was published in [83] (existence proof, as given here) and in [85] (constructive proof). NP-hardness of checking strong solvability (Theorem 210) is an easy consequence of the same complexity result for the problem of checking regularity of interval matrices (Theorem ...), but because of the layout of this chapter it had to be proved independently.

Section 5.2.7. Characterization of strong feasibility of equations (Theorem 212) was published in [68] as a part of study of the interval linear programming problem. Many unsuccessful attempts by the author through the following years to find a characterization of strong feasibility that would not be inherently exponential finally led to the NP-hardness conjecture and to the proof of it in [96] (part 2 of the proof).

Section 5.2.9. Gerlach [14] initiated study of systems of interval linear inequalities by proving Theorem 217 as a follow-up of the Oettli-Prager theorem. NP-hardness of checking weak solvability of inequalities was proved in technical report [90] and has not been published in journal form.

Section 5.2.10. The result of Theorem 220 is obvious and is included here for completeness.

Section 5.2.11. Both Theorems 221 and 222 are due to Rohn and Kreslová [101]. The contrast between the complexity results for strong solvability of interval linear equations (Theorem 210) and inequalities (Theorem 221) is striking and reveals that classical solvability-preserving reductions between linear equations and linear inequalities are no longer in force when inexact data are present. In fact, a system of linear equations $Ax = b$ can be equivalently written as a system if linear inequalities $Ax \leq b$, $-Ax \leq -b$ and solved as such. But in case of interval data, the sets of weak solutions of $\mathbf{A}x = \mathbf{b}$ and of $\mathbf{A}x \leq \mathbf{b}$, $-\mathbf{A}x \leq -\mathbf{b}$ are generally not identical since the latter family contains systems of inequalities of type $Ax \leq b$, $-\tilde{A}x \leq -\tilde{b}$ ($A, \tilde{A} \in \mathbf{A}$, $b, \tilde{b} \in \mathbf{b}$) that may possess solutions which do not satisfy $Ax = b$ for any $A \in \mathbf{A}$, $b \in \mathbf{b}$. Existence of strong solutions in case of strong solvability (Theorem 222) is a nontrivial fact which can be expected to find some applications, although none of them have been known to date.

Section 5.2.12. Theorem 224 is again obvious.

Section 5.4.1. Shary presented his idea of (Z, z)-solutions, which he called "∀∃-solutions", at a conference in Wuppertal in 1995 and published it in [110]. His formulation of Theorem 228 contained, however, interval arithmetic operations. A proof not using these operations and based on the Oettli-Prager theorem was given in this author's letter to Shary and Lakeyev [64]. The final step towards utmost simplicity by employing the Hadamard product was done by Lakeyev in [39].

Section 5.4.2. Introduction of the notion of tolerance solutions was motivated by considerations concerning crane construction (Nuding and Wilhelm [56]) and input-output planning with inexact data of the socialist economy of former Czechoslovakia

[66]. Descriptions (ii), (iii) of tolerance solutions in Theorem 230 were proved in [74]. Tolerance solutions were studied since by Neumaier [54], Deif [8], Kelling and Oelschlägel [33], Kelling [31], [32], Shaydurov and Shary [113], Shary [105], [107], [108], [109] and Lakeyev and Noskov [41].

Section 5.4.3. Control solutions were introduced by Shary [106] and further studied by him in [109], [111]. The description (5.110) in Theorem 231 is due to Lakeyev and Noskov [41] who in the same paper also proved NP-hardness of checking existence of control solutions, as well as of algebraic solutions. For other possible types of solutions see the survey paper by Shary [112].

Section 5.4.5. Algebraic solutions were introduced by Ratschek and Sauer [61], although for the case $m = 1$ only. The condition (5.125) was proved in [72]. The topic makes more sense when the problem is formulated as $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$, where $\mathbf{x}$ is an interval vector and multiplication is performed in interval arithmetic. A solution of this problem in full generality is not known so far; for a partial solution see [84].

Chapter 6

Interval linear programming

6.1 Introduction

6.2 Summary of facts: Duality in linear programming

We shall now switch to optimization problems. Given $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$, the problem

$$\text{minimize } c^T x \tag{6.1}$$

subject to (s.t.)

$$Ax = b, \ x \geq 0 \tag{6.2}$$

is called a linear programming problem, or simply a linear program. We shall write the problem (6.1), (6.2) briefly as

$$\text{Min}\{c^T x; \ Ax = b, x \geq 0\} \tag{6.3}$$

(notice the use of the upper case in "Min" to denote a problem in contrast to "min" which denotes minimum when applicable). A vector x satisfying (6.2) is called a *feasible solution* of (6.3). A problem (6.3) having a feasible solution is said to be *feasible*, and *infeasible* in the opposite case. Hence, the problem (6.3) is feasible if and only if the system $Ax = b$ is feasible in terminology of Section 5.2.2.

For a given linear program (6.3) we introduce the value

$$f(A, b, c) = \inf\{c^T x; \ Ax = b, x \geq 0\} \tag{6.4}$$

and we shall call it the *optimal value* of (6.3)[1]. The optimal value can be computed by any linear programming technique, as e.g. the simplex method by Dantzig [7], or the polynomial-time algorithms by Khachiyan [34], Karmarkar [30] and others (see Padberg [58]). Exactly one of the following three cases may occur:

(a) If $f(A, b, c)$ is finite, then, as proved in part (a) of the proof of Theorem 236 below, the infimum in (6.4) is attained as minimum, so that there exists a feasible solution x^* of (6.3) satisfying $f(A, b, c) = c^T x^*$. Such an x^* is called an *optimal solution* of (6.3). In this case we say that the problem (6.3) has an optimal solution.

(b) If $f(A, b, c) = -\infty$, then the set of feasible solutions of (6.3) contains a half-line along which the value of $c^T x$ tends to $-\infty$ (see part (b) of the proof of Theorem 236); in this case we call the problem (6.3) *unbounded*.

(c) If $f(A, b, c) = \infty$, then the set of feasible solutions of (6.3) is empty, hence the problem (6.3) is infeasible.

Given a problem (6.3) (called "*primal*" in this context), we can formulate its *dual problem* as

$$\text{maximize } b^T p \tag{6.5}$$

s.t.

$$A^T p \leq c, \tag{6.6}$$

[1]in linear programming only finite value of $f(A, b, c)$ is accepted as the optimal value; we use this formulation for the sake of utmost generality of later results.

or briefly

$$\text{Max}\{b^T p;\ A^T p \leq c\} \tag{6.7}$$

(notice that the nonnegativity constraint is missing in (6.6)). The dual problem is called *solvable* if the system $A^T p \leq c$ is solvable[2], and *unsolvable* in the opposite case. In analogy with the primal problem, we introduce for the dual problem the value

$$g(A, b, c) = \sup\{b^T p;\ A^T p \leq c\}.$$

A solution p^* of $A^T p \leq c$ is called an optimal solution of (6.7) if $g(A, b, c) = b^T p^*$; if $g(A, b, c) = -\infty$, then the problem (6.7) is unsolvable, and if $g(A, b, c) = \infty$, then it is called *unbounded*. The primal and the dual problem are connected by the following important result whose proof is included here for the sake of completeness:

Theorem 236. *[Duality theorem] If $f(A, b, c) < \infty$ or $g(A, b, c) > -\infty$, then*

$$f(A, b, c) = g(A, b, c). \tag{6.8}$$

Comment. The formal equality (6.8) covers three qualitative issues: (i) if one of the problems (6.3), (6.7) has an optimal solution, then so does the second one and the optimal values of both problems are equal, (ii) if the primal problem (6.3) is unbounded, then the dual problem (6.7) is unsolvable, (iii) if the dual problem (6.7) is unbounded, then the primal problem (6.3) is infeasible. If the assumptions of the theorem are not met, then (6.3) is infeasible and (6.7) is unsolvable, in which case nothing more can be said.

Proof: Three possibilities may occur under our assumptions: (a) $f(A, b, c) < \infty$ and $g(A, b, c) > -\infty$, (b) $f(A, b, c) < \infty$ and $g(A, b, c) = -\infty$, (c) $f(A, b, c) = \infty$ and $g(A, b, c) > -\infty$.

(a) Let $f(A, b, c) < \infty$ and $g(A, b, c) > -\infty$. We shall first prove that the system

$$Ax = b,\ x \geq 0, \tag{6.9}$$

$$A^T p \leq c, \tag{6.10}$$

$$c^T x \leq b^T p \tag{6.11}$$

has a solution. Introducing artificial variables, we can write it in the form

$$\begin{pmatrix} A & 0 & 0 & 0 & 0 \\ 0 & A^T & -A^T & I & 0 \\ c^T & -b^T & b^T & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ p_1 \\ p_2 \\ p_3 \\ \xi \end{pmatrix} = \begin{pmatrix} b \\ c \\ 0 \end{pmatrix}, \tag{6.12}$$

[2]at this point we must depart from traditional linear programming terminology where (6.7) is called feasible if $A^T p \leq c$ has a solution; but for us feasibility means nonnegative solvability, so that we cannot use this term here and we must stick to terminology introduced in Section 5.2.2.

where all the variables are nonnegative. Now we can apply Farkas theorem which says that (6.12) has a nonnegative solution if and only if for each t_1, t_2 and τ,

$$\begin{pmatrix} A^T & 0 & c \\ 0 & A & -b \\ 0 & -A & b \\ 0 & I & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ \tau \end{pmatrix} \geq 0 \quad \text{implies} \quad \begin{pmatrix} b \\ c \\ 0 \end{pmatrix}^T \begin{pmatrix} t_1 \\ t_2 \\ \tau \end{pmatrix} \geq 0, \tag{6.13}$$

which means that

$$A^T t_1 + \tau c \geq 0, \tag{6.14}$$

$$At_2 = \tau b, \tag{6.15}$$

$$t_2 \geq 0,\ \tau \geq 0 \tag{6.16}$$

should imply

$$b^T t_1 + c^T t_2 \geq 0. \tag{6.17}$$

To prove the last statement, in view of nonnegativity of τ we can consider two cases. If $\tau > 0$, then we have $b = \frac{1}{\tau} A t_2$, hence

$$b^T t_1 + c^T t_2 = \tfrac{1}{\tau} t_2^T A^T t_1 + c^T t_2 = \tfrac{1}{\tau} t_2^T (A^T t_1 + \tau c) \geq 0$$

because of (6.14), (6.16), which is (6.17). If $\tau = 0$, then (6.14)-(6.16) turn into

$$A^T t_1 \geq 0, \tag{6.18}$$

$$At_2 = 0, \tag{6.19}$$

$$t_2 \geq 0. \tag{6.20}$$

Since $f(A, b, c) < \infty$, the system $Ax = b$ is feasible and (6.18) by Farkas theorem implies $b^T t_1 \geq 0$; since $g(A, b, c) > -\infty$, the system $A^T p \leq c$ is solvable and (6.19), (6.20) by Theorem 201 imply $c^T t_2 \geq 0$, hence (6.17) again holds. In this way we have proved the implication (6.13), which in turn guarantees existence of a solution x^*, p^* of the system (6.9)-(6.11). From (6.9), (6.10) we obtain

$$c^T x^* = x^{*T} c \geq x^{*T} A^T p^* = (Ax^*)^T p^* = b^T p^*,$$

which together with (6.11) gives $c^T x^* = b^T p^*$. Summing up, we have proved that there exist x^*, p^* satisfying

$$Ax^* = b,\ x^* \geq 0,$$
$$A^T p^* \leq c,$$
$$c^T x^* = b^T p^*.$$

Now, for each feasible solution x of the primal problem we have

$$c^T x = x^T c \geq x^T A^T p^* = (Ax)^T p^* = b^T p^* = c^T x^*,$$

which means that

$$c^T x^* = \min\{c^T x;\ Ax = b, x \geq 0\} = f(A,b,c),$$

and similarly for each solution p of the system of constraints $A^T p \leq c$ of the dual problem we have

$$b^T p = p^T b = p^T A x^* = (A^T p)^T x^* \leq c^T x^* = b^T p^*,$$

which gives that

$$b^T p^* = \max\{b^T p;\ A^T p \leq c\} = g(A,b,c),$$

and finally

$$f(A,b,c) = c^T x^* = b^T p^* = g(A,b,c),$$

which is (6.8).

(b) Let $f(A,b,c) < \infty$ and $g(A,b,c) = -\infty$. Then the primal problem has a feasible solution, say x_1, and the dual problem is unsolvable, so that the system $A^T p \leq c$ has no solution, hence according to Theorem 201 there exists an x_0 satisfying $Ax_0 = 0$, $x_0 \geq 0$ and $c^T x_0 < 0$. Then for each $\alpha \in \mathbb{R}$, $\alpha \geq 0$ we have $A(x_1 + \alpha x_0) = Ax_1 = b$ and $x_1 + \alpha x_0 \geq 0$, hence $x_1 + \alpha x_0$ is a feasible solution of the primal problem for each $\alpha \geq 0$ and

$$\lim_{\alpha\to\infty} c^T(x_1 + \alpha x_0) = \lim_{\alpha\to\infty} (c^T x_1 + \alpha c^T x_0) = -\infty$$

because of $c^T x_0 < 0$, hence

$$f(A,b,c) = \inf\{c^T x;\ Ax = b, x \geq 0\} = -\infty = g(A,b,c),$$

which is (6.8).

(c) Let $f(A,b,c) = \infty$ and $g(A,b,c) > -\infty$. Then the primal problem is infeasible and the system of constraints $A^T p \leq c$ of the dual problem has a solution, say p_1. Since the system $Ax = b$ is not feasible, according to Farkas theorem there exists a p_0 satisfying $A^T p_0 \geq 0$ and $b^T p_0 < 0$. Then for each $\alpha \in \mathbb{R}$, $\alpha \geq 0$ we have $A^T(p_1 - \alpha p_0) \leq A^T p_1 \leq c$ and

$$\lim_{\alpha\to\infty} b^T(p_1 - \alpha p_0) = \lim_{\alpha\to\infty} (b^T p_1 - \alpha b^T p_0) = \infty$$

because of $b^T p_0 < 0$, hence

$$g(A,b,c) = \sup\{b^T p;\ A^T p \leq c\} = \infty = f(A,b,c),$$

which is (6.8).

We have proved that in all three cases (a), (b), (c) the equality (6.8) holds. This concludes the proof. □

6.3 The interval linear programming problem

6.3.1 Definition

Let $\mathbf{A} = [\underline{A}, \overline{A}] = [A_c - \Delta, A_c + \Delta]$ be an $m \times n$ interval matrix and let $\mathbf{b} = [\underline{b}, \overline{b}] = [b_c - \delta, b_c + \delta]$ and $\mathbf{c} = [\underline{c}, \overline{c}] = [c_c - \gamma, c_c + \gamma]$ be an m-dimensional and n-dimensional interval vector, respectively. The *family* of linear programming problems

$$\mathrm{Min}\{c^T x;\ Ax = b, x \geq 0\} \tag{6.21}$$

with data satisfying

$$A \in \mathbf{A},\ b \in \mathbf{b},\ c \in \mathbf{c} \tag{6.22}$$

is called an *interval linear programming problem*. Since for each linear programming problem (6.21) we have a uniquely determined optimal value $f(A, b, c)$, it is natural to consider its range over the data (6.22) by introducing the values

$$\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = \inf\{f(A, b, c);\ A \in \mathbf{A}, b \in \mathbf{b}, c \in \mathbf{c}\},$$

$$\overline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = \sup\{f(A, b, c);\ A \in \mathbf{A}, b \in \mathbf{b}, c \in \mathbf{c}\}.$$

The interval $[\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}), \overline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c})]$, whose bounds may be infinite, is called the *range of the optimal value* of the interval linear programming problem (6.21), (6.22). In the next section we shall derive formulae for computing the range that will become the cornerstone point of our approach.

6.3.2 Range of the optimal value

The following theorem gives explicit formulae for computing the bounds of the range. Notice that the result holds without any additional assumptions.

Theorem 237. *We have*

$$\underline{f}(\mathbf{A},\mathbf{b},\mathbf{c}) = \inf\{\underline{c}^T x;\ \underline{A}x \le \overline{b}, \overline{A}x \ge \underline{b}, x \ge 0\}, \tag{6.23}$$

$$\overline{f}(\mathbf{A},\mathbf{b},\mathbf{c}) = \sup_{y \in Y_m} f(A_{ye}, b_y, \overline{c}). \tag{6.24}$$

Comment. Hence, solving only one linear programming problem is needed to evaluate $\underline{f}(\mathbf{A},\mathbf{b},\mathbf{c})$, whereas up to 2^m of them are to be solved to compute $\overline{f}(\mathbf{A},\mathbf{b},\mathbf{c})$ according to (6.24). Although the set Y_m is finite, we use "sup" here because some of the values may be infinite.

Proof: For given $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$ denote $\underline{f} := \underline{f}(\mathbf{A},\mathbf{b},\mathbf{c})$, $\overline{f} := \overline{f}(\mathbf{A},\mathbf{b},\mathbf{c})$.

(a) To prove (6.23), put

$$\underline{\varphi} = \inf\{\underline{c}^T x;\ \underline{A}x \le \overline{b}, \overline{A}x \ge \underline{b}, x \ge 0\}.$$

(a.1) First we prove $\underline{f} \le \underline{\varphi}$. This is obvious if $\underline{\varphi} = \infty$. If $\underline{\varphi} < \infty$, then the linear system

$$\underline{A}x \le \overline{b},\ \overline{A}x \ge \underline{b} \tag{6.25}$$

is feasible. Let x be a nonnegative solution of it. Then in view of Theorem 208, x is a nonnegative weak solution of $\mathbf{A}x = \mathbf{b}$, hence there exist $A \in \mathbf{A}$, $b \in \mathbf{b}$ such that $Ax = b$ holds. Then $\underline{f} \le f(A, b, \underline{c}) \le \underline{c}^T x$, and since x is an arbitrary nonnegative solution of (6.25), we obtain $\underline{f} \le \underline{\varphi}$.

(a.2) Second we prove $\underline{\varphi} \le \underline{f}$ by showing that

$$\underline{\varphi} \le f(A, b, c) \tag{6.26}$$

holds for each $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$. This is obvious if $f(A, b, c) = \infty$. If $f(A, b, c) < \infty$, then the linear programming problem

$$\text{Min}\{c^T x; Ax = b, x \ge 0\} \tag{6.27}$$

is feasible. Let x be any feasible solution of it. Then, according to Theorem 208, x is also a nonnegative solution of the system (6.25), hence $\underline{\varphi} \le \underline{c}^T x \le c^T x$, which implies (6.26). Thus (6.26) holds for each $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$, which means that $\underline{\varphi} \le \underline{f}$. Hence, from (a.1) and (a.2) we obtain $\underline{f} = \underline{\varphi}$, which is (6.23).

(b) To prove (6.24), put

$$\overline{\varphi} = \sup_{y \in Y_m} f(A_{ye}, b_y, \overline{c}).$$

(b.1) Since $A_{ye} \in \mathbf{A}$, $b_y \in \mathbf{b}$ for each $y \in Y_m$ and $\bar{c} \in \mathbf{c}$, we immediately obtain that

$$\overline{\varphi} \leq \sup\{f(A,b,c);\ A \in \mathbf{A}, b \in \mathbf{b}, c \in \mathbf{c}\} = \overline{f}.$$

(b.2) Finally we prove $\overline{f} \leq \overline{\varphi}$ by showing that

$$f(A,b,c) \leq \overline{\varphi} \tag{6.28}$$

holds for each $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$. This is obvious if $f(A,b,c) = -\infty$. If $f(A,b,c) = \infty$, then the linear programming problem (6.27) is infeasible, hence the system $\mathbf{A}x = \mathbf{b}$ is not strongly feasible, which in view of Theorem 212 means that a system $A_{ye}x = b_y$ is not feasible for some $y \in Y_m$, so that $f(A_{ye}, b_y, \bar{c}) = \infty$, hence $\overline{\varphi} = \infty$ and (6.28) holds. Thus we are left with the case of $f(A,b,c)$ finite. Then by the duality theorem the dual problem to (6.27)

$$\mathrm{Max}\{b^T p;\ A^T p \leq c\}$$

has an optimal solution p^* and $f(A,b,c) = b^T p^*$ holds. Put $y = \operatorname{sgn} p^*$, then $y \in Y_m$ and $|p^*| = T_y p^*$. Consider the linear programming problem

$$\mathrm{Min}\{\bar{c}^T x;\ A_{ye}x = b_y, x \geq 0\} \tag{6.29}$$

and its dual problem

$$\mathrm{Max}\{b_y^T p;\ A_{ye}^T p \leq \bar{c}\}. \tag{6.30}$$

The dual problem (6.30) is solvable because p^* solves $A_{ye}^T p \leq \bar{c}$: in fact, since $|(A - A_c)^T p^*| \leq \Delta^T |p^*|$, we have

$$\begin{aligned} A_{ye}^T p^* &= (A_c - T_y \Delta)^T p^* = (A_c^T - \Delta^T T_y) p^* = A_c^T p^* - \Delta^T |p^*| \\ &\leq (A_c + A - A_c)^T p^* = A^T p^* \leq c \leq \bar{c}. \end{aligned}$$

Now, if the primal problem (6.29) is infeasible, then $f(A_{ye}, b_y, \bar{c}) = \infty$, hence $\overline{\varphi} = \infty$ and (6.28) holds. If it is feasible, then $f(A,b,c) < \infty$ and $g(A,b,c) > -\infty$, and by the duality theorem the dual problem (6.30) has an optimal solution $\hat{p}$ satisfying $f(A_{ye}, b_y, \bar{c}) = b_y^T \hat{p}$, hence

$$\begin{aligned} f(A,b,c) &= b^T p^* = (b_c + b - b_c)^T p^* \leq b_c^T p^* + \delta^T |p^*| = (b_c^T + \delta^T T_y) p^* \\ &= (b_c + T_y \delta)^T p^* = b_y^T p^* \leq b_y^T \hat{p} = f(A_{ye}, b_y, \bar{c}) \leq \overline{\varphi}, \end{aligned}$$

which is (6.28). This proves that (6.28) holds for each $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$, implying $\overline{f} \leq \overline{\varphi}$. Hence, (b.1) and (b.2) together give $\overline{f} = \overline{\varphi}$, which proves (6.24). This completes the proof. □

In Section 5.2.7 we presented an algorithm for checking feasibility of the systems

$$A_{ye}x = b_y$$

for all $y \in Y_m$. Since we are now facing a very close problem of solving

$$\mathrm{Min}\{\overline{c}^T x;\ A_{ye}x = b_y, x \geq 0\}$$

for all $y \in Y_m$, we can adapt the previous algorithm for the current purpose. Let us reorder the equations of $\mathbf{A}x = \mathbf{b}$ so that those containing at least one nondegenerate interval coefficient go first. Let q be the number of them, so that after reordering the last $m - q$ equations consist only of real (i.e., noninterval) data. Then the following algorithm, where $z \in \mathbb{R}^q$ and $y \in \mathbb{R}^q$, does the job:

```
compute f̲ by (6.23);
z := 0; y := e;
A := A̲; b := b̄; f̄ := f(A, b, c̄);
while z ≠ e & f̄ < ∞
    k := min{i; z_i = 0};
    for i := 1 to k − 1, z_i := 0; end
    z_k := 1; y_k := −y_k;
    if y_k = 1 then A_k· := A̲_k·; b_k := b̄_k; else A_k· := Ā_k·; b_k := b̲_k; end
    f̄ := max{f̄, f(A, b, c̄)};
end
% [f̲, f̄] is the range of the optimal value.
```

Example 238 Let

$$c_c = (-1, -2, 3, 4)^T,$$

$$A_c = \begin{pmatrix} 5 & 6 & -7 & 8 \\ 10 & -11 & 12 & 13 \end{pmatrix}, \quad b_c = \begin{pmatrix} -9 \\ 14 \end{pmatrix}$$

(the pattern of the absolute values of coefficients is obvious). For each $\varepsilon > 0$ consider the interval data

$$\mathbf{A}_\varepsilon = [A_c - \varepsilon ee^T, A_c + \varepsilon ee^T], \quad \mathbf{b}_\varepsilon = [b_c - \varepsilon e, b_c + \varepsilon e], \quad \mathbf{c}_\varepsilon = [c_c - \varepsilon e, c_c + \varepsilon e].$$

Using the algorithm, we have computed $\underline{f}(\mathbf{A}_\varepsilon, \mathbf{b}_\varepsilon, \mathbf{c}_\varepsilon)$ and $\overline{f}(\mathbf{A}_\varepsilon, \mathbf{b}_\varepsilon, \mathbf{c}_\varepsilon)$ for $\varepsilon := 0.00, 0.01, \ldots, 0.24$ with MATLAB 6.0, where we employed the procedure QP.M for evaluating $f(A, b, c)$. The results, rounded to four decimal places, are summed up in the following table (the last column brings the values of $\overline{f}(\mathbf{A}_\varepsilon, \mathbf{b}_\varepsilon, \mathbf{c}_\varepsilon) - \underline{f}(\mathbf{A}_\varepsilon, \mathbf{b}_\varepsilon, \mathbf{c}_\varepsilon)$, denoted for short as $\overline{f} - \underline{f}$):

ε	$\underline{f}(\mathbf{A}_\varepsilon, \mathbf{b}_\varepsilon, \mathbf{c}_\varepsilon)$	$\overline{f}(\mathbf{A}_\varepsilon, \mathbf{b}_\varepsilon, \mathbf{c}_\varepsilon)$	$\overline{f} - \underline{f}$
0.00	5.0000	5.0000	0.0000
0.01	4.8085	5.2228	0.4143
0.02	4.6424	5.4847	0.8423
0.03	4.4971	5.7965	1.2994
0.04	4.3692	6.1735	1.8043
0.05	4.2559	6.6375	2.3816
0.06	4.1550	7.2217	3.0667
0.07	4.0647	7.9784	3.9137
0.08	3.9836	8.9955	5.0119
0.09	3.9104	10.4327	6.5223
0.10	3.8442	12.6143	8.7701
0.11	3.7841	16.3131	12.5290
0.12	3.7294	23.9388	20.2094
0.13	3.6796	48.7450	45.0654
0.14	3.6340	∞	∞
0.15	3.5923	∞	∞
0.16	3.5541	∞	∞
0.17	3.5189	∞	∞
0.18	3.4866	∞	∞
0.19	3.4569	∞	∞
0.20	3.4295	∞	∞
0.21	3.4043	∞	∞
0.22	3.3810	∞	∞
0.23	3.3396	∞	∞
0.24	$-\infty$	∞	∞

As we can see, all the linear programming problems in the family have optimal solutions for ε up to 0.13, infeasible problems appear from $\varepsilon = 0.14$ on and the family contains infeasible and unbounded problems (as well as those having optimal solutions) from $\varepsilon = 0.24$ on.

In the next two sections we shall study separately properties of the two bounds.

6.3.3 The lower bound

In this section we shall derive some consequences of the formula for the lower bound

$$\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = \inf\{\underline{c}^T x;\ \underline{A}x \leq \overline{b}, \overline{A}x \geq \underline{b}, x \geq 0\}$$

in Theorem 237. If $\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = \infty$, then each problem in the family is infeasible. Let us consider the case $\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = -\infty$.

Theorem 239. *If $\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = -\infty$, then there exists an $A_0 \in \mathbf{A}$ such that*

$$f(A_0, b, \underline{c}) \in \{-\infty, \infty\} \tag{6.31}$$

holds for each $b \in \mathbf{b}$.

Comment. In other words, none of the problems

$$\text{Min}\{\underline{c}^T x;\ A_0 x = b, x \geq 0\}, \quad b \in \mathbf{b}$$

has an optimal solution.

Proof: If $\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = -\infty$, then by Theorem 237 the linear programming problem

$$\text{Min}\{\underline{c}^T x;\ \underline{A}x \leq \overline{b}, \overline{A}x \geq \underline{b}, x \geq 0\}$$

is unbounded and by the duality theorem its dual problem

$$\text{Max}\{\underline{b}^T p_1 - \overline{b}^T p_2;\ \overline{A}^T p_1 - \underline{A}^T p_2 \leq \underline{c}, p_1 \geq 0, p_2 \geq 0\}$$

is infeasible, hence the system

$$\overline{A}^T p_1 - \underline{A}^T p_2 \leq \underline{c}$$

is infeasible and Theorem 202 assures existence of an x_0 that satisfies

$$\underline{A}x_0 \leq 0,\ \overline{A}x_0 \geq 0,\ x_0 \geq 0,\ \underline{c}^T x_0 < 0.$$

Then Theorem 208 gives that x_0 is a nonnegative weak solution of the system $[\underline{A}, \overline{A}]x = [0, 0]$, hence there exists a matrix $A_0 \in \mathbf{A}$ such that

$$A_0 x_0 = 0,\ x_0 \geq 0,\ \underline{c}^T x_0 < 0. \tag{6.32}$$

Now consider the problem

$$\text{Min}\{\underline{c}^T x;\ A_0 x = b, x \geq 0\} \tag{6.33}$$

for a $b \in \mathbf{b}$. If it is infeasible, then $f(A_0, b, \underline{c}) = \infty$. If it has a feasible solution x_1, then from (6.32) it follows that $x_1 + \alpha x_0$ is a feasible solution of (6.33) for each $\alpha \geq 0$ and

$$\lim_{\alpha\to\infty} \underline{c}^T(x_1 + \alpha x_0) = \lim_{\alpha\to\infty} (\underline{c}^T x_1 + \alpha \underline{c}^T x_0) = -\infty$$

due to (6.32), hence the problem (6.33) is unbounded and $f(A_0, b, \underline{c}) = -\infty$. Thus for each $b \in \mathbf{b}$ we have (6.31), which concludes the proof. □

However, in case of $\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = -\infty$ the family need *not* contain an unbounded problem.

Example 240 *Let*

$$\mathbf{A} = [0, 1], \quad \mathbf{b} = [1, 1], \quad \mathbf{c} = [-1, -1]$$

(i.e., $m = n = 1$). Then each problem in the family is of the form

$$\text{Min}\{-x;\, ax = 1,\, x \geq 0\},$$

it is infeasible for $a = 0$ and its optimal value is equal to $-1/a$ for $a \in (0, 1]$, hence $\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = -\infty$ but no problem in the family is unbounded.

If the lower bound $\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is finite, then it can be expected that it is attained as the optimal value of some problem in the family. The following theorem shows a constructive way how to find the data of such a problem.

Theorem 241. *Let $\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ be finite and let x^* be an optimal solution of the problem*

$$\text{Min}\{\underline{c}^T x;\, \underline{A}x \leq \overline{b}, \overline{A}x \geq \underline{b}, x \geq 0\}. \tag{6.34}$$

Then

$$\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = f(A_c - T_y\Delta, b_c + T_y\delta, \underline{c}), \tag{6.35}$$

where

$$y_i = \begin{cases} \frac{(A_c x^* - b_c)_i}{(\Delta x^* + \delta)_i} & \text{if } (\Delta x^* + \delta)_i > 0, \\ 1 & \text{if } (\Delta x^* + \delta)_i = 0 \end{cases} \qquad (i = 1, \ldots, m). \tag{6.36}$$

Proof: If $\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is finite, then according to (6.23) it is equal to the optimal value of the problem (6.34), hence $\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = \underline{c}^T x^*$, where x^* is an arbitrary optimal solution of (6.34). Hence x^* satisfies

$$\underline{A}x^* \leq \overline{b},\ \overline{A}x^* \geq \underline{b},\ x^* \geq 0,$$

which can be equivalently written as

$$|A_c x^* - b_c| \leq \Delta x^* + \delta,\ x^* \geq 0. \tag{6.37}$$

Now Proposition 205 gives that $(A_c - T_y\Delta)x^* = b_c + T_y\delta$, where y is given by (6.36). Then $|y| \le e$ because of (6.37), hence $A_c - T_y\Delta \in \mathbf{A}$ and $b_c + T_y\delta \in \mathbf{b}$, and we have

$$\underline{c}^T x^* = \underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) \le f(A_c - T_y\Delta, b_c + T_y\delta, \underline{c}) \le \underline{c}^T x^*,$$

which proves (6.35). □

Notice that the vector y defined by (6.36) satisfies $y \notin Y_m$ in general (this is why we wrote $A_c - T_y\Delta$, $b_c + T_y\delta$ instead of A_{ye}, b_y in (6.35) because A_{ye}, b_y are defined for $y \in Y_m$ only, see Section 3.2.2). But y can be enforced to belong to Y_m under an additional assumption:

Theorem 242. *Let the problem*

$$\mathrm{Max}\{\underline{b}^T p_1 - \overline{b}^T p_2;\ \overline{A}^T p_1 - \underline{A}^T p_2 \le \underline{c}, p_1 \ge 0, p_2 \ge 0\} \tag{6.38}$$

have an optimal solution p_1^*, p_2^* satisfying

$$p_1^* + p_2^* > 0. \tag{6.39}$$

Then

$$\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = f(A_{ye}, b_y, \underline{c}), \tag{6.40}$$

where

$$y_i = \begin{cases} 1 & \text{if } (p_2^*)_i > 0, \\ -1 & \text{if } (p_2^*)_i = 0 \end{cases} \qquad (i = 1, \ldots, m). \tag{6.41}$$

Proof: If the problem (6.38) has an optimal solution p_1^*, p_2^*, then its primal problem

$$\mathrm{Min}\{\underline{c}^T x;\ \underline{A}x \le \overline{b}, \overline{A}x \ge \underline{b}, x \ge 0\}$$

has an optimal solution x^*, $\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = \underline{c}^T x^*$ holds by Theorem 237 and the complementary slackness conditions of linear programming [7] give

$$p_1^{*T}(\overline{A}x^* - \underline{b}) = p_2^{*T}(\overline{b} - \underline{A}x^*) = 0.$$

Since all four vectors p_1^*, $\overline{A}x^* - \underline{b}$, p_2^* and $\overline{b} - \underline{A}x^*$ are nonnegative, it must be

$$(p_1^*)_i(\overline{A}x^* - \underline{b})_i = (p_2^*)_i(\overline{b} - \underline{A}x^*)_i = 0 \tag{6.42}$$

for $i = 1, \ldots, m$. Now, if $(p_2^*)_i > 0$, then (6.42) gives $(\underline{A}x^*)_i = \overline{b}_i$; if $(p_2^*)_i = 0$, then $(p_1^*)_i > 0$ by (6.39) and (6.42) implies $(\overline{A}x^*)_i = \underline{b}_i$ $(i = 1, \ldots, m)$. Hence for the vector y defined by (6.41) we have $y \in Y_m$ and $A_{ye}x^* = b_y$, where $A_{ye} \in \mathbf{A}$ and $b_y \in \mathbf{b}$. Then

$$f(A_{ye}, b_y, \underline{c}) \le \underline{c}^T x^* = \underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) \le f(A_{ye}, b_y, \underline{c}),$$

which gives (6.40). □

Finally we shall prove a kind of duality theorem for $\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ which shows that this value, if finite, can be reached via optimization over strong solutions of $\mathbf{A}^T p \le \mathbf{c}$ only (see Theorems 222 and 223).

Theorem 243. *If $\underline{f}(\mathbf{A},\mathbf{b},\mathbf{c})$ is finite, then*

$$\underline{f}(\mathbf{A},\mathbf{b},\mathbf{c}) = \max\{\min_{b\in\mathbf{b}} b^T p;\ A^T p \le c \text{ for each } A\in\mathbf{A}, c\in\mathbf{c}\}. \tag{6.43}$$

Proof: The proof consists of three steps.

(a) First we shall prove a technical result: for each $p\in\mathbb{R}^m$ there holds

$$\overline{A}^T p^+ - \underline{A}^T p^- = A_{ye}^T p, \tag{6.44}$$

$$\underline{b}^T p^+ - \overline{b}^T p^- = b_y^T p = \min_{b\in\mathbf{b}} b^T p, \tag{6.45}$$

where $y = -\operatorname{sgn} p$. Indeed, since $|p| = -T_y p$, we have

$$\begin{aligned}\overline{A}^T p^+ - \underline{A}^T p^- &= (A_c+\Delta)^T p^+ - (A_c-\Delta)^T p^- \\ &= A_c^T(p^+ - p^-) + \Delta^T(p^+ + p^-) \\ &= A_c^T p + \Delta^T |p| = A_c^T p - \Delta^T T_y p \\ &= (A_c - T_y\Delta)^T p = A_{ye}^T p,\end{aligned}$$

which is (6.44), and

$$\begin{aligned}b^T p &= b^T(p^+ - p^-) \ge \underline{b}^T p^+ - \overline{b}^T p^- = b_c^T(p^+ - p^-) - \delta^T(p^+ + p^-) \\ &= b_c^T p - \delta^T|p| = b_c^T p + \delta^T T_y p = (b_c + T_y\delta)^T p = b_y^T p\end{aligned}$$

for each $b\in\mathbf{b}$, hence

$$\min_{b\in\mathbf{b}} b^T p \ge \underline{b}^T p^+ - \overline{b}^T p^- = b_y^T p \ge \min_{b\in\mathbf{b}} b^T p,$$

which is (6.45).

(b) If $\underline{f}(\mathbf{A},\mathbf{b},\mathbf{c})$ is finite, then by Theorem 237 and by the duality theorem we have

$$\underline{f}(\mathbf{A},\mathbf{b},\mathbf{c}) = \max\{\underline{b}^T p_1 - \overline{b}^T p_2;\ \overline{A}^T p_1 - \underline{A}^T p_2 \le \underline{c}, p_1\ge 0, p_2 \ge 0\}. \tag{6.46}$$

Let p satisfy $A^T p \le c$ for each $A\in\mathbf{A}$, $c\in\mathbf{c}$. Then in particular $A_{ye}^T p \le \underline{c}$, where $y = -\operatorname{sgn} p$, and (6.44) gives $\overline{A}^T p^+ - \underline{A}^T p^- = A_{ye}^T p \le \underline{c}$, hence from (6.45), (6.46) we obtain

$$\begin{aligned}\min_{b\in\mathbf{b}} b^T p &= \underline{b}^T p^+ - \overline{b}^T p^- \\ &\le \max\{\underline{b}^T p_1 - \overline{b}^T p_2;\ \overline{A}^T p_1 - \underline{A}^T p_2 \le \underline{c}, p_1 \ge 0, p_2\ge 0\} \\ &= \underline{f}(\mathbf{A},\mathbf{b},\mathbf{c})\end{aligned}$$

and consequently

$$\sup\{\min_{b\in\mathbf{b}} b^T p;\ A^T p \le c \text{ for each } A\in\mathbf{A}, c\in\mathbf{c}\} \le \underline{f}(\mathbf{A},\mathbf{b},\mathbf{c}). \tag{6.47}$$

(c) To prove that equality holds in (6.47), take any optimal solution p_1^*, p_2^* of the problem

$$\text{Max}\{\underline{b}^T p_1 - \overline{b}^T p_2;\ \overline{A}^T p_1 - \underline{A}^T p_2 \le \underline{c}, p_1 \ge 0, p_2 \ge 0\} \tag{6.48}$$

and put $\hat{p}_1 = p_1^* - d$, $\hat{p}_2 = p_2^* - d$, where $d = \min\{p_1^*, p_2^*\}$. Then $d \ge 0$, $\hat{p}_1 \ge 0$, $\hat{p}_2 \ge 0$ and $\hat{p}_1^T \hat{p}_2 = 0$. We shall show that $\hat{p}_1$, $\hat{p}_2$ is again an optimal solution of (6.48). In fact,

$$\begin{aligned} \overline{A}^T \hat{p}_1 - \underline{A}^T \hat{p}_2 &= \overline{A}^T (p_1^* - d) - \underline{A}^T (p_2^* - d) = \overline{A}^T p_1^* - \underline{A}^T p_2^* - (\overline{A} - \underline{A})^T d \\ &\le \overline{A}^T p_1^* - \underline{A}^T p_2^* \le \underline{c} \end{aligned} \tag{6.49}$$

since $(\overline{A} - \underline{A})^T d \ge 0$, and

$$\begin{aligned} \underline{b}^T \hat{p}_1 - \overline{b}^T \hat{p}_2 &= \underline{b}^T (p_1^* - d) - \overline{b}^T (p_2^* - d) = \underline{b}^T p_1^* - \overline{b}^T p_2^* + (\overline{b} - \underline{b})^T d \\ &\ge \underline{b}^T p_1^* - \overline{b}^T p_2^* \end{aligned}$$

since $(\overline{b} - \underline{b})^T d \ge 0$, hence it must be

$$\underline{b}^T \hat{p}_1 - \overline{b}^T \hat{p}_2 = \underline{b}^T p_1^* - \overline{b}^T p_2^* \tag{6.50}$$

and $\hat{p}_1$, $\hat{p}_2$ is an optimal solution of (6.48). Put $p = \hat{p}_1 - \hat{p}_2$. Since $\hat{p}_1^T \hat{p}_2 = 0$, it follows that $p^+ = \hat{p}_1$, $p^- = \hat{p}_2$, hence for each $A \in \mathbf{A}$, $c \in \mathbf{c}$ we have

$$A^T p = A^T (\hat{p}_1 - \hat{p}_2) \le \overline{A}^T \hat{p}_1 - \underline{A}^T \hat{p}_2 \le \underline{c} \le c$$

by (6.49), and

$$\min_{b \in \mathbf{b}} b^T p = \underline{b}^T p^+ - \overline{b}^T p^- = \underline{b}^T \hat{p}_1 - \overline{b}^T \hat{p}_2 = \underline{b}^T p_1^* - \overline{b}^T p_2^* = \underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c})$$

by (6.45), (6.50) and (6.46), hence the value $\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is attained in (6.47) and (6.43) holds. □

6.3.4 The upper bound

The formula

$$\overline{f}(\mathbf{A},\mathbf{b},\mathbf{c}) = \sup_{y\in Y_m} f(A_{ye}, b_y, \overline{c})$$

of Theorem 237 requires solving up to 2^m linear programs. In this section we shall show that the upper bound is closely connected with the optimal value of the nonlinear program

$$\text{maximize } b_c^T p + \delta^T |p| \tag{6.51}$$

s.t.

$$A_c^T p - \Delta^T |p| \le \overline{c}. \tag{6.52}$$

Let us denote

$$\overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c}) = \sup\{b_c^T p + \delta^T|p|;\ A_c^T p - \Delta^T|p| \le \overline{c}\}. \tag{6.53}$$

We shall consider separately the cases of $\overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c}) = -\infty$, $\overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c}) = \infty$ and $\overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c})$ finite.

Proposition 244 *If* $\overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c}) = -\infty$, *then* $\overline{f}(\mathbf{A},\mathbf{b},\mathbf{c}) \in \{-\infty,\infty\}$.

Proof: Assume that $A^T p \le c$ has a solution for some $A \in \mathbf{A}$, $c \in \mathbf{c}$. Then $A_c^T p - \Delta^T|p| \le A^T p \le c \le \overline{c}$, hence p solves (6.52) (cf. Theorem 217), which implies that $\overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c}) > -\infty$, a contradiction. Hence for each $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$ the dual problem

$$\text{Max}\{b^T p;\ A^T p \le c\}$$

is unsolvable, which means that each primal problem

$$\text{Min}\{c^T x;\ Ax = b, x \ge 0\}$$

is either infeasible or unbounded, so that $f(A,b,c) \in \{-\infty,\infty\}$ for each $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$ and consequently $\overline{f}(\mathbf{A},\mathbf{b},\mathbf{c}) \in \{-\infty,\infty\}$. □

Proposition 245 *If* $\overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c}) = \infty$, *then* $\overline{f}(\mathbf{A},\mathbf{b},\mathbf{c}) = \infty$.

Proof: If $\overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c}) = \sup\{b_c^T p + \delta^T|p|;\ A_c^T p - \Delta^T|p| \le \overline{c}\} = \infty$, then for each positive integer k there exists a $p_k \in \mathbb{R}^m$ such that

$$A_c^T p_k - \Delta^T|p_k| \le \overline{c} \tag{6.54}$$

and

$$b_c^T p_k + \delta^T|p_k| \ge k. \tag{6.55}$$

For each $k = 1, 2, \ldots$ put $y_k = \operatorname{sgn} p_k$. Since $y_k \in Y_m$ for each k and Y_m is finite, the sequence $\{y_k\}_{k=1}^{\infty}$ must contain a member which appears there infinitely many times, i.e., there exists a subsequence $\{k_j\}_{j=1}^{\infty}$ and a $y \in Y_m$ such that $\operatorname{sgn} p_{k_j} = y$ for each j. Then from (6.54), (6.55) we have

$$A_{ye}^T p_{k_j} = A_c^T p_{k_j} - \Delta^T T_y p_{k_j} = A_c^T p_{k_j} - \Delta^T |p_{k_j}| \le \overline{c},$$

$$b_y^T p_{k_j} = b_c^T p_{k_j} + \delta^T T_y p_{k_j} = b_c^T p_{k_j} + \delta^T |p_{k_j}| \ge k_j$$

for $j = 1, 2, \ldots$, hence the problem

$$\operatorname{Max}\{b_y^T p;\ A_{ye}^T p \le \overline{c}\}$$

is unbounded and by the duality theorem the respective primal problem

$$\operatorname{Min}\{\overline{c}^T x;\ A_{ye} x = b_y, x \ge 0\}$$

is infeasible, hence $f(A_{ye}, b_y, \overline{c}) = \infty$ and consequently $\overline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = \infty$. □

Theorem 246. *If $\overline{\varphi}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is finite, then*

$$\overline{\varphi}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = \max\{f(A, b, c);\ f(A, b, c) < \infty, A \in \mathbf{A}, b \in \mathbf{b}, c \in \mathbf{c}\}. \qquad (6.56)$$

Proof: (a) First we shall prove that if $f(A, b, c) < \infty$ for some $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$, then

$$f(A, b, c) \le \overline{\varphi}(\mathbf{A}, \mathbf{b}, \mathbf{c}). \qquad (6.57)$$

This is clearly the case if $f(A, b, c) = -\infty$. Thus let $f(A, b, c)$ be finite. Then $f(A, b, c) = b^T p^*$, where p^* is an optimal solution of the dual problem

$$\operatorname{Max}\{b^T p;\ A^T p \le c\}.$$

Since

$$A_c^T p^* - \Delta^T |p^*| \le A^T p^* \le c \le \overline{c},$$

we can see that p^* solves (6.52), hence

$$\begin{aligned} f(A, b, c) &= b^T p^* \le \sup\{b^T p;\ A_c^T p - \Delta^T |p| \le \overline{c}\} \\ &\le \sup\{b_c^T p + \delta^T |p|;\ A_c^T p - \Delta^T |p| \le \overline{c}\} = \overline{\varphi}(\mathbf{A}, \mathbf{b}, \mathbf{c}). \end{aligned}$$

This proves (6.57) and hence also

$$\sup\{f(A, b, c);\ f(A, b, c) < \infty, A \in \mathbf{A}, b \in \mathbf{b}, c \in \mathbf{c}\} \le \overline{\varphi}(\mathbf{A}, \mathbf{b}, \mathbf{c}). \qquad (6.58)$$

(b) To prove that the upper bound is attained in (6.58), we start from the fact that because of (6.53) for each positive integer k there exists a vector p_k satisfying

$$A_c^T p_k - \Delta^T |p_k| \le \overline{c},$$

$$\overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c}) - \tfrac{1}{k} < b_c^T p_k + \delta^T |p_k| \le \overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c}).$$

Arguing as in the proof of Proposition 245, we can assure existence of a $y \in Y_m$ satisfying $\operatorname{sgn} p_{k_j} = y$ for an infinite subsequence $\{k_j\}$. For each k_j we then have

$$A_{ye}^T p_{k_j} \le \overline{c}, \tag{6.59}$$

$$\overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c}) - \tfrac{1}{k_j} < b_y^T p_{k_j} \le \overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c}). \tag{6.60}$$

Now consider the problem

$$\operatorname{Max}\{b_y^T p;\ A_{ye}^T p \le \overline{c}\}.$$

From (6.53) we have that its optimal value is bounded by $\overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c})$, and (6.59), (6.60) show that this bound can be approximated with arbitrary accuracy by the value of the objective $b_y^T p$ over the solution set of $A_{ye}^T p \le \overline{c}$. This gives, by the duality theorem,

$$\overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c}) = \max\{b_y^T p;\ A_{ye}^T p \le \overline{c}\} = f(A_{ye}, b_y, \overline{c}),$$

hence the upper bound in (6.58) is attained and (6.56) holds. □

Now we arrive at an important consequence which justifies introduction of the value $\overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c})$:

Theorem 247. *If $\overline{f}(\mathbf{A},\mathbf{b},\mathbf{c})$ is finite, then*

$$\overline{f}(\mathbf{A},\mathbf{b},\mathbf{c}) = \overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c}). \tag{6.61}$$

Proof: Since the possibilities $\overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c}) = -\infty$ and $\overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c}) = \infty$ are precluded by Propositions 244 and 245, $\overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c})$ must be finite, and Theorem 246 gives

$$\begin{aligned}\overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c}) &= \max\{f(A,b,c);\ f(A,b,c) < \infty, A \in \mathbf{A}, b \in \mathbf{b}, c \in \mathbf{c}\}\\ &= \max\{f(A,b,c);\ A \in \mathbf{A}, b \in \mathbf{b}, c \in \mathbf{c}\} = \overline{f}(\mathbf{A},\mathbf{b},\mathbf{c}).\end{aligned}$$

□

Hence, if $\overline{f}(\mathbf{A},\mathbf{b},\mathbf{c})$ is finite, then it can be computed as the optimal value of a single nonlinear programming problem (6.51), (6.52) by nonlinear programming techniques. Moreover, the equality (6.61) yields a computable upper bound on $\overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c})$. Let us remind that if A has linearly independent rows, then the matrix AA^T is nonsingular and there holds

$$(A^+)^T = (A^T)^+ = (AA^T)^{-1}A,$$

where A^+ is the Moore-Penrose inverse of A.

Theorem 248. *If A_c has linearly independent rows and*

$$\varrho(\Delta|A_c^+|) < 1 \tag{6.62}$$

holds, then

$$\overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c}) \le \overline{c}^T|A_c^+|(I-\Delta|A_c^+|)^{-1}(|b_c|+\delta). \tag{6.63}$$

Proof: If the inequality (6.52) has no solution, then $\overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c}) = -\infty$ and (6.63) holds. Thus let p be a solution to (6.52). Then we have

$$\begin{aligned} |p| &= |(A_cA_c^T)^{-1}A_cA_c^Tp| = |(A_c^+)^TA_c^Tp| \le |A_c^+|^T|A_c^Tp| \\ &\le |A_c^+|^T(\Delta^T|p|+\overline{c}) \le (\Delta|A_c^+|)^T|p| + |A_c^+|^T\overline{c}, \end{aligned}$$

hence

$$(I-\Delta|A_c^+|)^T|p| \le |A_c^+|^T\overline{c}. \tag{6.64}$$

Because of (6.62) the matrix $I-\Delta|A_c^+|$ is nonnegatively invertible and premultiplying (6.64) by its transposed inverse gives

$$|p| \le ((I-\Delta|A_c^+|)^{-1})^T|A_c^+|^T\overline{c}$$

and

$$\begin{aligned} b_c^Tp+\delta^T|p| &\le (|b_c|+\delta)^T|p| \le (|b_c|+\delta)^T((I-\Delta|A_c^+|)^{-1})^T|A_c^+|^T\overline{c} \\ &= \overline{c}^T|A_c^+|(I-\Delta|A_c^+|)^{-1}(|b_c|+\delta), \end{aligned}$$

which yields (6.63). □

Theorem 247 can also be reformulated as a counterpart of Theorem 243 of Section 6.3.3:

Theorem 249. *If $\overline{f}(\mathbf{A},\mathbf{b},\mathbf{c})$ is finite, then*

$$\overline{f}(\mathbf{A},\mathbf{b},\mathbf{c}) = \max\{\max_{b\in\mathbf{b}} b^Tp;\ A^Tp \le c \text{ for some } A\in\mathbf{A}, c\in\mathbf{c}\}. \tag{6.65}$$

Proof: By Gerlach theorem 217, p satisfies (6.52) if and only if it is a weak solution of the system $[A_c^T-\Delta^T, A_c^T+\Delta^T]p \le [\underline{c},\overline{c}]$, i.e., if and only if it satisfies $A^Tp \le c$ for some $A\in\mathbf{A}$, $c\in\mathbf{c}$. Next, as in the part (a) of the proof of Theorem 243 we can show that

$$b_c^Tp+\delta^T|p| = b_y^Tp = \max_{b\in\mathbf{b}} b^Tp,$$

where $y = \operatorname{sgn} p$. The rest follows from Theorem 247. □

Observe the additional "duality" between quantifiers used in formulae (6.43) and (6.65): the optimization is performed over strong solutions in (6.43) and over weak solutions in (6.65).

Finally, we have this complexity result:

Theorem 250. *Computing the upper bound $\overline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is NP-hard.*

Proof: Given a symmetric M-matrix A, consider the interval linear programming problem with $A_c = (A, -A)$, $\Delta = (0, 0)$, $b_c = 0$, $\delta = e$, $c_c = (e^T, e^T)^T$ and $\gamma = 0$. Then each primal problem in the family has the form

$$\text{Min}\{e^T x_1 + e^T x_2;\ A(x_1 - x_2) = b, x_1 \geq 0, x_2 \geq 0\}$$

and its dual problem is of the form

$$\text{Max}\{b^T p;\ -e \leq Ap \leq e\}$$

(because A is symmetric by assumption). Each dual problem is solvable ($p = 0$ solves the system) and each solution p of $-e \leq Ap \leq e$ satisfies $|p| = |A^{-1}Ap| \leq |A^{-1}|e$, hence $|b^T p| \leq e^T |A^{-1}|e$, so that each dual problem has an optimal solution and thus also each primal problem has an optimal solution and the absolute value of its optimal value is bounded by $e^T |A^{-1}|e$, hence $\overline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is finite. Then by Theorem 247 we have

$$\overline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = \overline{\varphi}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = \max\{e^T |p|;\ -e \leq Ap \leq e\},$$

hence

$$\overline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) \geq 1 \tag{6.66}$$

holds if and only if the system

$$-e \leq Ap \leq e,$$
$$e^T |p| \geq 1$$

has a solution. Since the latter problem is NP-complete by Theorem 22, the problem of deciding whether (6.66) holds is NP-hard, hence computing $\overline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is NP-hard. □

Summing up, we arrive at the following conclusion: computing the lower bound of the range of the optimal value $[\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}), \overline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c})]$ can be performed in polynomial time, whereas computing the upper bound is NP-hard.

6.3.5 Finite range

In applications we are mostly interested in linear programming problems having optimal solutions. Therefore for problems with inexact data the case when all problems in the family have optimal solutions is of particular interest. Several equivalent conditions are listed in the following theorem.

Theorem 251. *For an interval linear programming problem with data* $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$ *the following assertions are equivalent:*

(i) for each $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$ *the problem*

$$\text{Min}\{c^T x;\ Ax = b, x \geq 0\} \tag{6.67}$$

has an optimal solution,

(ii) both $\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ *and* $\overline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ *are finite,*

(iii) both $\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ *and* $\overline{\varphi}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ *are finite,*

(iv) the system

$$\overline{A}^T p_1 - \underline{A}^T p_2 \leq \underline{c} \tag{6.68}$$

is feasible and $\overline{\varphi}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ *is finite.*

In each case the range of the optimal value is given by

$$[\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}), \overline{\varphi}(\mathbf{A}, \mathbf{b}, \mathbf{c})].$$

Proof: We shall prove (i)⇒(ii)⇒(iii)⇒(iv)⇒(i).

(i)⇒(ii): Since each problem (6.67) has an optimal solution, it must be $\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) < \infty$, and the possibility of $\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = -\infty$ is precluded by Theorem 239. Hence $\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is finite, and Theorem 237 implies that $\overline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is also finite.

(ii)⇒(iii) follows directly from Theorem 247.

(iii)⇒(iv): If $\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is finite, then, as shown in the part (b) of the proof of Theorem 243, equation (6.46), there holds

$$\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = \max\{\underline{b}^T p_1 - \overline{b}^T p_2;\ \overline{A}^T p_1 - \underline{A}^T p_2 \leq \underline{c}, p_1 \geq 0, p_2 \geq 0\},$$

hence the system (6.68) is feasible.

(iv)⇒(i): Let the system (6.68) have a nonnegative solution p_1, p_2 and let $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$. Then we have

$$A^T(p_1 - p_2) \leq \overline{A}^T p_1 - \underline{A}^T p_2 \leq \underline{c} \leq c,$$

so that the dual problem to (6.67)

$$\text{Max}\{b^T p;\ A^T p \leq c\} \tag{6.69}$$

is solvable, and for each solution p of $A^T p \le c$ there holds

$$\begin{aligned} b^T p &\le \sup\{b^T p;\ A^T p \le c\} \le \sup\{b_c^T p + \delta^T |p|;\ A_c^T p - \Delta^T |p| \le \overline{c}\} \\ &= \overline{\varphi}(\mathbf{A}, \mathbf{b}, \mathbf{c}) < \infty, \end{aligned}$$

hence the objective is bounded, so that the problem (6.69) has an optimal solution and by the duality theorem the problem (6.67) also has an optimal solution.

Since in all four cases $\overline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is finite, we have $\overline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = \overline{\varphi}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ by Theorem 247 and the range of the optimal value is equal to

$$[\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}), \overline{\varphi}(\mathbf{A}, \mathbf{b}, \mathbf{c})],$$

which concludes the proof. □

Finally, we have this complexity result:

Theorem 252. *Checking whether each problem (6.67) with data satisfying $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$ has an optimal solution is NP-hard.*

Proof: Since a system $Ax = b$ is feasible if and only if the problem

$$\mathrm{Min}\{e^T x;\ Ax = b, x \ge 0\}$$

has an optimal solution, we have that a system of interval linear equations $\mathbf{A}x = \mathbf{b}$ is strongly feasible if and only if each problem (6.67) with data satisfying $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in [e, e]$ has an optimal solution. Since the former problem is NP-hard by Theorem 213, the latter one is NP-hard as well. □

6.3.6 An algorithm for computing the range

Summing up Proposition 245 and Theorem 251, we can formulate the following alternative algorithm for computing the range which, in contrast to the algorithm of Section 6.3.2, requires solving two optimization problems only:

compute the optimal value $\underline{f}$ of the problem
    $\mathrm{Min}\{\underline{c}^T x;\ \underline{A}x \leq \overline{b}, \overline{A}x \geq \underline{b}, x \geq 0\}$;
compute the optimal value $\overline{\varphi}$ of the problem
    $\mathrm{Max}\{b_c^T p + \delta^T |p|;\ A_c^T p - \Delta^T |p| \leq \overline{c}\}$;
if $\underline{f}$ is finite or $\overline{\varphi} = \infty$
then $[\underline{f}, \overline{\varphi}]$ is the range of the optimal value
end

6.3.7 The set of optimal solutions

6.3.8 Basis stability

6.3.9 Radii of (in)feasibility and (un)boundedness

6.4 Condition number for the optimal value

$$\gamma_\alpha(A,b,c) = \max\left\{\left|\frac{f(A',b',c') - f(A,b,c)}{f(A,b,c)}\right|;\ |A'-A| \le \alpha|A|, |b'-b| \le \alpha|b|, |c'-c| \le \alpha|c|\right\}$$

$$\gamma(A,b,c) = \lim_{\alpha \to 0_+} \frac{\gamma_\alpha(A,b,c)}{\alpha}$$

Theorem 253. *Let the basic optimal solution x^* of ... be nondegenerate, let the nonbasic relative cost coefficients be positive and let $c^T x^* \neq 0$. Then there holds*

$$\gamma(A,b,c) = \frac{|c|^T x^* + |b|^T |y^*| + |y^*|^T |A| x^*}{|c^T x^*|},$$

where y^* is the (unique) dual optimal solution.

Proof: [80], p. 106. □

6.5 Notes and references

In the last section we again give some additional notes and references to the material of the chapter.

Section 6.2. Duality theorem was published by Gale, Kuhn and Tucker [12] in 1951. The notion of it appeared earlier in an unpublished manuscript by J. von Neumann [117] which had evolved from his discussions with G. Dantzig in the autumn of 1947. Our formulation using functions $f(A,b,c)$ and $g(A,b,c)$ that may attain infinite values is untypical, but it allows to formulate the duality theorem as well as two its consequences in the form of a single equality (6.8).

Section 6.3.1. Although sensitivity analysis forms a standard part of linear programming textbooks, the interval linear programming problem was seemingly pioneered only in 1970 by Machost in his report [42]. His attempt to perform the simplex algorithm by replacing standard arithmetic operations by their interval arithmetic counterparts proved, however, to be ineffective; moreover, the report contained some errors ([5], p. 8). The first paper that handled the interval linear programming problem systematically was due to Krawczyk [37], followed by the state-of-the-art report by Beeck [5].

Section 6.3.2. As we have seen, Theorem 237 which gives formulae for computing $\underline{f}(\mathbf{A},\mathbf{b},\mathbf{c})$ and $\overline{f}(\mathbf{A},\mathbf{b},\mathbf{c})$ forms the cornerstone of our approach. The formula (6.23) for computing $\underline{f}(\mathbf{A},\mathbf{b},\mathbf{c})$, which is an easy consequence of description of the set of nonnegative weak solutions of $\mathbf{A}x=\mathbf{b}$ by the system of inequalities (5.40), (5.41), appeared in [65]. The formula (6.24) for $\overline{f}(\mathbf{A},\mathbf{b},\mathbf{c})$ was proved in the report [69] (although for finite values only) and republished by Mráz in his survey paper [50]. The general treatment which allows for infinite values of $\overline{f}(\mathbf{A},\mathbf{b},\mathbf{c})$ presented here is new.

Section 6.3.3. The lower bound $\underline{f}(\mathbf{A},\mathbf{b},\mathbf{c})$ can be computed as the optimal value of the problem

$$\mathrm{Min}\{\underline{c}^T x;\ \underline{A}x \le \overline{b}, \overline{A}x \ge \underline{b}, x \ge 0\},$$

where the number of constraints is doubled compared to the original problem. But Theorems 241 and 242 suggest that one might also succeed with solving a problem of the original size with properly parameterized constraints. Mráz's report [45] is dedicated to this question; these and related results are summed up in his *habilitationsschrift* [48]. The "duality theorem" 243 for $\underline{f}(\mathbf{A},\mathbf{b},\mathbf{c})$ was published in [67] using a burdensome notation that obscured its actual contents (i.e., optimization over strong solutions of $\mathbf{A}^T p \le \mathbf{c}$).

Section 6.3.4. While computing $\underline{f}(\mathbf{A},\mathbf{b},\mathbf{c})$ is easy, computation of $\overline{f}(\mathbf{A},\mathbf{b},\mathbf{c})$ is much more involved. Some partial results were achieved by Mráz in [46], [47], [49]. The treatment via $\overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c})$, as presented in this section, is new. NP-hardness of computing $\overline{f}(\mathbf{A},\mathbf{b},\mathbf{c})$ was proved in technical report [90].

Section 6.3.5. The problem of finite range was first addressed in [68]; the conditions given in this section are easy consequences of our previous results. The NP-hardness result (Theorem 252) was proved in [96]. Based on the ideas outlined in this section, a "condition number" for linear programs was proposed in [80].

Section 6.3.6. The algorithm, which reduces the complicated formula (6.24) to solving one nonlinear program, perhaps shows a promising way; but at the time this text was being written there was limited computational experience at our disposal only.

In our exposition we have left aside the difficult problem of determining (or bounding) the set of optimal solutions of all the linear programming problems contained in the family. A general treatment was done by Jansson [26], [27], [28], and computational aspects were studied by Jansson and Rump [29]. A special class is formed by so-called basis stable problems (where each problem in the family has a unique nondegenerate basic optimal solution with the same basis index set B) that were introduced by Krawczyk [37], characterized in [88], [69] and further studied by Koníčková [35], [36]. Basis stable problems are much more easy to handle; but checking basis stability was proved to be NP-hard in an unpublished manuscript by Rohn.

Related works include Bauch et al. [18], Filipowski [11], Nedoma [52], Ramík [59], [60], Renegar [62], Vatolin [115] and Vera [116].

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