Literatur

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Correction of Coefficients of the Input-Output Model

An economic system described by the input-output model

$$(E - A_0)\, x = y \tag{0}$$

(see e.g. [1]) is fully determined by the matrix $A_0 = (a^0_{ij})_{n \times n}$, but in practice it is very difficult to find out the exact values of its coefficients. Here we suppose that these values are unknown but $a^0_{ij} \in \langle \underline{a}_{ij}, \bar{a}_{ij} \rangle$, where $\underline{a}_{ij}$, $\bar{a}_{ij}$ are given real numbers, $\underline{a}_{ij} \leqq \bar{a}_{ij}$, for $i, j = 1, \dots, n$. Suppose it is known that a ***final demand y*** corresponds to a ***gross output x*** in the sense of model (0). The main problem of this paper is how to use this information for obtaining more precise estimations of a^0_{ij}, i.e. for the sharpening of intervals $\langle \underline{a}_{ij}, \bar{a}_{ij} \rangle$, $i, j = 1, \dots, n$. The presented results are contained in [2].

Denote $\underline{A} = (\underline{a}_{ij})_{n \times n}$, $\bar{A} = (\bar{a}_{ij})_{n \times n}$, $\langle \underline{A}, \bar{A} \rangle = \{A \mid \underline{A} \leqq A \leqq \bar{A}\}$ and let $M = \{A \mid A \in \langle \underline{A}, \bar{A} \rangle, (E - A)\, x = y\}$. Now the problem under consideration can be formulated as the problem of computing numbers

$$\tilde{a}_{ij} = \max\{(A)_{ij} \mid A \in M\}\,, \qquad \underset{\sim}{a}_{ij} = \min\{(A)_{ij} \mid A \in M\}\,, \qquad i, j = 1, \dots, n\,.$$

Let $\underset{\sim}{A} = (\underset{\sim}{a}_{ij})_{n \times n}$, $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$. It is obvious that $A_0 \in \langle \underset{\sim}{A}, \tilde{A} \rangle \subset \langle \underline{A}, \bar{A} \rangle$.

In the sequel we assume that the following two conditions are satisfied:

(i) $x > 0$,

(ii) $(E - \bar{A})\, x \leqq y \leqq (E - \underline{A})\, x$.

Condition (i) requires all the sectors of the system to be producing and (ii) holds if and only if $M \neq \emptyset$. The "if" part is clear, because if $A \in M$, then $(E - \bar{A})\, x \leqq (E - A)\, x = y \leqq (E - \underline{A})\, x$, the "only if" part is proved below.

Theorem 1: *Let* (i) *and* (ii) *be satisfied. Then*

$$\tilde{a}_{ij} = \min\left(\bar{a}_{ij}; \frac{1}{x_j}\Big(x_i - y_i - \sum_{k \neq j} \underline{a}_{ik} x_k\Big)\right) \tag{1}$$

$$\underset{\sim}{a}_{ij} = \max\left(\underline{a}_{ij}; \frac{1}{x_j}\Big(x_i - y_i - \sum_{k \neq j} \bar{a}_{ik} x_k\Big)\right), \qquad i, j = 1, \dots, n\,. \tag{2}$$

Proof: We shall prove the statement (1) only, because the proof of (2) is quite analogous. For $l = 1, \dots, n$ put $t_l = \dfrac{((E - \underline{A})\, x - y)_l}{((\bar{A} - \underline{A})\, x)_l}$ if $((\bar{A} - \underline{A})\, x)_l \neq 0$ and $t_l = 0$ otherwise. It follows from (ii) that $t_l \in \langle 0, 1 \rangle$ for each l. Let $a^*_{lk} = \underline{a}_{lk} + t_l(\bar{a}_{lk} - \underline{a}_{lk})$, $l, k = 1, \dots, n$, and let $A^* = (a^*_{lk})_{n \times n}$. Then $A^* \in \langle \underline{A}, \bar{A} \rangle$ and $(E - A^*)\, x = y$, so that $A^* \in M$. Hence $M \neq \emptyset$.

Let $i, j \in \{1, \dots, n\}$. Put $\alpha_{ij} = \dfrac{1}{x_j}\Big(x_i - y_i - \sum_{k \neq j} \underline{a}_{ik} x_k\Big)$. Let $A = (a_{lk})_{n \times n} \in M$; then $(E - A)\, x = y$, thus $a_{ij} = \dfrac{1}{x_j}\Big(x_i - y_i - \sum_{k \neq j} a_{ik} x_k\Big) \leqq \alpha_{ij}$. Since $a_{ij} \leqq \bar{a}_{ij}$, it implies that $a_{ij} \leqq \min(\bar{a}_{ij}; \alpha_{ij})$, hence also

$$\tilde{a}_{ij} = \max\{(A)_{ij} \mid A \in M\} \leqq \min(\bar{a}_{ij}; \alpha_{ij})\,.$$

To complete the proof, it will suffice to find a matrix $A \in M$ such that $(A)_{ij} = \min(\bar{a}_{ij}; \alpha_{ij})$. We distinguish two cases:

1) Let $\alpha_{ij} \leqq \bar{a}_{ij}$. It follows from (ii) that $\underline{a}_{ij} \leqq \alpha_{ij}$, so that $\alpha_{ij} \in \langle \underline{a}_{ij}, \bar{a}_{ij} \rangle$. Put $a^1_{ij} = \alpha_{ij}$, $a^1_{ik} = \underline{a}_{ik}$ for $k \neq j$, $a^1_{lk} = a^*_{lk}$ for $l \neq i$, $k = 1, \dots, n$, and let $A^1 = (a^1_{lk})_{n \times n}$. Then $A^1 \in M$ and $(A^1)_{ij} = \alpha_{ij} = \min(\bar{a}_{ij}; \alpha_{ij})$.

2) Let $\bar{a}_{ij} < \alpha_{ij}$. For $t \in \langle 0, 1 \rangle$ define the real function

$$\varphi_{ij}(t) = \bar{a}_{ij} x_j + \sum_{k \neq j} \big(\underline{a}_{ik} + t(\bar{a}_{ik} - \underline{a}_{ik})\big)\, x_k\,.$$

Then $\varphi_{ij}(0) < x_i - y_i$, $\varphi_{ij}(1) = (\bar{A}x)_i \geqq x_i - y_i$, thus there exists a $\tau_{ij} \in \langle 0, 1 \rangle$ such that $\varphi_{ij}(\tau_{ij}) = x_i - y_i$. Now put $a^2_{ij} = \bar{a}_{ij}$, $a^2_{ik} = \underline{a}_{ik} + \tau_{ij}(\bar{a}_{ik} - \underline{a}_{ik})$ for $k \neq j$, $a^2_{lk} = a^*_{lk}$ for $l \neq i$, $k = 1, \dots, n$, $A^2 = (a^2_{lk})_{n \times n}$. Then $A^2 \in M$ and $(A^2)_{ij} = \bar{a}_{ij} = \min(\bar{a}_{ij}; \alpha_{ij})$, Q. E. D.

We say the ij-th coefficient (of the matrix A_0) is *corrected* if $a_{ij} < \underline{a}_{ij}$ or $\tilde{a}_{ij} < a_{ij}$, i.e. if $\tilde{a}_{ij} - \underline{a}_{ij} < a_{ij} - \underline{a}_{ij}$. Taking $p_{ij} = (a_{ij} - \underline{a}_{ij})\, x_j$, $i, j = 1, \ldots, n$, it follows from (1) and (2) that

$$\tilde{a}_{ij} < \bar{a}_{ij} \quad \text{if and only if} \quad p_{ij} > \big((E - \underline{A})\, x - y)\big)_i\,, \tag{3}$$

$$\underline{a}_{ij} < \underline{a}_{ij} \quad \text{if and only if} \quad p_{ij} > \big(y - (E - \bar{A})\, x\big)_i\,, \tag{4}$$

so that the ij-th coefficient is corrected if and only if

$$p_{ij} > \min\big(((E - \underline{A})\, x - y)_i, \qquad (y - (E - A)\, x)_i\big)\,. \tag{5}$$

Theorem 2: *Let* (i), (ii) *hold. Then*

a) *if* $\tilde{a}_{ij} < \bar{a}_{ij}$ *for some* i, j, *then* $\underline{a}_{ik} = \underline{a}_{ik}$ *for all* $k \neq j$,

b) *if* $\underline{a}_{ij} < \underline{a}_{ij}$ *for some* i, j, *then* $\tilde{a}_{ik} = a_{ik}$ *for all* $k \neq j$.

Proof: Suppose there exist $i, j, k, j \neq k$, such that $\tilde{a}_{ij} < a_{ij}$, $a_{ik} < \underline{a}_{ik}$. Using (3) and (4), we obtain

$$p_{ij} + p_{ik} > \big((\bar{A} - \underline{A})x\big)_i = \sum_l p_{il} \geqq p_{ij} + p_{ik}\,,$$

which is a contradiction.

Theorem 3: *Let* (i), (ii) *be satisfied. Then for* $i, j, k = 1, \ldots, n$,

$$0 < p_{ij} \leqq p_{ik} \Rightarrow \frac{\tilde{a}_{ik} - \underline{a}_{ik}}{a_{ik} - \underline{a}_{ik}} \leqq \frac{\tilde{a}_{ij} - \underline{a}_{ij}}{a_{ij} - \underline{a}_{ij}}. \tag{6}$$

Proof: We may suppose that $k \neq j$ and $\tilde{a}_{ij} - \underline{a}_{ij} < a_{ij} - \underline{a}_{ij}$, because the other cases are trivial. Then two cases are possible:

1) $\tilde{a}_{ij} < a_{ij}$. Then $p_{ik} \geqq p_{ij} > ((E - A)\, x - y)_i$, thus $\tilde{a}_{ik} < a_{ik}$ and according to Theorem 2, $\underline{a}_{ij} = \underline{a}_{ij}$, $\underline{a}_{ik} = \underline{a}_{ik}$. Now we have $(\tilde{a}_{ij} - \underline{a}_{ij})\, x_j = \big((E - A)\, x - y\big)_i = (\tilde{a}_{ik} - \underline{a}_{ik})\, x_k$, so that

$$\frac{\tilde{a}_{ik} - \underline{a}_{ik}}{a_{ik} - \underline{a}_{ik}} = \frac{((E - A)\, x - y)_i}{p_{ik}} \leqq \frac{((E - \underline{A})\, x - y)_i}{p_{ij}} = \frac{\tilde{a}_{ij} - \underline{a}_{ij}}{a_{ij} - \underline{a}_{ij}}.$$

2) $\underline{a}_{ij} < \underline{a}_{ij}$. In the similar manner we obtain $(\tilde{a}_{ij} - \underline{a}_{ij})\, x_j = \big(y - (E - \bar{A})\, x\big)_i = (\tilde{a}_{ik} - \underline{a}_{ik}) x_k$ and it again implies (6).

Note that the ij-th coefficient is corrected if and only if $\dfrac{\tilde{a}_{ij} - \underline{a}_{ij}}{a_{ij} - \underline{a}_{ij}} < 1$. Thus Theorem 3 implies that if $0 < p_{ij} \leqq p_{ik}$ and the ij-th coefficient is corrected, then the ik-th one is also corrected. This result can be used also in the following way: let

$$p_{ij_i} = \max_j p_{ij}, \qquad i = 1, \ldots, n;$$

then if the ij_i-th coefficient is not corrected, then no coefficient of the i-th row is corrected. Note also that if $0 < p_{ij} = p_{ik}$, then $\dfrac{\tilde{a}_{ij} - \underline{a}_{ij}}{a_{ij} - \underline{a}_{ij}} = \dfrac{\tilde{a}_{ik} - \underline{a}_{ik}}{a_{ik} - a_{ik}}$.

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Multicommodity Location Problems

Recently A. Warszawski has developed a procedure for the *multicommodity location problem with no capacity constraints on the supply sources*. The purpose of this contribution is to describe how this procedure can be extended to the problems with *nontrivial capacity constraints*.

The *single-commodity problem* involves locating of one or more supply sources within a set of given possible sites so as to minimize the sum of the setup costs and the transportation costs to a set of destinations with given demands. Denoting by m the number of possible sites, n the number of destinations, g_i the setup costs associated with placing the supply source at site i, A_i the capacity of the supply source if placed at site i, b_j the demand at