DUALITY IN INTERVAL LINEAR PROGRAMMING

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I. INTRODUCTION

An interval linear programming problem is a problem of the form

$$\max \left\{ c^T x \mid A^I x = b^I,\ x \geq 0 \right\}, \qquad (P)$$

where $A^I = \{ A \mid \underline{A} \leq A \leq \overline{A} \}$, $\underline{A}$ and $\overline{A}$ being m by n matrices satisfying $\underline{A} \leq \overline{A}$, whose rows are denoted by $\underline{a}_i$, $\overline{a}_i$ $(i = 1,\ldots, m)$, and $b^I = \{ b \mid \underline{b} \leq b \leq \overline{b} \}$, $\underline{b} = (\underline{b}_i)$ and $\overline{b} = (\overline{b}_i)$ being m-vectors with $\underline{b} \leq \overline{b}$. An n-vector x is said to be a solution to $A^I x = b^I$ if there are $A \in A^I$ and $b \in b^I$ such that $Ax = b$ holds. It is well-known (1) that nonnegative solutions of $A^I x = b^I$ can be described as nonnegative solutions of the

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system $\underline{A}x \leq \overline{b}$, $\overline{A}x \geq \underline{b}$, so that (P) is equivalent to the linear programming problem

$$\max\left\{ c^T x \mid \underline{A}x \leq \overline{b},\ -\overline{A}x \leq -\underline{b},\ x \geq 0\right\}. \qquad (P_o)$$

For recent results concerning (P), see (2). In the present paper, we give a duality theorem for interval linear programming and various forms of optimality criteria.

First we introduce some notations. Let $y = (y_i)$ be an m-vector; then by A_y we denote the m by n matrix whose i-th row is equal to $\underline{a}_i$ if $y_i \geq 0$ and is equal to $\overline{a}_i$ if $y_i < 0$ $(i = 1,\dots,m)$. Similarly, we denote by b_y the m-vector defined by $(b_y)_i = \overline{b}_i$ if $y_i \geq 0$ and $(b_y)_i = \underline{b}_i$ otherwise. It is easy to verify that $A^T y \geq A_y^T y$ for any $A \in A^I$ and $b^T y \leq b_y^T y$ for any $b \in b^I$. Moreover, for the m-vectors y^+, y^- defined by $(y^+)_i = \max\{y_i, 0\}$, $(y^-)_i = \max\{-y_i, 0\}$ $(i = 1,\dots,m)$, we have $A_y^T y = \underline{A}^T y^+ - \overline{A}^T y^-$ and $b_y^T y = \overline{b}^T y^+ - \underline{b}^T y^-$. In the sequel, we use the usual linear programming terminology, see (3).

II. DUALITY THEOREM

The dual problem to (P_o)

$$\min\left\{ \overline{b}^T u - \underline{b}^T v \mid \underline{A}^T u - \overline{A}^T v \geq c,\ u \geq 0,\ v \geq 0\right\} \qquad (D_o)$$

is closely connected with the problem

$$\min \left\{ b_y^T y \mid A_y^T y \geqslant c \right\}, \qquad (D)$$

as the following lemma shows:

<u>Lemma 2.1.</u> We have (i)-(v):

(i) (D_o) is feasible if and only if (D) is feasible.

(ii) (D_o) is unbounded if and only if (D) is unbounded.

(iii) If u,v is an optimal solution to (D_o), then $y = u - v$ is an optimal solution to (D).

(iv) If y is an optimal solution to (D), then $u = y^+$, $v = y^-$ is an optimal solution to (D_o).

(v) If both (D_o) and (D) have optimal solutions, then they have a common optimal value.

<u>Proof.</u> Let u,v be a feasible solution of (D_o) and let $y = u - v$. Then, we have $A_y^T y \geqslant \underline{A}^T u - \overline{A}^T v \geqslant c$, hence y is a feasible solution of (D) with $b_y^T y \leqslant \overline{b}^T u - \underline{b}^T v$, which proves the "only if" parts of (i), (ii). Conversely, let y_1 be a feasible solution of (D) and let $u_1 = y_1^+$, $v_1 = y_1^-$. Then $\underline{A}^T u_1 - \overline{A}^T v_1 = A_{y_1}^T y_1 \geqslant c$, hence u_1, v_1 is a feasible solution of (D_o) with $\overline{b}^T u_1 - \underline{b}^T v_1 = b_{y_1}^T y_1$, which completes the proof of (i), (ii). Thus if one of the problems (D_o), (D) has an optimal solution, then so does the second one. Let u,v and y_1 be optimal solutions of (D_o) and (D), respectively and let y and u_1, v_1 be defined as above. Then, we have $\overline{b}^T u_1 - \underline{b}^T v_1 = b_{y_1}^T y_1 \leqslant b_y^T y \leqslant \overline{b}^T u - \underline{b}^T v$ and the optimality of u,v implies $\overline{b}^T u_1 - \underline{b}^T v_1 = \overline{b}^T u - \underline{b}^T v$, hence u_1, v_1 is an optimal solution to (D_o), y is an optimal solution to (D) and (D_o), (D) have a common optimal value, which proves (iii), (iv), (v), Q.E.D.

With the help of the Lemma 2.1, we can prove the following duality theorem for the problems (P), (D):

<u>Theorem 2.1.</u> If both (P) and (D) are feasible, then they both have optimal solutions and have a common optimal value. If one of the problems (P), (D) is infeasible, then the second one is either infeasible, or unbounded.

<u>Proof.</u> If both (P) and (D) are feasible, then (P_o) and (D_o) are also feasible and the classical duality theorem (3) as applied to (P_o), (D_o) gives that they both have optimal solutions and have a common optimal value, hence so do (P) and (D) due to Lemma 2.1. If (P) is infeasible, then so is (P_o) and the duality theorem states that (D_o) is infeasible or unbounded, hence so is (D) in accordance with Lemma 2.1. For (D) infeasible, an analogous argument applies. Q.E.D.

<u>Note 2.1.</u> Theorem 2.1 implies that if x,y are feasible solutions of (P), (D), then $c^T x \leqslant b_y^T y$ and if $c^T x = b_y^T y$, then x, y are optimal solutions of the respective problems.

A Farkas-type theorem can be derived directly from the duality theorem.

<u>Theorem 2.2.</u> A system $A^I x = b^I$ has a nonnegative solution if and only if $A_y^T y \geqslant 0$ implies $b_y^T y \geqslant 0$ for any y.

<u>Proof.</u> Consider the two problems

$$\max \left\{ 0^T x \mid A^I x = b^I,\ x \geqslant 0 \right\} \qquad (P^o)$$

$$\min \left\{ b_y^T y \mid A_y^T y \geqslant 0 \right\}. \qquad (D^o)$$

If $A^I x = b^I$ has a nonnegative solution, then both (P^o) and (D^o) are feasible (0 is a feasible solution to (D^o)), hence the optimal value of (D^o) is 0 due to Theorem 2.1, so that any y with $A_y^T y \geqslant 0$ satisfies $b_y^T y \geqslant 0$. Conversely, if the condition of the theorem is met, then (D^o) is bounded, hence (P^o) is feasible, Q.E.D.

III. OPTIMALITY CRITERIA

In this section, we give various forms of optimality criteria for the problems (P), (D). First, we have an analogue to the classical case:

Theorem 3.1. Let x, y be feasible solutions of (P), (D). Then, they are optimal solutions of the respective problems if and only if

$$\begin{aligned} x^T(A_y^T y - c) &= 0 \\ y^T(A_y x - b_y) &= 0 \end{aligned} \qquad (1)$$

hold.

Proof. Define u,v by $u = y^+$, $v = y^-$. Then in view of the Lemma 2.1., x and y are optimal solutions of (P), (D) if and only if x and u,v are optimal solutions of (P_o), (D_o), the latter case being equivalent (see (3)) to

$$x^T(\underline{A}^T u - \overline{A}^T v - c) = 0$$
$$u^T(\underline{A}x - \overline{b}) + v^T(\underline{b} - \overline{A}x) = 0. \tag{2}$$

Thus to complete the proof, it suffices to verify that

$$x^T(\underline{A}^T u - \overline{A}^T v - c) = x^T(A_y^T y - c)$$
$$u^T(\underline{A}x - \overline{b}) + v^T(\underline{b} - \overline{A}x) = y^T(A_y x - b_y)$$

hold.

Note 3.1. The conditions (1) can be rewritten in an equivalent form

$$x_j > 0 \text{ implies } (A_y^T y)_j = c_j \quad (j = 1,\dots,n)$$
$$y_i \neq 0 \text{ implies } (A_y x)_i = (b_y)_i \quad (i = 1,\dots,m).$$

In fact, the first implication is obvious. If $y_i > 0$, then $u_i = (y^+)_i > 0$, and since x is a feasible solution of (P_o), (2) implies $(\underline{A}x)_i = \overline{b}_i$; if $y_i < 0$, then $v_i = (y^-)_i > 0$, hence $(\overline{A}x)_i = \underline{b}_i$, so that $(A_y x)_i = (b_y)_i$ in both the cases.

Theorem 3.2. Let $\underline{b} < \overline{b}$ and let both (P) and (D) be feasible. Then, they have a pair of optimal solutions x, y satisfying

$$x_j > 0 \text{ if and only if } (A_y^T y)_j = c_j \quad (j = 1,\dots,n) \tag{3.1}$$
$$y_i \neq 0 \text{ if and only if } (A_y x)_i = (b_y)_i \quad (i = 1,\dots,m). \tag{3.2}$$

__Proof.__ Since both (P_o) and (D_o) are feasible, according to a well-known theorem (see (3)) they have a pair of optimal solutions x and u,v satisfying: $x_j > 0$ iff $(\underline{A}^T u - \overline{A}^T v)_j = c_j$ $(j = 1,\dots,n)$, $u_i > 0$ iff $(\underline{A}x)_i = \overline{b}_i$ and $v_i > 0$ iff $(\overline{A}x)_i = \underline{b}_i$ $(i = 1,\dots,m)$. Assume that $u_i v_i > 0$ for some i; then, we have $0 \leqslant ((\overline{A} - \underline{A})x)_i = \underline{b}_i - \overline{b}_i < 0$, a contradiction. Hence the vector $y = u - v$ is an optimal solution to (D) satisfying $A_y^T y = \underline{A}^T u - \overline{A}^T v$, which immediately proves (3.1). If $y_i = 0$, then $u_i = v_i = 0$, hence $(\underline{A}x)_i < \overline{b}_i$, which means $(A_y x)_i < (b_y)_i$; the "only if" part of (3.2) follows from the previous note. Q.E.D.

Now we shall turn to another sort of optimality criteria. If $A \in A^I$ and $b \in b^I$, then the problem

$$\max\left\{ c^T x \,\middle|\, Ax = b,\ x \geqslant 0\right\} \tag{P_s}$$

is called a <u>subproblem</u> of the problem (P). Obviously, (P_s) has a dual problem

$$\min\left\{ b^T y \,\middle|\, A^T y \geqslant c\right\} . \tag{D_s}$$

A subproblem whose system of constraints has the form $A_z x = b_z$ for some m-vector z is called an <u>extremal subproblem</u> of (P). Thus the i-th row of an extremal subproblem of (P) has either the form $\underline{a}_i x = \overline{b}_i$ or the form $\overline{a}_i x = \underline{b}_i$ $(i = 1,\dots,m)$, hence (P) has at most 2^m mutually different extremal subproblems.

Theorem 3.3. Let x be an optimal solution of a subproblem (P_s) of (P). Then, it is also an optimal solution to (P) if and only if (D_s) has an optimal solution y satisfying $A_y^T y \geqslant c$ and $b^T y = b_y^T y$.

Proof. If x is an optimal solution to (P), then taking an arbitrary optimal solution y of (D) (which exists due to Theorem 2.1), we have $A^T y \geqslant A_y^T y \geqslant c$ and $b^T y \geqslant c^T x = b_y^T y \geqslant b^T y$, which means that y is an optimal solution of (D_s) with $b^T y = b_y^T y$. Conversely, if an optimal solution of (D_s) satisfies $A_y^T y \geqslant c$ and $b^T y = b_y^T y$, then it is a feasible solution of (D) with $c^T x = b_y^T y$, hence x is an optimal solution to (P) due to Note 2.1. Q.E.D.

Theorem 3.4. Let $\underline{b} < \overline{b}$ and let x be an optimal solution of an extremal subproblem (P_s) of (P). Then, x is also an optimal solution to (P) if and only if (D_s) has an optimal solution y satisfying $b^T y = b_y^T y$.

Proof. In view of Theorem 3.3, it will suffice to show that for any optimal solution y of (D_s), $b^T y = b_y^T y$ implies $A_y^T y \geqslant c$. In fact, if $y_i > 0$ for some i, then $b_i = \overline{b}_i$, hence the i-th row of A is $\underline{a}_i$; if $y_i < 0$, then $b_i = \underline{b}_i$ and the i-th row of A is $\overline{a}_i$. This gives $A_y^T y = A^T y \geqslant c$, Q.E.D.

A subproblem (P_s) of (P) is said to be an equivalent one if its set of optimal solutions is equal to that of (P). We have this characterization:

Theorem 3.5. Let $\underline{b} < \overline{b}$ and let (P) have an optimal solution. Then, (P) has an equivalent extremal subproblem if and only if (D) has an optimal solution y such that $y_i \neq 0$ $(i = 1,\dots,m)$.

Proof. Let x, u, v and y be defined as in the proof of Theorem 3.2. If $y_i = 0$ for some i, then $u_i = v_i = 0$, hence $(\underline{A}x)_i < \overline{b}_i$ and $(\overline{A}x)_i > \underline{b}_i$, so that the optimal solution x of (P) cannot be a feasible solution of any extremal subproblem of (P). Thus y is an optimal solution of (D) satisfying $y_i \neq 0$ $(i = 1,\dots,m)$. Conversely, if (D) has an optimal solution with this property, then Note 3.1. shows that $\max\{c^T x \mid A_y x = b_y,\ x \geqslant 0\}$ is the desired equivalent extremal subproblem of (P), which completes the proof.

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