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INPUT-OUTPUT MODEL WITH INTERVAL DATA[1]

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IN THIS PAPER, we deal with a modified input-output model of the form

$$(E-A^I)x^I=y^I,$$

where A^I is an interval matrix and x^I, y^I are interval vectors (i.e. their elements are intervals of real numbers). This model can be used, when some (or all) input-output coefficients are not exactly known and are estimated by intervals, to solve the two main problems: (α) given x^I, find y^I; (β) given y^I, find x^I. After giving some preliminary results, we show in Theorem 1 that the interval equation $(E-A^I)x^I=y^I$ can be in a certain sense replaced by a linear system. Using this result, in the next two theorems we give explicit solutions to the above problems (α), (β).

Throughout the paper, we shall use the following notation. If $\underline{B}=(\underline{b}_{ij})$, $\bar{B}=(\bar{b}_{ij})$ are matrices (or, especially, vectors) of the same size, we write $\underline{B}\leq\bar{B}$ if $\underline{b}_{ij}\leq\bar{b}_{ij}$ for all i,j. If $\underline{B}\leq\bar{B}$, then the set $[\underline{B},\bar{B}]=\{B|\underline{B}\leq B\leq\bar{B}\}$ is called matrix (vector) interval and is denoted by B^I. An interval $[\underline{B},\bar{B}]$ is said to be nonnegative if $\underline{B}\geq 0$, 0 being the zero matrix.

The aim of this paper is to study the basic input-output equation

$$(1)\qquad (E-A)x=y$$

(see [**1, 2**]) with data being given by intervals. We assume that the n by n matrix of input-output coefficients A is not exactly known and that instead only a matrix interval $A^I=[\underline{A},\bar{A}]$ containing A is given. Then the matrix $E-A$ lies in the matrix interval $[E-\bar{A}, E-\underline{A}]$, which will be denoted by $E-A^I$. Usually, the matrix A in (1) is assumed to be nonnegative and to have all the column sums less than 1. We shall require any matrix belonging to A^I to possess these two properties. This leads to the assumptions:

(i) $\qquad \underline{A}\geq 0,$

(ii) $\qquad \sum_{i=1}^{n}\bar{a}_{ij}<1 \qquad (j=1,\dots,n).$

In the sequel, we shall assume these two assumptions to be satisfied without mentioning them in the theorems. Using the matrix norm $\|A\|=\max_j\sum_i|a_{ij}|$, from (i), (ii) we have $\|\bar{A}\|<1$, so that, according to the well-known theorem, $(E-\bar{A})^{-1}=\sum_0^\infty\bar{A}^i\geq 0$. Thus for the matrix F, defined by

$$F=(E-\underline{A})(E-\bar{A})^{-1},$$

we have $F\geq E$, because $F=E+(\bar{A}-\underline{A})(E-\bar{A})^{-1}$. Let us denote by $\bar{A}_1$ the matrix formed from $\bar{A}$ by replacing its diagonal elements by zeros, and by $\bar{A}_2$ the diagonal matrix $\bar{A}-\bar{A}_1$. In the same way, we form the matrices $\underline{A}_1$, $\underline{A}_2$ from $\underline{A}$. Now, we can define the matrix

$$H=E-\underline{A}_2+F\bar{A}_1,$$

which will play an important role in the sequel. Since $\underline{a}_{jj}\leq\sum_{i=1}^n\bar{a}_{ij}<1$ for $j=1,\dots,n$, we have $E-\underline{A}_2\geq 0$, which, combined with the nonnegativity of F, yields $H\geq 0$.

Let x^I be a nonnegative gross output interval. Then

$$Y=\{y|y=(E-A)x,\ A\in A^I,\ x\in x^I\}$$

is the set of all possible net output vectors for A belonging to A^I and x varying in x^I. Since Y is generally not an interval, it cannot be simply equated to y^I. Therefore, we introduce the

[1] This paper was improved by the comments and suggestions made by two anonymous referees.

following definition: we say that nonnegative vector intervals x^I, y^I satisfy the equation

(2) $$(E-A^I)x^I = y^I$$

if y^I is the smallest interval containing Y (or, in other words, y^I is the interval hull of Y). In the following theorem, we give a description of solutions to (2).

THEOREM 1: *Nonnegative intervals $x^I=[\underline{x},\bar{x}]$, $y^I=[\underline{y},\bar{y}]$ satisfy (2) if and only if they have the form*

(3.1) $$\underline{y}\geq 0,$$

(3.2) $$\bar{y}=F\underline{y}+v,$$

(3.3) $$\underline{x}=(E-\bar{A})^{-1}(\underline{y}+\bar{A}_1u),$$

(3.4) $$\bar{x}=\underline{x}+u,$$

where, u, v are nonnnegative vectors satisfying the equation

(4) $$Hu=v.$$

PROOF: (a) First, we shall prove the "only if" part of the theorem. Let $x^I=[\underline{x},\bar{x}]$, $y^I=[\underline{y},\bar{y}]$ satisfy (2). Then, due to the definition, for $i=1,\ldots,n$ we have

(5.1) $$\underline{y}_i=\min\{((E-A)x)_i|A\in A^I,\ x\in x^I\}$$
$$=(1-\bar{a}_{ii})\underline{x}_i-\sum_{j\neq i}\bar{a}_{ij}\bar{x}_j=((E-\bar{A}_2)\underline{x}-\bar{A}_1\bar{x})_i$$

(5.2) $$\bar{y}_i=\max\{((E-A)x)_i|A\in A^I,\ x\in x^I\}$$
$$=(1-\underline{a}_{ii})\bar{x}_i-\sum_{j\neq i}\underline{a}_{ij}\underline{x}_j=((E-\underline{A}_2)\bar{x}-\underline{A}_1\underline{x})_i$$

which gives

(6.1) $$\underline{y}=(E-\bar{A}_2)\underline{x}-\bar{A}_1\bar{x},$$

(6.2) $$\bar{y}=(E-\underline{A}_2)\bar{x}-\underline{A}_1\underline{x}.$$

Take $u=\bar{x}-\underline{x}$, $v=\bar{y}-F\underline{y}$. Then u is nonnegative, (3.1), (3.2), (3.4) are clearly met and putting $\bar{x}=\underline{x}+u$, $\bar{y}=F\underline{y}+v$ into (6.1), (6.2), we obtain (3.3) and (4). The nonnegativity of v follows from (4) in view of the nonnegativity of H and u.

(b) To prove the "if" part of the theorem, we first observe that (3.1)–(3.4) imply $0\leq\underline{x}\leq\bar{x}$ and $0\leq\underline{y}\leq\bar{y}$ (since $F\geq E$), so that x^I, y^I are nonnegative intervals. Then, by substitution of the expressions $u=\bar{x}-\underline{x}$, $v=\bar{y}-F\underline{y}$ from (3.4), (3.2) into (3.3), (4), we obtain (6.1), (6.2), which in the light of (5.1), (5.2) means that x^I, y^I satisfy (2), and the proof is complete.

REMARK: The theorem shows that the interval equation (2) can be in a certain sense replaced by the linear system (4) for the "parameters" u, v of intervals x^I, y^I. Note that in case $\underline{A}=\bar{A}=A$ the matrix H is formed by absolute values of elements of $E-A$.

Now, we shall turn to the problems (α), (β) mentioned at the beginning of the paper. First, suppose that an interval x^I is given; the following theorem gives a necessary and sufficient condition for the nonnegative solvability of (2) in y^I.

THEOREM 2: *Let $x^I=[\underline{x},\bar{x}]$ be a nonnegative interval. Then (2) has a nonnegative solution y^I if and only if*

(7) $$(E-\bar{A}_2)\underline{x}\geq\bar{A}_1\bar{x}$$

holds. If this condition is met, then the solution $y^I = [\underline{y}, \bar{y}]$ *is given by*

(8.1) $\underline{y} = (E - \bar{A})\underline{x} - \bar{A}_1(\bar{x} - \underline{x}),$

(8.2) $\bar{y} = F\underline{y} + H(\bar{x} - \underline{x}).$

PROOF: If $y^I = [\underline{y}, \bar{y}]$ satisfies (2), then Theorem 1, (3.3) gives $(E - \bar{A})\underline{x} = \underline{y} - \bar{A}_1 u \geq \bar{A}_1(\bar{x} - \underline{x})$, which implies (7). Conversely, if (7) holds, then $\underline{y}$ defined by (8.1) is nonnegative and the interval $y^I = [\underline{y}, \bar{y}]$ satisfies (2) according to Theorem 1. *Q.E.D.*

Note that when $\underline{A} = \bar{A} = A$, $\underline{x} = \bar{x} = x$, condition (7) is reduced to the usual condition $(E - A)x \geq 0$. For the nonnegative solvability of (2) in the case when y^I is given, we need an additional assumption of invertibility of H:

THEOREM 3: *Let H be nonsingular and let $y^I = [\underline{y}, \bar{y}]$ be a nonnegative interval. Then (2) has a nonnegative solution $x^I = [\underline{x}, \bar{x}]$ if and only if*

(9) $H^{-1}\bar{y} \geq H^{-1}F\underline{y}$

holds. If this condition is met, then the solution is given by

(10.1) $\underline{x} = (E - \bar{A})^{-1}(\underline{y} + \bar{A}_1 H^{-1}(\bar{y} - F\underline{y})),$

(10.2) $\bar{x} = \underline{x} + H^{-1}(\bar{y} - F\underline{y}).$

PROOF: If (2) has a nonnegative solution x^I subject to y^I, then Theorem 1, (3.2) implies $H^{-1}\bar{y} = H^{-1}F\underline{y} + u \geq H^{-1}F\underline{y}$, which is (9). Conversely, if (9) holds, then taking $v = \bar{y} - F\underline{y}$, we have $u = H^{-1}v \geq 0$, hence also $v = Hu \geq 0$ and defining $\underline{x}, \bar{x}$ by (10.1), (10.2), we obtain a nonnegative solution $x^I = [\underline{x}, \bar{x}]$ of (2) according to Theorem 1. *Q.E.D.*

Note that condition (9) is stronger than $\bar{y} \geq \underline{y}$ since premultiplying it by the nonnegative matrix H yields $\bar{y} \geq F\underline{y}$ which in view of $F \geq E$ implies $\bar{y} \geq \underline{y}$. If $\underline{A} = \bar{A} = A$, $\underline{y} = \bar{y} = y$, then (9) holds trivially, because both sides of it are equal to $H^{-1}y$ in this case.

Some other results concerning the input-output model with interval data are given in **[3, 4]**.

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