Interval Linear Systems with Prescribed Column Sums

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ABSTRACT

Nonnegative solutions of an interval linear system $A^I x = b^I$ (A^I being an interval matrix and b^I an interval vector) with additional column sum restrictions of the type $\sum_i \alpha_{ij} a_{ij} \in [\underline{c}_j, \bar{c}_j]$ $\forall j$ are described by a system of linear inequalities with auxiliary variables.

INTRODUCTION

An interval linear system is a system of the form

$$A^I x = b^I \tag{0}$$

with $A^I = \{A \mid \underline{A} \leqslant A \leqslant \bar{A}\}$, $b^I = \{b \mid \underline{b} \leqslant b \leqslant \bar{b}\}$, where $\underline{A} = (\underline{a}_{ij})$, $\bar{A} = (\bar{a}_{ij})$ are m by n matrices, $\underline{b} = (\underline{b}_i)$, $\bar{b} = (\bar{b}_i)$ are m-vectors, and $\underline{A} \leqslant \bar{A}$, $\underline{b} \leqslant \bar{b}$ (the relation $\leqslant$ is to be understood componentwise). A nonnegative x is said to be a nonnegative solution to (0) if there are $A \in A^I$, $b \in b^I$ such that $Ax = b$ holds. Oettli and Prager [1] gave a general result concerning the solutions of (0) which implies as a special case that nonnegative solutions of (0) are precisely the nonnegative solutions of the system

$$\underline{A} x \leqslant \bar{b},$$

$$\bar{A} x \geqslant \underline{b}$$

(for a simple proof, see [2]). In the above definition of solution, the matrix A is not subject to any additional restriction, which, however, is not the case in some practical problems.

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EXAMPLE. In the input-output model

$$(E-A)x=y, \tag{M}$$

the number

$$1-\sum_i a_{ij},$$

which is equal to the sum of elements of the jth column of $E-A$, represents the value added per unit output in the jth sector of a national economy described by (M) (see [3]). Thus if the values of coefficients of A are not exactly known and (M) is to be treated as an interval linear system, then the bounds on values added must be taken into account and additional column sum restrictions of the type

$$1-\sum_i a_{ij}\in[\underline{c}_j,\bar{c}_j] \qquad \forall j$$

must be introduced.

In this paper, we shall consider the problem (0) with additional weighted sum constraints being imposed on the columns of the respective matrix, i.e. the problem of the form

$$\begin{aligned} &A^I x=b^I,\\ &\sum_{i=1}^{m}\alpha_{ij}a_{ij}\in[\underline{c}_j,\bar{c}_j] \qquad (j=1,\ldots,n), \end{aligned} \tag{1}$$

where the α_{ij}'s, $\underline{c}_j$'s, and $\bar{c}_j$'s are given (arbitrary) parameters and $\underline{c}_j\leqslant\bar{c}_j$ $(j=1,\ldots,n)$. Accordingly, a nonnegative x is said to be a nonnegative solution to (1) if there is a matrix $A\in A^I$, $A=(a_{ij})$, satisfying $\sum_i\alpha_{ij}a_{ij}\in[\underline{c}_j,\bar{c}_j]$ $(j=1,\ldots,n)$ and a vector $b\in b^I$ such that $Ax=b$ holds. In the next section, we describe the nonnegative solutions of (1) by a system of linear inequalities with auxiliary variables.

DESCRIPTION OF SOLUTIONS

Denote

$$P_j=\{i\,|\,\alpha_{ij}\geqslant 0,\ 1\leqslant i\leqslant m\},$$

$$N_j=\{i\,|\,\alpha_{ij}<0,\ 1\leqslant i\leqslant m\} \qquad (j=1,\ldots,n).$$

Then it can be easily seen that the conditions (we write $\Sigma_{P_j}, \Sigma_{N_j}$ instead of $\Sigma_{i\in P_j}, \Sigma_{i\in N_j}$, respectively)

$$\sum_{P_j} \alpha_{ij}\underline{a}_{ij} + \sum_{N_j} \alpha_{ij}\bar{a}_{ij} \leqslant \bar{c}_j,$$
$$\sum_{P_j} \alpha_{ij}\bar{a}_{ij} + \sum_{N_j} \alpha_{ij}\underline{a}_{ij} \geqslant \underline{c}_j \qquad (j=1,\dots,n) \tag{2}$$

are necessary for the existence of a nonnegative solution of (1). On the other hand, if for some j the inequalities

$$\sum_{P_j} \alpha_{ij}\bar{a}_{ij} + \sum_{N_j} \alpha_{ij}\underline{a}_{ij} \leqslant \bar{c}_j,$$
$$\sum_{P_j} \alpha_{ij}\underline{a}_{ij} + \sum_{N_j} \alpha_{ij}\bar{a}_{ij} \geqslant \underline{c}_j \tag{3}$$

hold, then the jth column sum condition does not impose any restriction, since we then have $\Sigma_i \alpha_{ij}a_{ij} \in [\underline{c}_j, \bar{c}_j]$ for any $A \in A^I$. Thus denote by L the set of indices j for which at least one of the inequalities in (3) does not hold. Further, for $i=1,\dots,m$, $j=1,\dots,n$ put

$$\delta_{ij} = \bar{a}_{ij} - \underline{a}_{ij},$$
$$\beta_{ij} = \alpha_{ij}\delta_{ij},$$
$$\underline{\gamma}_j = \sum_i \alpha_{ij}\underline{a}_{ij} - \bar{c}_j,$$
$$\bar{\gamma}_j = \sum_i \alpha_{ij}\underline{a}_{ij} - \underline{c}_j,$$

so that the δ_{ij}'s are nonnegative and $\underline{\gamma}_j \leqslant \bar{\gamma}_j$ for each j. Finally, let

$$K = \{(i,j) | j \in L, \beta_{ij} \neq 0\},$$
$$K^i = \{j | (i,j) \in K\} \qquad (i=1,\dots,m),$$
$$K_j = \{i | (i,j) \in K\} \qquad (j=1,\dots,n).$$

In the following theorem, nonnegative solutions of (1) are described by a system of linear inequalities involving some auxiliary variables y_{ij}, $(i,j) \in K$.

THEOREM. *Let the conditions* (2) *hold. Then a vector x is a nonnegative solution of* (1) *if and only if it is the x-part of a nonnegative solution of the system*

$$\left.\begin{aligned} \sum_{j=1}^{n} \underline{a}_{ij}x_j+\sum_{j\in K^i}\delta_{ij}y_{ij}&\leqslant \bar{b}_i,\\ \sum_{j\in K^i}\underline{a}_{ij}x_j+\sum_{j\notin K^i}\bar{a}_{ij}x_j+\sum_{j\in K^i}\delta_{ij}y_{ij}&\geqslant \underline{b}_i \qquad (i=1,\dots,m),\end{aligned}\right\} \tag{S1}$$

$$\left.\begin{aligned} \underline{\gamma}_j x_j+\sum_{i\in K_j}\beta_{ij}y_{ij}&\leqslant 0,\\ \bar{\gamma}_j x_j+\sum_{i\in K_j}\beta_{ij}y_{ij}&\geqslant 0 \qquad (j\in L),\end{aligned}\right\} \tag{S2}$$

$$y_{ij}\leqslant x_j \qquad [(i,j)\in K]. \tag{S3}$$

Proof.

(a) Let x be a nonnegative solution of $Ax=b$ for some $A\in A^I$, $b\in b^I$, the matrix $A=(a_{ij})$ satisfying the column sum conditions. For $(i,j)\in K$ define

$$y_{ij}=\frac{a_{ij}-\underline{a}_{ij}}{\delta_{ij}}x_j$$

($\delta_{ij}>0$, since $\beta_{ij}\neq 0$). Then the y_{ij}'s are nonnegative and (S3) obviously holds. Further, we have

$$a_{ij}x_j=\underline{a}_{ij}x_j+\delta_{ij}y_{ij} \tag{4}$$

for any $(i,j)\in K$. This gives

$$\sum_j \underline{a}_{ij}x_j+\sum_{j\in K^i}\delta_{ij}y_{ij}\leqslant \sum_j a_{ij}x_j\leqslant \bar{b}_i$$

and

$$\sum_{j\in K^i}\underline{a}_{ij}x_j+\sum_{j\notin K^i}\bar{a}_{ij}x_j+\sum_{j\in K^i}\delta_{ij}y_{ij}\geqslant \sum_j a_{ij}x_j\geqslant \underline{b}_i$$

for $i=1,\dots,m$, which is (S1). Now, let $j\in L$; if $(i,j)\notin K$, then either $\alpha_{ij}=0$ or $a_{ij}=\underline{a}_{ij}=\bar{a}_{ij}$, in both cases $\alpha_{ij}\underline{a}_{ij}=\alpha_{ij}a_{ij}$. Thus we have

$$\sum_i \alpha_{ij}\underline{a}_{ij}x_j+\sum_{i\in K_j}\beta_{ij}y_{ij}=\sum_i \alpha_{ij}a_{ij}x_j\in[\underline{c}_j x_j,\bar{c}_j x_j],$$

which gives (S2). Hence x and the y_{ij}'s satisfy (S1)–(S3).

(b) Conversely, let (S1)–(S3) hold for some nonnegative x and y_{ij} $[(i,j)\in K]$. We shall show that x is a solution to (1). The conditions (2) imply the existence of a matrix $A^0\in A^I$, $A^0=(a_{ij}^0)$, with $\sum_i \alpha_{ij}a_{ij}^0\in[\underline{c}_j,\bar{c}_j]$ $(j=1,\dots,n)$. Define matrices $\underline{A}^1=(\underline{a}_{ij}^1)$, $\bar{A}^1=(\bar{a}_{ij}^1)$ as follows:

$$\underline{a}_{ij}^1=\bar{a}_{ij}^1=a_{ij}^0 \qquad \text{if} \quad x_j=0,$$

$$\underline{a}_{ij}^1=\bar{a}_{ij}^1=\underline{a}_{ij}+\frac{y_{ij}}{x_j}\delta_{ij} \qquad \text{if} \quad x_j>0 \text{ and } (i,j)\in K,$$

$$\underline{a}_{ij}^1=\underline{a}_{ij},\ \bar{a}_{ij}^1=\bar{a}_{ij} \qquad \text{if} \quad x_j>0 \text{ and } (i,j)\notin K.$$

Then both $\underline{A}^1$ and $\bar{A}^1$ belong to A^I due to (S3), $\underline{A}^1\leqslant\bar{A}^1$ and (S1) can be rewritten as

$$\underline{A}^1x\leqslant\bar{b},$$

$$\bar{A}^1x\geqslant\underline{b},$$

hence there is a matrix $A=(a_{ij})$, $\underline{A}^1\leqslant A\leqslant\bar{A}^1$, and a vector $b\in b^I$ such that $Ax=b$ holds. To complete the proof, it will suffice to show that

$$\sum_i \alpha_{ij}a_{ij}\in\left[\underline{c}_j,\bar{c}_j\right] \qquad (j=1,\dots,n). \tag{5}$$

If $j\notin L$, then (5) holds due to the definition of L. If $j\in L$ and $x_j=0$, then we have $\sum_i \alpha_{ij}a_{ij}=\sum_i \alpha_{ij}a_{ij}^0\in[\underline{c}_j,\bar{c}_j]$; finally, if $j\in L$ and $x_j>0$, then

$$\sum_i \alpha_{ij}a_{ij}x_j=\sum_i \alpha_{ij}\underline{a}_{ij}x_j+\sum_{i\in K_j}\beta_{ij}y_{ij}\in\left[\underline{c}_jx_j,\bar{c}_jx_j\right]$$

due to (S2), which implies (5). Hence x is a solution to (1). ∎

Corollary. *The set of nonnegative solutions of* (1) *is a convex polytope.*

Proof. Follows immediately from the fact that the set of nonnegative solutions of (1) is a projection of that of (S1)–(S3), which is a convex polytope. ∎

CONCLUDING REMARKS

(1) The system (S1)–(S3) has $2m+2|L|+|K|$ rows and $n+|K|$ columns ($|\cdots|$ denoting the number of elements), hence the size of $|K|$ is significant.

(2) Let $\underline{c}_j=\bar{c}_j$ for some $j\in L$. Then $\underline{\gamma}_j=\bar{\gamma}_j$, and denoting the common value by γ_j, the two corresponding inequalities in (S2) can be replaced by a single equation

$$\gamma_j x_j + \sum_{i\in K_j} \beta_{ij} y_{ij} = 0. \tag{6}$$

Moreover, if $\gamma_j=0$ and $\beta_{ij}>0$ for each $i\in K_j$, then $y_{ij}=0$ for $i\in K_j$, so that the system (S1)–(S3) can be reduced by dropping out the equation (6), the inequalities in (S3) with right-hand sides x_j, and the columns corresponding to variables y_{ij} $(i\in K_j)$, which can be formally done by putting

$$L:=L\setminus\{j\}$$

$$K:=K\setminus(K_j\times\{j\}).$$

(3) An optimization problem

$$\max_M c^T x,$$

where M is the set of nonnegative solutions of (1), can be solved as a linear programming problem

$$\max\{c^T x + 0^T y \,|\, (x,y) \text{ nonnegative solution of (S1)–(S3)}\}.$$

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