Dual Complementarity in Interval Linear Programming Problems

JIŘÍ ROHN

In certain practical linear programming problems, coefficients of the system of constraints are not exactly known. Assuming that interval estimations of these coefficients are given, we obtain an interval linear programming problem

$$\text{(P)} \qquad \max \{c^T x \mid A^I x = b^I, x \geqq 0\},$$

where $A^I = \{A \mid \underline{A} \leqq A \leqq \bar{A}\}$ and $b^I = \{b \mid \underline{b} \leqq b \leqq \bar{b}\}$, $\underline{A}$ and $\bar{A}$ being m by n matrices satisfying $\underline{A} \leqq \bar{A}$ and $\underline{b}$, $\bar{b}$ being two m-vectors with $\underline{b} \leqq \bar{b}$ (the inequality is to be understood componentwise). A nonnegative x is said to be a feasible solution of (P) if there are $A \in A^I$ and $b \in b^I$ such that $Ax = b$ holds. It is known that (P) is equivalent to the problem

$$(\mathrm{P}_0) \qquad \max \{c^T x \mid \underline{A}x \leqq \bar{b}, -\bar{A}x \leqq -\underline{b}, x \geqq 0\},$$

since the sets of feasible solutions of (P) and (P_0) are identical (see [2], [4]). The problem (P_0) has a dual problem

$$(\mathrm{D}_0) \qquad \min \{\bar{b}^T u - \underline{b}^T v \mid \underline{A}^T u - \bar{A}^T v \geqq c, u \geqq 0, v \geqq 0\}.$$

This paper is aimed at giving some rather theoretical results concerning complementarity properties of optimal solutions of (D_0); a feasible solution (u, v) of (D_0) is called a complementary one if $u^T v = 0$ holds. Below, after some preliminary results, we prove that if (D_0) has a finite optimum, then it has at least one complementary optimal solution (Theorem 5) and we examine necessary and sufficient conditions for any optimal solution of (D_0) to be a complementary one (Theorems 6, 8 and 9).

First, we give some notations. The i-th row of the nonnegative matrix $\bar{A} - \underline{A}$ is denoted by d_i and the i-th unit vector (whose i-th entry is 1 and the others are 0) is denoted by e_i $(i = 1, \ldots, m)$. For two m-vectors $u = (u_i)$, $v = (v_i)$, we define $\min(u, v)$ as the vector whose i-th entry is equal to $\min\{u_i, v_i\}$ $(i = 1, \ldots, m)$, and similarly for $\max(u, v)$. The set of optimal solutions of (D_0) is denoted by W.

Theorem 1. Let $(u, v) \in W$ and let vectors u^i, v^i be given by

$$\text{(1)} \qquad \begin{aligned} u^i &= \max(u - v_i e_i, 0) \\ v^i &= \max(v - u_i e_i, 0) \quad (i = 1, \ldots, m). \end{aligned}$$

Then, we have $(u^i, v^i) \in W$ for $i = 1, \ldots, m$.

Proof. Put $p = \min(u, v)$, $p = (p_i)$ and take an arbitrary i, $1 \leqq i \leqq m$. Then, we have $\underline{A}^T u^i - \bar{A}^T v^i = \underline{A}^T u - \bar{A}^T v + p_i d_i^T \geqq \underline{A}^T u - \bar{A}^T v \geqq c$, hence (u^i, v^i) is a feasible solution of (D_0) satisfying $\bar{b}^T u^i - \underline{b}^T v^i = \bar{b}^T u - \underline{b}^T v - p_i(\bar{b}_i - \underline{b}_i) \leqq \bar{b}^T u - \underline{b}^T v$, which in view of the optimality of (u, v) implies $(u^i, v^i) \in W$, Q.E.D.

Notice that if $u^T v > 0$, then $u_i v_i > 0$ for some i and we have $(u^i, v^i) \neq (u, v)$, hence Theorem 1 yields at least one new optimal solution of (D_0) in this case. On the other hand, if $u^T v = 0$, then $(u^i, v^i) = (u, v)$ for $i = 1, \ldots, m$.

In the following theorem, $|J|$ denotes the number of elements of J.

Theorem 2. Let $(u, v) \in W$ and let $J = \{i \mid u_i v_i > 0, 1 \leqq i \leqq m\}$. Then, we have $\dim W \geqq |J|$.

Proof. If $J = \emptyset$, then the conclusion is evident. Thus let $J \neq \emptyset$ and let $p = \min(u, v)$. Then for vectors u^i, v^i defined by (1) we have $(u^i, v^i) - (u, v) = -p_i(e_i, e_i)$, hence the vectors $(u^i, v^i) - (u, v)$, $i \in J$, are linearly independent which means that the convex hull of the $|J| + 1$ optimal solutions (u, v), (u^i, v^i), $i \in J$, which is a part of W, has dimension $|J|$. This gives $\dim W \geqq |J|$, Q.E.D.

Theorem 3. Let (D_0) have a unique optimal solution. Then it is a complementary one.

Proof. Assume that the unique optimal solution (u, v) of (D_0) satisfies $u^T v > 0$. Then Theorem 2 implies $\dim W \geqq 1$, which contradicts the uniqueness of optimal solution. Q.E.D.

The following theorem shows that any optimal solution of (D_0) can be easily transformed into a complementary one.

Theorem 4. Let $(u, v) \in W$ and let u_0, v_0 be given by

$$u_0 = \max(u - v, 0) \tag{2}$$

$$v_0 = \max(v - u, 0)\,.$$

Then, we have $(u_0, v_0) \in W$.

Proof. Define vectors ${}^i u, {}^i v$ $(i = 0, \ldots, m)$ with the help of (1) by induction as follows: ${}^0 u = u$, ${}^0 v = v$ and ${}^i u = ({}^{i-1}u)^i$, ${}^i v = ({}^{i-1}v)^i$, $(i = 1, \ldots, m)$. Then Theorem 1 gives by induction that $({}^i u, {}^i v) \in W$ for $i = 0, \ldots, m$. Since $({}^m u, {}^m v) = (u_0, v_0)$, we obtain $(u_0, v_0) \in W$, which completes the proof.

Theorem 5. Let (D_0) have a finite optimum. Then it has a complementary optimal solution.

The *proof* follows immediately from Theorem 4 since u_0, v_0 defined by (2) satisfy $u_0^T v_0 = 0$.

Next, we shall examine conditions for (D_0) to have complementary optimal solutions only. First, we shall give conditions in terms of dual optimal solutions (Theorem 6), then, after an auxiliary result, in terms of primal optimal solutions (Theorem 8).

Theorem 6. The following (i)–(iii) are mutually equivalent:

(i) Each optimal solution of (D_0) is a complementary one.
(ii) $(u')^T v'' = 0$ for any $(u', v') \in W$, $(u'', v'') \in W$.

(iii) For any i, $1 \leqq i \leqq m$, either (a) or (b) holds:

(a) $u_i = 0$ for any $(u, v) \in W$,

(b) $v_i = 0$ for any $(u, v) \in W$.

Proof. We prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): Assume that $(u')^T v'' > 0$ for some $(u', v') \in W$, $(u'', v'') \in W$ and put $u = \frac{1}{2}(u' + u'')$, $v = \frac{1}{2}(v' + v'')$. Then $(u, v) \in W$ and $u^T v \geqq \frac{1}{4}(u')^T v'' > 0$, which contradicts (i).

(ii) $\Rightarrow$ (iii): Assume that neither (a), nor (b) holds for some i. Then there is a $(u', v') \in W$ with $u'_i > 0$ and a $(u'', v'') \in W$ with $v''_i > 0$, hence $(u')^T v'' \geqq u'_i v''_i > 0$, a contradiction.

(iii) $\Rightarrow$ (i) is obvious.

Theorem 7. Let $W \neq \emptyset$ and $1 \leqq i \leqq m$. Then (i), (ii) are equivalent:

(i) $u_i v_i > 0$ for some $(u, v) \in W$.

(ii) $\underline{b}_i = \bar{b}_i$ and $d_i x = 0$ for any optimal solution x of (P_0).

Proof. (i) $\Rightarrow$ (ii): Let (i) hold and let x be an optimal solution of (P_0). Then, according to usual optimality conditions, we have $u^T(\underline{A}x - \bar{b}) = 0$ and $v^T(\bar{A}x - \underline{b}) = 0$, which implies $(\underline{A}x)_i = \bar{b}_i$ and $(\bar{A}x)_i = \underline{b}_i$, hence $d_i x = \underline{b}_i - \bar{b}_i$, the left-hand side of the last equation being nonnegative and the right-hand one being nonpositive. Thus $d_i x = 0$ and $\underline{b}_i = \bar{b}_i$, which proves (ii).

(ii) $\Rightarrow$ (i): According to a well-known theorem (see [1], p. 138), (P_0) and (D_0) have a pair of optimal solutions x and u, v such that $u_j > 0$ if and only if $(\underline{A}x)_j = \bar{b}_j$ and $v_j > 0$ if and only if $(\bar{A}x)_j = \underline{b}_j$ $(j = 1, \ldots, m)$. Let (ii) hold; then in view of the fact that x is a feasible solution of (P_0), we have $\bar{b}_i \geqq (\underline{A}x)_i = (\bar{A}x)_i \geqq \underline{b}_i = \bar{b}_i$, hence $(\underline{A}x)_i = \bar{b}_i$ and $(\bar{A}x)_i = \underline{b}_i$, which implies $u_i > 0$ and $v_i > 0$ and the proof is complete.

Theorem 8. Let $W \neq \emptyset$. Then, the following (i), (ii) are equivalent:

(i) Each optimal solution of (D_0) is a complementary one.

(ii) For any i with $\underline{b}_i = \bar{b}_i$, there is an optimal solution x of (P_0) with $d_i x > 0$.

The *proof* is obvious.

Theorem 8 gives two simple sufficient conditions.

Theorem 9. Let either (a) or (b) hold:

(a) $\underline{A} < \bar{A}$ and $0 \notin b^I$,

(b) $\underline{b} < \bar{b}$.

Then each optimal solution of (D_0) is a complementary one.

Proof. For $W = \emptyset$, there is nothing to prove. If $W \neq \emptyset$, then each of the conditions (a), (b) ensures the condition (ii) of Theorem 8 to be satisfied. Thus each optimal solution of (D_0) is a complementary one, Q.E.D.

Finally, we give a property of optimal solutions of (P_0) based on Theorem 7.

Theorem 10. Let

$$K = \{i \mid u_i v_i > 0 \text{ for some } (u, v) \in W,\ 1 \leqq i \leqq m\} .$$

Then each optimal solution of (P_0) belongs to the orthocomplement of the set $\{d_i \mid i \in K\}$.

Proof. The assertion obviously holds if $K = \emptyset$. Thus let $K \neq \emptyset$ and let x be an optimal solution of (P_0). Then according to Theorem 7 we have $d_i x = 0$ for any $i \in K$, which means that x belongs to the orthocomplement of the set $\{d_i \mid i \in K\}$. Q.E.D.

For some related results, see [3].

References

[1] Nikaido, H.: Convex Structures and Economic Theory. Academic Press, New York and London 1968.

[2] Oettli, W.: On the Solution Set of a Linear System with Inaccurate Coefficients. *J. SIAM Numer. Anal., Ser. B*, Vol. **2**, 1965, No. 1.

[3] Rohn, J.: Duality in Interval Linear Programming. (To appear in the proceedings of the Interval-Symposium 1980, Freiburg 1980.)

[4] Rohn, J.: Soustavy lineárních rovnic s intervalově zadanými koeficienty. *Ekonomicko-matematický obzor* **12**, 1976, pp. 311–315.

May 5, 1980.

RNDr. Jiří Rohn, CSc., Matematicko-fyzikální fakulta University Karlovy, Malostranské nám. 25, 118 00 Praha 1.

Resumé

DUÁLNÍ KOMPLEMENTARITA V ÚLOHÁCH INTERVALOVÉHO LINEÁRNÍHO PROGRAMOVÁNÍ

Jiří Rohn

V článku jsou uvedeny některé výsledky týkající se komplementarity optimálních řešení duální úlohy intervalového lineárního programování

$$(D_0) \qquad \min \{\bar{b}^T u - \underline{b}^T v \mid \underline{A}^T u - \bar{A}^T v \geqq c,\ u \geqq 0,\ v \geqq 0\}$$

(kde $\underline{b} \leqq \bar{b}$, $\underline{A} \leqq \bar{A}$ a řešení (u, v) se nazývá komplementární, jestliže $u^T v = 0$).

Mj. je ukázáno, že má-li úloha (D_0) optimální řešení, potom má i komplementární optimální řešení a jsou uvedeny nutné a postačující podmínky pro to, aby každé optimální řešení úlohy (D_0) bylo komplementární.