AN ALGORITHM FOR SOLVING INTERVAL LINEAR SYSTEMS AND INVERTING INTERVAL MATRICES

by

J. Rohn, Prague

1. Introduction and notations

Presented here is an algorithm for solving an interval linear system $A^I x = b^I$ with an $n \times n$ interval matrix $A^I = [A_c - \Delta, A_c + \Delta]$. The algorithm is performable if the spectral radius of the matrix $|A_c^{-1}|\Delta$ is less than 1 and computes the interval solution (or: the interval hull of the solution set) with arbitrary accuracy by solving iteratively p nonlinear $n \times n$ systems of the type $x = D_y|x| + d_y$, where $2 \leq p \leq 2^n$. If all the coefficients of A_c^{-1} are nonzero and A^I is sufficiently narrow, then $p \leq 2n$. The proof of the underlying theorem is placed at the end of the paper. The algorithm, if rearranged, gives an algorithm for computing the inverse of an interval matrix with arbitrary accuracy.

Several notations used throughout the paper are introduced here. The inequalities "$\leq$", "$<$" are to be understood componentwise. The ij-th coefficient of a matrix A is denoted by A_{ij} and $|A|$ stands for a matrix formed by the absolute values of the coefficients of A. Matrix and vector intervals are defined by

$$[A_c-\Delta, A_c+\Delta] = \{A \mid |A - A_c| \leq \Delta\}$$
$$[b_c-\delta, b_c+\delta] = \{b \mid |b - b_c| \leq \delta\}$$

respectively ($\Delta \geq 0$, $\delta \geq 0$). If X is a compact subset of R^n, we put $-X = \{-x \mid x \in X\}$ and define vectors min X, max X by

$$(\min X)_i = \min\{x_i \mid x \in X\}$$
$$(\max X)_i = \max\{x_i \mid x \in X\} \qquad (i=1,\ldots,n),$$

so that [min X, max X] is the narrowest interval containing X. Similarly if X is a set of matrices. If $y \in R^n$, then T_y denotes the diagonal matrix with diagonal y, i.e. $(T_y)_{jj} = y_j$ $(j=1,\ldots,n)$ and $(T_y)_{ij} = 0$ if $i \neq j$. The n-dimensional vector whose each entry is equal to 1 is denoted by e; thus $E = T_e$ is the unit matrix. For $x \in R^n$, we define a vector sgn x by

$$(\operatorname{sgn} x)_i = \begin{cases} 1 \text{ if } x_i \geq 0 \\ -1 \text{ otherwise} \end{cases} \qquad (i=1,\ldots,n),$$

so that $|\operatorname{sgn} x| = e$. If $z = \operatorname{sgn} x$, then $|x| = T_z x$.

2. Algorithm for solving interval linear systems

Consider an $n \times n$ interval linear system

$$A^I x = b^I \qquad (1)$$

where $A^I = [A_c-\Delta, A_c+\Delta]$, $b^I = [b_c-\delta, b_c+\delta]$, $\Delta \geq 0$, $\delta \geq 0$. Throughout the paper, we shall assume that the nonnegative matrix

$$D = |A_c^{-1}|\Delta \qquad (2)$$

satisfies

$$\rho(D) < 1, \qquad (3)$$

where ρ denotes the spectral radius. Under this assumption, A^I is regular (i.e. each $A \in A^I$ is regular) and the solution set of (1)

$$X = \{x \mid Ax = b,\ A \in A^I,\ b \in b^I\}$$

is compact (see [4], [6]). In this section, we give an algorithm for computing the interval solution $[\underline{x}, \overline{x}]$ of (1), where $\underline{x}$ and $\overline{x}$ are defined by

$$\underline{x} = \min X$$
$$\overline{x} = \max X.$$

Before doing so, we shall describe a construction of an auxiliary vector index set Y_o. Due to (3), the matrix $E - D$ is nonnegatively invertible. Define

$$C = D(E - D)^{-1} \tag{4}$$
$$\underset{\sim}{B} = A_c^{-1} - C|A_c^{-1}| \tag{5}$$
$$\widetilde{B} = A_c^{-1} + C|A_c^{-1}|, \tag{6}$$

then $C \geq 0$ and $\underset{\sim}{B} \leq \widetilde{B}$. Further define a matrix S by

$$S_{ij} = \begin{cases} 1 \text{ if } \underset{\sim}{B}_{ij} > 0 \\ -1 \text{ if } \widetilde{B}_{ij} < 0 \\ 0 \text{ otherwise} \end{cases} \qquad (i,j=1,\dots,n). \tag{7}$$

This being done, let

$$Y_i = \{y \in R^n \mid y_j = S_{ij}\ (S_{ij} \neq 0),\ |y_j| = 1\ (S_{ij}=0)\} \tag{8}$$

$(i=1,\dots,n)$, so that Y_i has 2^{p_i} elements, where p_i is the number of zeros in the i-th row of S. Finally, let

$$Y_o = \bigcup_{i=1}^{n} (Y_i \cup (-Y_i)). \tag{9}$$

Notice that Y_o depends only upon A^I, $Y_o = -Y_o$ and $|y| = e$ for each $y \in Y_o$, thus the number p of elements of Y_o is even and $p \leq 2^n$. For the description of the algorithm, we shall also need the vector

$$x_c = A_c^{-1} b_c \tag{10}$$

and matrices D_y and vectors d_y defined for each $y \in Y_o$ by

$$D_y = A_c^{-1} T_y \Delta$$
$$d_y = x_c + A_c^{-1} T_y \delta$$

(recall that T_y is the diagonal matrix with diagonal y).

We have this result:

<u>Theorem 1.</u> Let (3) hold. Then for each $y \in Y_o$, the sequence $\{x_y^m\}_{m=o}^{\infty}$ given by

$$\begin{aligned} x_y^o &= d_y \\ x_y^m &= D_y |x_y^{m-1}| + d_y \qquad (m=1,2,\ldots) \end{aligned} \tag{11}$$

tends to a point $x_y \in X$ and we have

$$\begin{aligned} \underline{x} &= \min\{x_y \mid y \in Y_o\} \\ \overline{x} &= \max\{x_y \mid y \in Y_o\}. \end{aligned} \tag{12}$$

The proof of this theorem will be carried out in section 4. In the sequel, we use the matrix norm $\|A\| = \max_i \sum_j |A_{ij}|$ and the vector norm $\|x\| = \max_i |x_i|$. Let

$$\gamma = \|C\| . \tag{13}$$

If $\gamma = 0$, then $\Delta = 0$ and (12) becomes simply

$$\underline{x} = \min\{d_y \mid y \in Y_o\}$$
$$\overline{x} = \max\{d_y \mid y \in Y_o\} .$$

Thus assume $\gamma > 0$. As an immediate consequence of Theorem 1, we obtain this algorithm ($\varepsilon > 0$ is a precision wanted):

<u>Algorithm 1</u> (solving interval linear systems).

1 Compute A_c^{-1} and $D, C, \underset{\sim}{B}, \widetilde{B}, S$ by (2), (4)-(7).

2 Construct Y_o by (8), (9). Let $Y_o = \{y^1, y^2, \ldots, y^p\}$.

3 $i:=1$, $\underline{x}:=x_c$, $\overline{x}:=x_c$ (10).

4 $y:=y^i$

5 Compute x_y^m $(m=0,1,\dots)$ by (11) until $\|x_y^m - x_y^{m-1}\| < \frac{\varepsilon}{\gamma}$ (13).

6 Set $\underline{x}:=\min\{\underline{x}, x_y^m\}$, $\overline{x}:=\max\{\overline{x}, x_y^m\}$.

7 $i:=i+1$

8 If $i \leq p$, go to 4. Otherwise stop! $[\underline{x}, \overline{x}]$ is the interval solution to (1) computed with precision ε (if the round-off errors are not taken into account).

We add a few remarks.

(a) For practical purposes, the condition (3) may be replaced by a more easily verifiable condition

$$\|D\| < 1 \qquad (14)$$

(since $\varrho(D) \leq \|D\|$). All the examples from the literature the author has tested satisfied (14).

(b) In some cases, the absolute value can be removed from (11). In fact, as it will be shown in the proof of Theorem 1, the inequality

$$|x_y^{m+p} - x_y^m| \leq C|x_y^m - x_y^{m-1}| \qquad (15)$$

holds for each $y \in Y_o$, $m \geq 1$, $p \geq 0$. Taking $m = 1$, we have $|x_y^{p+1} - d_y| \leq (C + E)|x_y^1 - x_y^o| \leq C|d_y|$ for each $p \geq 0$. Thus if d_y satisfies

$$C|d_y| < |d_y|,$$

then each x_y^p belongs to the same orthant as d_y, hence $|x_y^p| = T_z x_y^p$ where $z = \operatorname{sgn} d_y$ and (11) may be replaced by

$$x_y^o = d_y$$
$$x_y^m = D_y' x_y^{m-1} + d_y,$$

where $D_y' = D_y T_z$.

(c) Also from (15), taking $p \to \infty$, we get $|x_y - x_y^m| \leq C|x_y^m - x_y^{m-1}|$ which implies $\|x_y - x_y^m\| \leq \gamma \|x_y^m - x_y^{m-1}\|$. Thus the stopping rule used in the step 5 of the algorithm ensures $\|x_y - x_y^m\| < \varepsilon$.

(d) Contrary to $x_y \in X$, as stated in Theorem 1, we have generally $x_y^m \notin X$. In fact, it can be proved using the Oettli-Prager result [14] that $x_y^m \in X$ if and only if $\Delta|x_y^{m-1}| \leq \Delta|x_y^m|$ ($y \in Y_o$, $m \geqslant 1$).

(e) Assume that D satisfies a more strong condition than (14):

$$\|D\| < \frac{q}{q+Q} \qquad (16)$$

where $q = \min_{i,j} |A_c^{-1}|_{ij}$, $Q = \max_{i,j} |A_c^{-1}|_{ij}$. Then $C|A_c^{-1}| < |A_c^{-1}|$ and from (5)-(7) we get $S_{ij} \neq 0$ for each i,j, thus each Y_i consists of a single vector, namely $Y_i = \{(S_{i1},\dots,S_{in})^T\}$ (the upper T denotes transposition), where $S_{ij} = \operatorname{sgn}(A_c^{-1})_{ij}$ for each i,j. Hence $p \leq 2n$, so that at most 2n sequences are to be generated in the algorithm. The condition (16) is satisfied provided all the coefficients of A_c^{-1} are nonzero and A^I is "sufficiently narrow".

(f) Let (16) hold and let $A_c^{-1} > 0$. Then $CA_c^{-1} < A_c^{-1}$, which gives $(A_c - \Delta)^{-1} = \widetilde{B} > 0$ and $(A_c + \Delta)^{-1} \geqslant \underset{\sim}{B} > 0$, hence A^I is positively invertible due to the Kuttler's result [10,p. 240] . Moreover, $\underset{\sim}{B} > 0$ implies $S_{ij} = 1$ for each i,j. Thus we have $Y_1 = \dots = Y_n = \{e\}$, hence $Y_o = \{-e,e\}$, so that only two sequences are to be generated. It will be shown in section 4 that $\underline{x} = x_{-e}$, $\overline{x} = x_e$, hence $\underline{x}$ and $\overline{x}$ satisfy

$$\begin{aligned} \underline{x} &= x_c - A_c^{-1}(\Delta|\underline{x}| + \delta) \\ \overline{x} &= x_c + A_c^{-1}(\Delta|\overline{x}| + \delta). \end{aligned} \qquad (17)$$

For example, if $b_c - \delta \geqslant 0$, then $\underline{x} \geqslant 0$, $\overline{x} \geqslant 0$ and from (17) we obtain $\underline{x} = (A_c + \Delta)^{-1}(b_c - \delta)$, $\overline{x} = (A_c - \Delta)^{-1}(b_c + \delta)$; similarly if $b_c + \delta \leq 0$ or $|b_c| \leq \delta$ (cf. [3], [5]).

(g) Notice that $D_{-y} = -D_y$ and $d_{-y} = 2x_c - d_y$ for each $y \in Y_o$, which may be of some help in computations.

<u>Example 1.</u> We shall demonstrate the algorithm on the 2×2 example by Alefeld and Herzberger [2, p. 107], which in our notations has the form

$$A_c = \frac{1}{16}\begin{pmatrix} 24 & 3 \\ 8 & 19 \end{pmatrix}$$

$$\Delta = \frac{1}{16}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

$b_c = \frac{7}{8}e$ and $\delta = \frac{1}{8}e$. Here we have

$$A_c^{-1} = \frac{1}{27}\begin{pmatrix} 19 & -3 \\ -8 & 24 \end{pmatrix}$$

and $\|D\| = \frac{2}{27} < \frac{3}{27} = \frac{q}{q+Q}$, hence the condition (16) is met and from the sign pattern of A_c^{-1} we conclude $Y_o = \{y^1, y^2\}$, where $y^1 = (1,-1)^T$ and $y^2 = (-1,1)^T$. According to (11), the sequence $\{x^m_{y^1}\}$ is given by

$$x^o_{y^1} = \frac{1}{108}(67,40)^T$$

$$x^m_{y^1} = \frac{1}{216}\begin{pmatrix} 0 & 11 \\ 0 & -16 \end{pmatrix}\left|x^{m-1}_{y^1}\right| + x^o_{y^1} \qquad (m=1,2,\ldots)$$

and the sequence $\{x^m_{y^2}\}$ by

$$x^o_{y^2} = \frac{1}{108}(45,72)^T$$

$$x^m_{y^2} = \frac{1}{216}\begin{pmatrix} 0 & -11 \\ 0 & 16 \end{pmatrix}\left|x^{m-1}_{y^2}\right| + x^o_{y^2} \qquad (m=1,2,\ldots).$$

On a pocket calculator which rounds to seven decimals we obtained

$$x^7_{y^1} = x^6_{y^1} = (0.6379309,\ 0.3448276)^T$$

$$x^7_{y^2} = x^6_{y^2} = (0.3800000,\ 0.7199999)^T$$

which gives

$$\underline{x} = (0.3800000,\ 0.3448276)^T$$

$$\overline{x} = (0.6379309,\ 0.7199999)^T.$$

The exact values, given in [2], are $\underline{x} = (\frac{19}{50}, \frac{10}{29})^T$, $\overline{x} = (\frac{37}{58}, \frac{18}{25})^T$.

Examples 2. For the 2×2 examples by Barth and Nuding [3, p.118] and Hansen [8, p. 36] we have $\underset{\sim}{B} < 0$, $\widetilde{B} > 0$, hence $S = 0$. The 2×2 example by Nickel [12, p. 33] satisfies $S_{22} = 1$ and $S_{ij} = 0$ otherwise. In all three examples we thus have $p = 4$ and $Y_o = \{(1,1)^T, (1,-1)^T, (-1,1)^T, (-1,-1)^T\}$. This is due to the fact that the intervals are too wide compared with the values of coefficients of A_c^{-1}.

Example 3. The 4×4 example by J. Albrecht [1], studied also by Oettli [13] and Hansen [9] , satisfies (16) and we have $S_{14} = S_{41} = -1$ and $S_{ij} = 1$ otherwise, thus Y_o consists of 6 vectors $(1,1,1,-1)^T$, $(1,1,1,1)^T$, $(-1,1,1,1)^T$, $(-1,-1,-1,1)^T$, $(-1,-1,-1,-1)^T$, $(1,-1,-1,-1)^T$. We do not go into computational details here.

3. Algorithm for inverting interval matrices

Let $A^I = [A_c - \Delta\ , A_c + \Delta]$ be again an $n \times n$ interval matrix satisfying (3). The interval matrix $B^I = [\underline{B}, \overline{B}]$ defined by

$$\underline{B} = \min \{A^{-1} \mid A \in A^I\}$$
$$\overline{B} = \max \{A^{-1} \mid A \in A^I\}$$

is called the interval inverse of A^I. With Y_o being defined as before, from Theorem 1 we obtain this result (to be again proved in section 4):

Theorem 2. Let A^I satisfy (3). Then for each $y \in Y_o$, the sequence $\{B_y^m\}_{m=o}^{\infty}$ given by

$$B_y^o = A_c^{-1}$$
$$B_y^m = D_y |B_y^{m-1}| + A_c^{-1} \qquad (m=1,2,\dots) \qquad (18)$$

tends to a matrix B_y and we have

$$\underline{B} = \min \{ B_y \mid y \in Y_o \}$$
$$\overline{B} = \max \{ B_y \mid y \in Y_o \} . \qquad (19)$$

So we have

Algorithm 2 (inverting interval matrices).

1 Compute A_c^{-1} and $D, C, \underset{\sim}{B}, \widetilde{B}, S$ by (2), (4)-(7).

2 Construct Y_o by (8), (9). Let $Y_o = \{y^1, y^2, \dots, y^p\}$.

3 $i:=1$, $\underline{B}:=A_c^{-1}$, $\overline{B}:=A_c^{-1}$.

4 $y:=y^i$

5 Compute B_y^m $(m=0,1,\dots)$ by (18) until $\| B_y^m - B_y^{m-1} \| < \frac{\varepsilon}{\gamma}$ (13).

6 Set $\underline{B} := \min\{\underline{B}, B_y^m\}$, $\overline{B} := \max\{\overline{B}, B_y^m\}$.

7 $i:=i+1$

8 If $i \leq p$, go to 4. Otherwise stop! $[\underline{B}, \overline{B}]$ is the interval inverse of A^I computed with precision ε (if the round-off errors are not taken into account).

The stopping rule used in step 5 ensures $\| B_y - B_y^m \| < \varepsilon$ as before. If the condition (16) is satisfied, then again at most 2n sequences are to be generated.

Example 4. The 2×2 matrix by Moore [11, p.52], although a narrow one, does not satisfy (16) since A_c^{-1} contains zeros. We have $p = 4$, but the convergence is a fast one.

Example 5. The 3×3 matrix by Hansen [7, p.315] satisfies (16) and $Y_o = \{(1,-1,1)^T, (-1,1,-1)^T\}$, so that only two procedures are to be used.

4. Proofs

Proofs of both theorems are given in this section.

Proof of Theorem 1. Put $Y = \{ y\in R^n \mid |y| = e\}$, so that $Y_o \subset Y$, and extend the definitions of D_y, d_y and $\{x_y^m\}$ to the whole of Y. Let $y \in Y$. From (11), for each $m \geq 1$ we have $|x_y^{m+1} - x_y^m| \leq D|x_y^m - x_y^{m-1}|$ and by induction for each $p \geq 0$, $|x_y^{m+p} - x_y^m| \leq (D+\dots+D^p)|x_y^m - x_y^{m-1}| \leq C|x_y^m - x_y^{m-1}| \leq CD^m|d_y|$. Since $D^m \to 0$ due to (3), the sequence $\{x_y^m\}$ is a cauchian one, thus convergent; let $x_y^m \to x_y$. Taking $m \to \infty$ in (11), we see that x_y satisfies

$$x_y = D_y|x_y| + d_y \ . \tag{20}$$

Put $z = \operatorname{sgn} x_y$, so that $|x_y| = T_z x_y$. Then (20) gives

$$\begin{aligned} (A_c - T_y \Delta T_z)x_y &= b_c + T_y \delta \\ T_z x_y &\geq 0. \end{aligned} \tag{21}$$

Since $|T_y \Delta T_z| = \Delta$ and $|T_y \delta| = \delta$, we have $A_c - T_y \Delta T_z \in A^I$ and $b_c + T_y \delta \in b^I$, hence $x_y \in X$. Furthermore, (21) in the notations of [15] means that x_y solves the system $A_{-y,z}x = b_{-y}$, $x^z \geq 0$. Thus the assumptions of Theorem 1 of [15] are met and we obtain

$$\begin{aligned} \underline{x} &= \min \{ x_y \mid y\in Y\} \\ \overline{x} &= \max \{ x_y \mid y \in Y\} \ . \end{aligned} \tag{22}$$

Next we shall prove that for each i, i=1,...,n, we have

$$\bar{x}_i = (x_y)_i \quad \text{for some } y \in Y_i. \tag{23}$$

First consider the case $\delta > 0$. According to (22), $\bar{x}_i = (x_y)_i$ for some $y \in Y$. Assume for contrary $y \notin Y_i$, so that $S_{ij} = -y_j$ for some j, e.g. $S_{ij} = 1$, $y_j = -1$. From (21) we have $Ax_y = b_c + T_y\delta$ for some $A \in A^I$. Since $A^{-1} \in [\underset{\sim}{B}, \widetilde{B}]$ (as it can be easily verified), $S_{ij} = 1$ implies $A^{-1}_{ij} > 0$. Defining y' by $y'_j = -y_j$, $y'_k = y_k$ $(k \neq j)$, for the vector $x' = A^{-1}(b_c + T_{y'}\delta)$ we obtain $x' \in X$ and $x'_i > (x_y)_i = \bar{x}_i$, a contradiction. Similarly if $S_{ij} = -1$, $y_j = 1$. Hence $y \in Y_i$, which proves (23).

Second we prove (23) for the case of an arbitrary nonnegative δ. For $k = 1,2,\ldots$ define $\delta_k = \delta + \frac{1}{k}e$ (so that $\delta_k > 0$), $b^I_k = [b_c - \delta_k, b_c + \delta_k]$ and consider the interval solution $[\underline{x}^k, \bar{x}^k]$ of the system $A^I x = b^I_k$. According to what we have just proved we have $\bar{x}^k_i = x^k_i$, where x^k satisfies

$$x^k = D_{y_k}|x^k| + d_{y_k} + \frac{1}{k}A_c^{-1}T_{y_k}e \tag{24}$$

for some $y_k \in Y_i$. Since Y_i is finite, there exists a $y \in Y_i$ such that $y_k = y$ for infinitely many k. Thus the sequence $\{x^k\}_{y_k=y}$, whose all elements belong to the compact solution set of the system $A^I x = b^I_1$, contains a convergent subsequence $\{x^{k_j}\}$, $x^{k_j} \longrightarrow x^*$. Taking $k_j \longrightarrow \infty$, from (24) we obtain $x^* = D_y|x^*| + d_y$, which, if combined with (20), gives $|x^* - x_y| \le D|x^* - x_y|$, hence $(E - D)|x^* - x_y| \le 0$, and the nonnegative invertibility of $E - D$ implies $x^* = x_y$. Since $x^k_i \rightarrow \bar{x}_i$, we have $\bar{x}_i = (x_y)_i$, which gives (23).

Since $x_y \in X$ for each $y \in Y$, (23) implies

$$\bar{x}_i = \max\{(x_y)_i \mid y \in Y_i\}. \tag{25}$$

In a similar manner we would obtain

$$\underline{x}_i = \min\{(x_y)_i \mid y \in -Y_i\}. \tag{26}$$

Since $Y_i \cup (-Y_i) \subset Y_o$ for each i, (25) and (26) imply (12), which concludes the proof. ■

As the reader can see, we have proved slightly more than stated in the theorem. (17) is a consequence of (25), (26).

For the purposes of the <u>proof of Theorem 2</u>, let $(A)_j$ denote the j-th column of a matrix A. Let $1 \leq j \leq n$; obviously, we have $(\underline{B})_j = \underline{x}^j$, $(\overline{B})_j = \overline{x}^j$, where $[\underline{x}^j, \overline{x}^j]$ is the interval solution of the system

$$A^I x = e^I_j , \qquad (27)$$

where $e^I_j = [e_j, e_j]$, $e_j = (E)_j$. Let $(x^m_y)^j$, $(x_y)^j$ denote the vectors x^m_y, x_y from Theorem 1 for the system (27). Then from (11) we have

$$(x^o_y)^j = (A_c^{-1})_j$$

$$(x^m_y)^j = D_y \left|(x^{m-1}_y)^j\right| + (A_c^{-1})_j \qquad (m=1,2,\dots)$$

which compared with (18) shows that $\left\{(x^m_y)^j\right\}^{\infty}_{m=o}$ and $\left\{(B^m_y)_j\right\}^{\infty}_{m=o}$ satisfy the same recurrence, hence $(x^m_y)^j = (B^m_y)_j$ and thus also $(x_y)^j = (B_y)_j$ for each m,y,j. This proves the convergence and from Theorem 1 we have

$$(\underline{B})_j = \underline{x}^j = \min \left\{(x_y)^j \mid y \in Y_o\right\} = \min \left\{(B_y)_j \mid y \in Y_o\right\} ,$$

which proves $\underline{B} = \min \left\{B_y \mid y \in Y_o\right\}$. Similarly for $\overline{B}$. ■

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Author's address:

J. Rohn

Faculty of Mathematics and Physics

Charles University

Malostranské nám. 25

11800 Prague

Czechoslovakia