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PROOFS TO "SOLVING INTERVAL LINEAR SYSTEMS"

by

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The sole purpose of this paper consists in presenting the proofs to eleven theorems given in the author's paper "Solving interval linear systems" [0]. The reader is assumed to be familiar with that paper; notations, formulae and references introduced there are used here without further explanations.

1. Theorem 0

Theorem 0. Let for each $y \in Y$ the equation

$$A_{ye}x^1 - A_{yf}x^2 = b_y$$

have a nonnegative solution x_y^1, x_y^2. Then for each $A \in A^I$ and $b \in b^I$, the equation $Ax = b$ has a solution belonging to $\mathrm{Conv}\{x_y^1 - x_y^2;\ y \in Y\}$.

Comment. As it will be seen from the proof, the theorem is valid for arbitrary $n \times m$ interval matrices (if A_{yz} is defined by $A_{yz} = A_c - T_y \Delta T_z$, $y \in Y_n$, $z \in Y_m$). The proof is constructive: an algorithm for computing a solution to $Ax = b$ directly from the vectors $x_y^1 - x_y^2$ ($y \in Y$) is given below. For its description, we give two definitions. First we define by induction an ordering for each set $Y_j = \{y \in R^j;\ |y_k| = 1\ (k = 1,\dots,j)\}$ ($j \in N$): (i) the ordering of Y_1 is 1, -1; (ii) if $y_1,\dots,y_{2^j}$ is the ordering of Y_j, then $(y_1,1),\dots,(y_{2^j},1)$,

AMS Subject Classifications: 65G10, 90C33

$(y_1,-1),\dots,(y_{2^j},-1)$ is the ordering of Y_{j+1}. Second, given a sequence $a_1,\dots,a_{2m}$, then each pair a_k, a_{m+k} $(k = 1,\dots,m)$ is called a conjugate pair.

Algorithm (computing a solution to $Ax = b$).

Step 0. For each $y \in Y$ set $x_y = x_y^1 - x_y^2$, $r_y = Ax_y - b$ and order the pairs (x_y, r_y) in the ordering of Y.

Step 1. Set $j = n$.

Step 2. For each conjugate pair (x_y, r_y), $(x_{y'}, r_{y'})$ in the current sequence compute

$$\lambda = \begin{cases} (r_{y'})_j/(r_{y'} - r_y)_j & \text{if } (r_{y'})_j \neq (r_y)_j \\ 1 & \text{otherwise} \end{cases}$$

and set

$$x_y = \lambda x_y + (1 - \lambda)x_{y'}$$
$$r_y = \lambda r_y + (1 - \lambda)r_{y'} \ .$$

Step 3. Drop out the second half of the sequence.

Step 4. If there remains a single pair (x_y, r_y), terminate. x_y solves $Ax = b$ (and $r_y = 0$).

Step 5. Otherwise set $j = j - 1$ and go to step 2.

Proof. For the purposes of the proof, we shall extend the pairs (x_y, r_y) to quadruples (x_y, r_y, x_y^1, x_y^2), where x_y^1, x_y^2 have their original meaning in step 0 and are updated in step 2 by

$$x_y^1 = \lambda x_y^1 + (1 - \lambda)x_{y'}^1$$
$$x_y^2 = \lambda x_y^2 + (1 - \lambda)x_{y'}^2 \ .$$

From this we see that $x_y = x_y^1 - x_y^2$, $r_y = Ax_y - b$ hold throughout the algorithm. Below we shall show that each $\lambda \in [0,1]$ so that x_y^1, x_y^2 remain nonnegative throughout. We shall prove by induction on $j = n,\dots,1$ that after completing step 2 there always holds

$$(A_{ye}x_y^1 - A_{yf}x_y^2)_i = (b_y)_i \qquad (i = 1,\dots,j-1) \qquad (a)$$
$$(Ax_y)_i = b_i \qquad (i = j,\dots,n).$$

If $j = n$, then at the beginning of step 2 we have

$$A_{ye}x_y^1 - A_{yf}x_y^2 = b_y \qquad (b)$$

for each $y \in Y$ by assumption; if $j < n$, then for each $y \in Y$ corresponding to a quadruple in the current sequence we have at the beginning of step 2

$$(A_{ye}x_y^1 - A_{yf}x_y^2)_i = (b_y)_i \qquad (i = 1,\dots,j) \qquad (c)$$
$$(Ax_y)_i = b_i \qquad (i = j+1,\dots,n)$$

due to the inductive assumption. Notice that (b) is a special case of (c) for $j = n$; thus for each j we may assume (c) to hold at the beginning of step 2. Since $y_i = y'_i$ for each $i \neq j$ (by ordering), the updated values x_y^*, x_y^{1*}, x_y^{2*} of x_y, x_y^1, x_y^2 satisfy

$$(A_{ye}x_y^{1*} - A_{yf}x_y^{2*})_i = (b_y)_i \qquad (i = 1,\dots,j-1) \qquad (d)$$
$$(Ax_y^*)_i = b_i \qquad (i = j+1,\dots,n).$$

Since $y_j = 1$, $y'_j = -1$, we have

$$(r_y)_j = (A(x_y^1 - x_y^2) - b)_j \geq (A_{ye}x_y^1 - A_{yf}x_y^2 - b_y)_j = 0$$
$$(r_{y'})_j = (A(x_{y'}^1 - x_{y'}^2) - b)_j \leq (A_{y'e}x_{y'}^1 - A_{y'f}x_{y'}^2 - b_{y'})_j = 0.$$

If $(r_y)_j \neq (r_{y'})_j$, then from

$$\lambda = (Ax_{y'} - b)_j/(Ax_{y'} - Ax_y)_j$$

we get

$$(Ax_y^*)_j = b_j \; ; \qquad (e)$$

if $(r_y)_j = (r_{y'})_j$, then both the values are 0 and from $(Ax_y)_j = (Ax_{y'})_j = b_j$ we again obtain (e), which together with (d) gives (a). Further, $(r_y)_j \geq 0 \geq (r_{y'})_j$ implies $\lambda \in [0,1]$, hence $x_y^{1*} \geq 0$, $x_y^{2*} \geq 0$ and x_y^* is a convex combination of x_y, $x_{y'}$. This concludes the inductive proof; hence from (a) for $j = 1$ we obtain $Ax_y = b$, thus justifying step 4. Since in step 0 we begin with vectors

$x_y^1 - x_y^2$ $(y \in Y)$ and at each step 2 a convex combination of two previously computed vectors is taken, the final result must belong to $\mathrm{Conv}\left\{x_y^1 - x_y^2 \; ; \; y \in Y\right\}$, which completes the proof.

2. Theorems 1 and 2

We shall first prove the lemma; notice that assertion (i) is generalized here.

Lemma 1. Let A be a regular n×m matrix and let D_j be an n×m matrix whose all rows except the j-th are zero. Let $\alpha = 1 + (D_j A^{-1})_{jj}$. Then we have:

(i) $A + D_j$ is regular if and only if $\alpha \neq 0$; in this case,
$$(A + D_j)^{-1} = A^{-1} - \frac{1}{\alpha} A^{-1} D_j A^{-1},$$

(ii) if $\alpha \leq 0$, then $A + tD_j$ is singular for some $t \in (0,1]$.

Proof. Let $G = D_j A^{-1}$, then $A + D_j = (E + G)A$ and $\det(E + G) = \alpha$. Hence $A + D_j$ is regular iff $\alpha \neq 0$. Since $G^2 = (\alpha - 1)G$, we have
$$(A + D_j)(A^{-1} - \frac{1}{\alpha} A^{-1} G) = E - \frac{1}{\alpha} G + G - \frac{1}{\alpha} G^2 = E,$$
which proves (i). If $\alpha \leq 0$, then there is a $t \in (0,1]$ with $1 + tG_{jj} = 0$. Then $A + tD_j = (E + tG)A$ is singular since $\det(E + tG) = 1 + tG_{jj} = 0$.

Before proving theorems 1 and 2, we state this

Theorem A. Let A^I be regular. Then for each $A_1, A_2 \in A^I$, both $A_1 A_2^{-1}$ and $A_1^{-1} A_2$ are P-matrices.

Proof. 1) First we prove that all leading principal minors $m_1, \dots, m_n$ of $A_1 A_2^{-1}$ are positive. Put $D = A_1 - A_2$, so that $A_1 A_2^{-1} = E + DA_2^{-1}$, and let D^j $(j \in N)$ be the matrix whose first j rows are identical with those of D and the remaining ones are zero. Then

$m_j = \det(E + D^j A_2^{-1})$ for each j. We shall prove by induction that $m_j > 0$ $(j \in N)$.

1.1) $j = 1$: since $m_1 = \det(E + D^1 A_2^{-1}) = 1 + (D^1 A_2^{-1})_{11}$, the lemma implies $m_1 > 0$ for otherwise $A_2 + tD^1$ would be singular for some $t \in (0,1]$, a contradiction.

1.2) Let $m_{j-1} > 0$, $2 \leq j \leq n$. Consider the matrix

$$(E + D^j A_2^{-1})(E + D^{j-1} A_2^{-1})^{-1} = E + (D^j - D^{j-1}) A_2^{-1} (E + D^{j-1} A_2^{-1})^{-1}.$$

Taking determinants on both sides, we obtain

$$\frac{m_j}{m_{j-1}} = 1 + ((D^j - D^{j-1}) A_2^{-1} (E + D^{j-1} A_2^{-1})^{-1})_{jj} .$$

If the right-hand side were nonpositive, then according to lemma 1 the matrix $A_2 + D^{j-1} + t(D^j - D^{j-1}) = (E + D^{j-1} A_2^{-1} + t(D^j - D^{j-1}) A_2^{-1}) A_2$ would be singular for some $t \in (0,1]$, a contradiction. Hence

$$\frac{m_j}{m_{j-1}} > 0,$$

so that $m_j > 0$ due to the inductive assumption.

2) Second we prove that each principal minor of $A_1 A_2^{-1}$ is positive. Consider a principal minor formed from rows and columns $k_1, \dots, k_r$. Let P be any permutation matrix with $P_{k_j j} = 1$ $(j = 1, \dots, r)$. Then the above minor is equal to the r-th leading principal minor of $P^T A_1 A_2^{-1} P = (P^T A_1 P)(P^T A_2 P)^{-1}$. Since the interval matrix $\{P^T A P \; ; \; A \in A^I\}$ is regular, all leading principal minors of $(P^T A_1 P)(P^T A_2 P)^{-1}$ are positive due to 1).

3) To prove that $A_1^{-1} A_2$ is also a P-matrix, consider the interval matrix $(A^I)^T = \{A^T ; \; A \in A^I\}$; according to 2), its regularity implies that $(A_2^T)(A_1^T)^{-1} = (A_1^{-1} A_2)^T$ is a P-matrix, hence so is $A_1^{-1} A_2$.

Theorems 1 and 2 are now easy consequences of theorems 0 and A.

<u>Theorem 1.</u> A^I is regular if and only if A_y is a P- matrix for each $y \in Y$.

<u>Proof.</u> "Only if": follows from theorem A. "If": take $y \in Y$, $j \in N$. Then according to the result by Samelson, Thrall and Wesler [12], the linear complementarity problem $x^+ = A_y x^- + A_{ye}^{-1} e_j$ has a solution x_y, hence $A_{ye} x_y^+ - A_{yf} x_y^- = e_j$. Now the regularity follows from theorem 0 since $Ax = e_j$ has a solution for each $A \in A^I$ $j \in N$.

<u>Theorem 2.</u> Let A^I be regular. Then for each $y \in Y$, the equation

$$A_{ye} x^+ - A_{yf} x^- = b_y \tag{f}$$

has exactly one solution x_y. Moreover, we have $x_y \in X$ for each $y \in Y$ and $\text{Conv } X = \text{Conv} \{x_y \ ; \ y \in Y\}$; especially,

$$\underline{x} = \min \{x_y \ ; \ y \in Y\}$$

$$\overline{x} = \max \{x_y \ ; \ y \in Y\} \ .$$

<u>Proof.</u> From theorem 1 and from the result by Samelson et al. it follows that for each $y \in Y$ the equation $x^+ = A_y x^- + w_y$ has exactly one solution x_y, thus satisfying (f). From the equivalent equation $A_{yz} x_y = b_y$, $z = \text{sgn } x_y$, we see that $x_y \in X$. Now, according to theorem 0, for each $A \in A^I$, $b \in b^I$ the (unique) solution to $Ax = b$ belongs to $X_1 = \text{Conv} \{ x_y \ ; \ y \in Y\}$, hence $X \subset X_1$ and $\text{Conv } X \subset X_1$, implying $\text{Conv } X = X_1$. So $\underline{x} = \min X = \min \text{Conv } X = \min \{x_y \ ; \ y \in Y\}$; similarly for $\overline{x}$.

3. Theorems 3 and 5

Since theorem 5 is a direct consequence of theorem 3, it is placed here just after this theorem, the proof of theorem 4 to be given in the next section. Theorem 3 is proved here in a slightly weaker form.

Theorem 3. Let A^I be regular. Then for each $i \in N$ we have:

(i) $\underline{x}_i = (x_y)_i$ for some $y \in Y$ satisfying $(A_{yz}^{-1}T_y)_{i.} \leq 0^T$, where $T_z x_y \geq 0$,

(ii) $\overline{x}_i = (x_y)_i$ for some $y \in Y$ satisfying $(A_{yz}^{-1}T_y)_{i.} \geq 0^T$, where $T_z x_y \geq 0$.

Proof. We prove (i) only; (ii) is analogous. Let $i \in N$.

1) First we prove the theorem for the case $\delta > 0$. Theorem 2 assures the existence of a $y \in Y$ such that $\underline{x}_i = (x_y)_i$. Take a $j \in N$, set $y' = y - 2y_j e_j$ (i.e. $y'_j = -y_j$ and $y'_k = y_k$ for $k \neq j$) and consider the system $A_{y',z}x' = b_{y'}$, $z = \operatorname{sgn} x_y$. Since $A_{y',z} = A_{yz} + 2y_j T_{e_j} \Delta T_z$, we may use lemma 1 for evaluating $A_{y',z}^{-1}$, which after a lengthy computation gives $x' = x_y - \frac{2}{\alpha}(\Delta|x_y| + \delta)_j y_j (A_{yz}^{-1})_{.j}$. Since $\alpha > 0$, $\delta > 0$ and $x'_i \geq \underline{x}_i = (x_y)_i$, we obtain $y_j(A_{yz}^{-1})_{ij} \leq 0$. Since j was arbitrary, we get $(A_{yz}^{-1}T_y)_{i.} \leq 0^T$.

2) Next let $\delta \geq 0$. For $k = 1,2,\ldots$, let $\delta_k = \delta + \frac{1}{k}e > 0$, $b_k^I = [b_c - \delta_k, b_c + \delta_k]$ and let $[\underline{x}^k, \overline{x}^k]$ be the interval solution to $A^I x = b_k^I$. According to 1), for each k we have $\underline{x}_i^k = (x_{y_k}^k)_i$ (where $x_{y_k}^k$ denotes the vector x_{y_k} for the system $A^I x = b_k^I$), $(A_{y_k z_k}^{-1} T_{y_k})_{i.} \leq 0^T$, $T_{z_k} x_{y_k}^k \geq 0$. Since Y is finite and each $x_{y_k}^k$ belongs to the compact solution set of $A^I x = b_1^I$, there exist $y \in Y$, $z \in Y$ and an infinite subsequence $\{k_j\}$ such that $y_{k_j} = y$, $z_{k_j} = z$ for

each k_j and $\{x_y^{k_j}\}$ is convergent, $x_y^{k_j} \to x$. Since $A_{yz}x_y^{k_j} = b_y + \frac{1}{k_j} T_y e$, $T_z x_y^{k_j} \geq 0$ for each k_j, taking $k_j \to \infty$ we obtain $A_{yz}x = b_y$, $T_z x \geq 0$, hence $x = x_y$. Since $(x_y^{k_j})_i = \underline{x}_i^{k_j} \to \underline{x}_i$ and $(x_y^{k_j})_i \to (x_y)_i$, we have $\underline{x}_i = (x_y)_i$, $(A_{yz}^{-1}T_y)_{i.} \leq 0^T$, $T_z x_y \geq 0$.

Theorem 5. Under the above [in [0]] notations, we have

$$\underline{x}_i = \min\left\{(x_y)_i \ ;\ y \in -Y_i\right\} \qquad (i \in N)$$
$$\overline{x}_i = \max\left\{(x_y)_i \ ;\ y \in Y_i\right\} \quad ,$$

hence also

$$\underline{x} = \min\left\{x_y \ ;\ y \in Y_o\right\}$$
$$\overline{x} = \max\left\{x_y \ ;\ y \in Y_o\right\} \quad .$$

Proof. We shall confine ourselves only to the proof of the formula for $\underline{x}_i$. According to theorem 3, $\underline{x}_i = (x_y)_i$ for some $y \in Y$ satisfying $(A_{yz}^{-1})_{ij} y_j \leq 0$ $(j \in N)$. If $\underset{\sim}{B}_{ij} > 0$, then $(A_{yz}^{-1})_{ij} > 0$, hence $y_j = -1$; if $\widetilde{B}_{ij} < 0$, then $y_j = 1$. Thus $y \in -Y_i$.

Next we prove three unnumbered statements following theorem 5 in [0] . Let (8) hold; then for each $A \in A^I$, using $\Delta_o = A_c - A$, $|\Delta_o| \leq \Delta$, we may expand A^{-1} into Neumann series

$$A^{-1} = (A_c - \Delta_o)^{-1} = \left(\sum_{j=0}^{\infty} (A_c^{-1}\Delta_o)^j\right) A_c^{-1} ,$$

implying

$$|A^{-1} - A_c^{-1}| \leq \left(\sum_{1}^{\infty} D^j\right) |A_c^{-1}| = C|A_c^{-1}| .$$

From this we have $A^{-1} \in [A_c^{-1} - C|A_c^{-1}|,\ A_c^{-1} + C|A_c^{-1}|]$, an estimation of the interval inverse. Second, if $C|A_c^{-1}| < |A_c^{-1}|$, then A^I is inverse-stable. Finally we prove that (8) holds for positively (even nonnegatively) invertible matrices. Assume for contrary that $r = \rho(D) = \rho(A_c^{-1}\Delta) \geq 1$. Then, due to the Perron-Frobenius theorem, $A_c^{-1}\Delta x = rx$ for some real $x \neq 0$, hence $(A_c - \frac{1}{r}\Delta)x = 0$, implying singularity; thus $r < 1$.

4. Theorem 4

We shall prove the finiteness of algorithm 1 in a slightly more general form, proposed by M. Baumann, with step 0 being replaced by

Step 0*. Select a $z \in Y$.

(in [0] , we set $z = e$; here, z is arbitrary).

Theorem 4. Let A^I be regular. Then the algorithm [with step 0*] is finite for each $y \in Y$.

Proof. Let z_o be the initial vector z in step 0 .

1) First assume that $z_o = e$. Consider what is going on in the current step of the algorithm. Let $A_{yz}x = b_y$; put $x^1 = \frac{1}{2}(T_z x + x)$, $x^2 = \frac{1}{2}(T_z x - x)$, then $(x^1)^T x^2 = 0$, $x = x^1 - x^2$, $T_z x = x^1 + x^2$ (but, generally, x^1 and x^2 need not be nonnegative). Then $A_{yz}x = A_{ye}x^1 - A_{yf}x^2 = b_y$. Since $x^1_j = 0$ $(z_j = -1)$, $x^2_j = 0$ $(z_j = 1)$, we can see that x^1, x^2 is a basic solution to the system $x^1 = A_y x^2 + w_y$ with basic variables x^1_j $(z_j = 1)$, x^2_j $(z_j = -1)$. Moreover, since $k = \min \{ j;\ z_j x_j < 0 \} = \min \{ j;\ x^1_j < 0 \text{ or } x^2_j < 0 \}$, in the next step x^1_k enters and x^2_k leaves the basis if $z_k = -1$ and conversely if $z_k = 1$. Since we started with $x^1 = w_y$, $x^2 = 0$, the algorithm in terms of x^1, x^2 is precisely Murty's algorithm [8] for solving the linear complementarity problem $x^+ = A_y x^- + w_y$. This algorithm, as proved in [8], terminates (since A_y is a P-matrix) in a finite number of steps with $x^1 \geq 0$, $x^2 \geq 0$; thus $T_z x = x^1 + x^2 \geq 0$.

2) Let z_o be arbitrary. Together with our algorithm, started with z_o, consider a parallel algorithm applied to the system $\tilde{A}^I x = b^I$, $\tilde{A}^I = [A_c T_{z_o} - \Delta, A_c T_{z_o} + \Delta]$, for the same y with the initial vector $z = e$. We shall prove by induction that at each step

the current values z, $\tilde{z}$ of both algorithms satisfy $T_z = T_{\tilde{z}} T_{z_o}$. This is clear for the initial step, when $z = z_o$, $\tilde{z} = e$. Assuming validity in certain step, for the current solutions x, $\tilde{x}$ we have $\tilde{A}_{y\tilde{z}}\tilde{x} = (A_c T_{z_o} - T_y \Delta T_z T_{z_o})\tilde{x} = A_{yz} T_{z_o} \tilde{x} = b_y = A_{yz} x$, hence $x = T_{z_o}\tilde{x}$, thus also $k = \min \{ j;\ z_j x_j < 0 \} = \min \{ j;\ \tilde{z}_j \tilde{x}_j < 0 \}$, and the updated values z', $\tilde{z}'$ again satisfy $T_{z'} = T_{\tilde{z}'} T_{z_o}$. Since $\tilde{A}^I$ is regular, the second algorithm terminates due to 1) in a finite number of steps with $T_{\tilde{z}}\tilde{x} \geq 0$. Then $T_z x = T_{\tilde{z}}\tilde{x} \geq 0$, and the first algorithm terminates at the same step.

For a verification of algorithm 2, let $z' = z - 2z_k e_k$ be the updated vector z. Then $A_{yz'} = A_{yz} + 2z_k T_y \Delta T_{e_k}$, where $T_y \Delta T_{e_k}$ has all <u>columns</u> except the k-th zero. We shall use an easily verifiable fact that lemma 1 holds in the same form also in this case if α is given by $\alpha = 1 + (A^{-1} D_j)_{jj}$. Then for $p = 2z_k A_{yz}^{-1} T_y \Delta_{.k} + e_k$ and $x' = A_{yz'}^{-1} b_y$ we have

$$(A_{yz'}^{-1})_{k.} = \frac{1}{p_k}(A_{yz}^{-1})_{k.}$$

$$(A_{yz'}^{-1})_{j.} = (A_{yz}^{-1})_{j.} - \frac{p_j}{p_k}(A_{yz}^{-1})_{k.} = (A_{yz}^{-1})_{j.} - p_j (A_{yz'}^{-1})_{k.} \qquad (j \neq k)$$

$$x'_k = \frac{1}{p_k} x_k$$

$$x'_j = x_j - \frac{p_j}{p_k} x_k = x_j - p_j x'_k \qquad (j \neq k),$$

hence pivoting on p_k in the tableau brings A_{yz}^{-1}, x to $A_{yz'}^{-1}$, x'. If $p_k \leq 0$, then $A_{yz} + t(A_{yz'} - A_{yz})$ is singular for some $t \in (0,1]$.

Next we prove the statement in parentheses following (8). Assume $D_{jj} \geq 1$ for some $j \in N$. Put $y = -\mathrm{sgn}(A_c^{-1})_{j.}$, then $1 + (A_c^{-1} T_y \Delta T_{e_j})_{jj} = 1 - D_{jj} \leq 0$, hence $A_c + tT_y \Delta T_{e_j} \in A^I$ is singular for some $t \in (0,1]$.

5. Theorems 6-10

Theorem 6. A^I is regular if and only if for each $y \in Y$ the matrix equation

$$B = D_y|B| + A_c^{-1} \qquad (g)$$

has a solution B_y. If this condition is met, then B_y is unique for each $y \in Y$. Moreover, for each $A \in A^I$ there exist nonnegative diagonal matrices L_y $(y \in Y)$ satisfying $\sum_{y \in Y} L_y = E$ such that

$$A^{-1} = \sum_{y \in Y} B_y L_y \qquad (h)$$

holds; especially, we have

$$\underline{B} = \min\{B_y \ ; \ y \in Y\}$$
$$\overline{B} = \max\{B_y \ ; \ y \in Y\}. \qquad (j)$$

Proof. Let A^I be regular. Then, according to theorem 2, for each $y \in Y$ and each $j \in N$ there exists a unique x_{yj} such that

$$A_{ye}x_{yj}^{+} - A_{yf}x_{yj}^{-} = e_j \ . \qquad (k)$$

Defining B_y by $(B_y)_{.j} = x_{yj}$, we obtain

$$A_{ye}B_y^{+} - A_{yf}B_y^{-} = E,$$

which can be easily rearranged to (g). Conversely, let (g) have a solution B_y for each $y \in Y$. Defining x_{yj} by $x_{yj} = (B_y)_{.j}$, we have (k). Thus, according to theorem 0, $Ax = e_j$ has a solution for each $A \in A^I$, $j \in N$, implying regularity of A^I. Furthermore, again from theorem 0, $(A^{-1})_{.j}$, being a solution to $A^I x = e_j$, can be expressed as

$$(A^{-1})_{.j} = \sum_{y \in Y} \lambda_{yj} x_{yj}$$

with $\lambda_{yj} \geqslant 0$, $\sum_{y \in Y} \lambda_{yj} = 1$. Now if we define L_y by $(L_y)_{jj} = \lambda_{yj}$ $(j \in N)$ and $(L_y)_{ij} = 0$ for $i \neq j$, we obtain (h). Finally,

$$\underline{B}_{.j} = \min\{x_{yj}; \ y \in Y\} = \min\{(B_y)_{.j}; \ y \in Y\} \ ,$$

which gives (j); similarly for $\overline{B}$.

Theorem 7. Let A^I be regular and let $i,j \in N$. Then, we have:

(i) $\underline{B}_{ij} = (A_{yz}^{-1})_{ij}$ for some $y,z \in Y$ satisfying $(A_{yz}^{-1}T_y)_{i.} \leq 0^T$, $(T_z A_{yz}^{-1})_{.j} \geq 0$,

(ii) $\overline{B}_{ij} = (A_{yz}^{-1})_{ij}$ for some $y,z \in Y$ satisfying $(A_{yz}^{-1}T_y)_{i.} \geq 0^T$, $(T_z A_{yz}^{-1})_{.j} \geq 0$.

Proof. Take $i,j \in N$. Since $[\underline{B}_{.j}, \overline{B}_{.j}]$ is the interval solution to $A^I x = e_j$, from theorem 3 we get $\underline{B}_{ij} = (x_y)_i = (A_{yz}^{-1})_{ij}$ for some y satisfying $(A_{yz}^{-1}T_y)_{i.} \leq 0^T$, where $T_z x_y \geq 0$, i.e. $(T_z A_{yz}^{-1})_{.j} \geq 0$; analogously for $\overline{B}_{ij}$.

Theorem 8. A^I is regular if and only if for each $y \in Y$ and each $j \in N$ there exists a $z \in Y$ such that $(T_z A_{yz}^{-1})_{.j} \geq 0$.

Proof. Let A^I be regular. Then for each $y \in Y$ and each $j \in N$ there exists a x_{yj} such that (k) holds. Putting $z = \operatorname{sgn} x_{yj}$, we have $A_{yz} x_{yj} = e_j$, hence $(T_z A_{yz}^{-1})_{.j} = T_z x_{yj} \geq 0$. Conversely, letting $(B_y)_{.j} = (A_{yz}^{-1})_{.j}$ $(y \in Y, j \in N)$, we see that B_y satisfies (g); hence A^I is regular due to theorem 6.

Unfortunately, theorem 9 is not valid in the form given in [0] ; its "only if" part is not true. In order to reformulate it correctly, let us introduce the type of a matrix A to be a matrix Z satisfying $Z_{ij} = 0$ if $A_{ij} = 0$ and $Z_{ij} = \operatorname{sgn} A_{ij}$ otherwise. Then, by definition, A^I is inverse-stable iff all the matrices A^{-1}, $A \in A^I$, have the same type Z, $|Z| > 0$.

Theorem 9. A^I is inverse-stable if and only if all the A_{yz}^{-1}'s are of the same type Z, $|Z| > 0$.

Proof. Only the "if" part is to be proved; put $z_j = Z_{.j}$ $(j \in N)$, then from $(T_{z_j} A_{yz_j}^{-1})_{.j} > 0$ $(y \in Y, j \in N)$ we obtain regularity of A^I in view of theorem 8. Next, as in the proof of theorem 6, for a given

$A \in A^I$ we have

$$(A^{-1})_{.j} = \sum_{y \in Y} \lambda_{yj} x_{yj}$$

for some $\lambda_{yj} \geq 0$, $\sum_{y \in Y} \lambda_{yj} = 1$. Since $x_{yj} = (A^{-1}_{yz_j})_{.j}$, there holds

$$T_{z_j}(A^{-1})_{.j} = \sum_{y \in Y} \lambda_{yj} T_{z_j}(A^{-1}_{yz_j})_{.j} > 0$$

for each $j \in N$, hence A^{-1} is of the type Z.

<u>Theorem 10.</u> A regular interval matrix A^I is positively invertible if and only if $A^{-1}_{ef} > 0$.

<u>Proof.</u> By assumption, $(A_c + \Delta)^{-1} > 0$. In the light of the well-known Kuttler's theorem, it will suffice to show that $(A_c - \Delta)^{-1} > 0$. For each $j = 0,1,\dots,n$, define $A_j \in A^I$ by

$$(A_j)_{i.} = \begin{cases} (A_c - \Delta)_{i.} & (i = 1,\dots,j) \\ (A_c + \Delta)_{i.} & (i = j+1,\dots,n). \end{cases}$$

We shall prove by induction that $A_j^{-1} > 0$ for each j. Since $A_0 = A_c + \Delta$, the first step follows from the assumption. Now let $A^{-1}_{j-1} > 0$ $(j \leq n)$. Let $D_j = A_j - A_{j-1}$, then all the rows of D_j are zero except the j-th which is equal to $-2\Delta_{j.}$. Hence lemma 1 gives

$$A_j^{-1} = (A_{j-1} + D_j)^{-1} = A^{-1}_{j-1} - \frac{1}{\alpha} A^{-1}_{j-1} D_j A^{-1}_{j-1}.$$

Since $A^{-1}_{j-1} > 0$, $D_j \leq 0$ and $\alpha > 0$, we have $A_j^{-1} > 0$, concluding the induction. Hence $(A_c - \Delta)^{-1} = A_n^{-1} > 0$.

Notice that the proof goes through also for nonnegatively invertible interval matrices ($A^{-1} \geq 0$ for each $A \in A^I$) if the condition is changed to "$A^{-1}_{ef} \geq 0$".

References

[0] J. Rohn, Solving interval linear systems, in: "Herrn Professor Dr. Karl Nickel zum 60. Geburtstag gewidmet" (J. Garloff, A. Neumaier, D. Norbert, A. Schäfer, Eds.), Freiburg 1984, 419-432

[1] - [12] as in [0] .

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