SOME RESULTS ON INTERVAL LINEAR SYSTEMS

by

J. Rohn, Prague

This article can be viewed a complement to our earlier papers [6],[7],[8]. Therefore we do not repeat here the basic notations which are the same as before.

In the first two sections we give new proofs of some theorems from [6],[8], especially of cornerstone theorems 2, 4 in [6] that were proved in [7] using rather strong results on P-matrices; the proof given here is quite elementary. The third section brings an LCP procedure, although slow and lacking proper theoretical support, for computing $\underline{x}_i$, $\overline{x}_i$ directly without using the x_y's. Several new necessary and sufficient regularity conditions for interval matrices are presented in section 4, while the last section 5 deals with the problem of testing effectively singularity of interval matrices; two descent algorithms are described there, none of which, however, is general.

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1. Other proofs of some theorems

In this section, we give elementary proofs to some important theorems presented in [6] - [8].

We begin with a nonconstructive proof of theorem 0 in [6], based on this lemma, which is of independent interest (it deals only with real matrices and vectors):

Lemma 1.1. Let $A \in R^{m \times n}$, $b \in R^n$ and let for each $J \subset N$ there exist an $x_J \in R^n$ such that

$$(Ax_J)_j \leqslant b_j \quad (j \in J)$$
$$(Ax_J)_j \geqslant b_j \quad (j \notin J) .$$

Then the equation $Ax = b$ has a solution which is a convex combination of vectors x_J , $J \subset N$.

Proof. We shall prove using Farkas lemma that the system $A(\sum_{J\subset N} \lambda_J x_J) = b$, $\sum_{J\subset N} \lambda_J = 1$ has a nonnegative solution λ_J , $J \subset N$. In fact, let $p^T Ax_J + p_0 \geqslant 0$ for some $p \in R^m$, $p_0 \in R^1$ and each $J \subset N$. Define J_0 by $J_0 = \{j;\ p_j \geqslant 0\ ,\ j \in N\}$. Then $p^T b + p_0 \geqslant p^T Ax_{J_0} + p_0 \geqslant 0$, and we are done.■

Now, theprem 0 of [6] is a simple consequence of the lemma :

<u>Theorem 1.1.</u> Let for each $y \in Y$ the equation $A_{ye}x^1 - A_{yf}x^2 = b_y$ have a nonnegative solution x_y^1, x_y^2. Then for each $A \in A^I$ and $b \in b^I$, the equation $Ax = b$ has a solution belonging to $\mathrm{Conv}\{x_y^1 - x_y^2 \,;\, y \in Y\}$.

<u>Proof.</u> Let $A \in A^I$, $b \in b^I$, $J \subset N$. Set $x_J = x_y^1 - x_y^2$, where $y_j = -1$ if $j \in J$ and $y_j = 1$ otherwise. Then $(Ax_J - b)_j \leqslant (\overline{A}x_y^1 - \underline{A}x_y^2 - \underline{b})_j = (A_{ye}x_y^1 - A_{yf}x_y^2 - b_y)_j = 0$ for $j \in J$, and similarly $(Ax_J - b)_j \geqslant 0$ for $j \notin J$. Hence the assertion follows from lemma 1.1. ■

Our next aim is an elementary proof of theorems 2 and 4 in [6], again preceded by a lemma :

<u>Lemma 1.2.</u> Let A^I be regular and let $A^1x^1 = A^2x^2$ for some $A^1, A^2 \in A^I$, $x^1, x^2 \in R^n$. Then either (a) or (b) holds :

(a) $x_i^1 x_i^2 > 0$ for some i with $A^1_{.i} \neq A^2_{.i}$,

(b) $x_i^1 = x_i^2 = 0$ for each i with $A^1_{.i} \neq A^2_{.i}$, and $x_i^1 = x_i^2$ otherwise.

<u>Proof.</u> Assume for contrary that neither (a) nor (b) holds for some A^1, A^2, x^1, x^2, so that $x_i^1 x_i^2 \leqslant 0$ for each i with $A^1_{.i} \neq A^2_{.i}$ and there is a j such that $A^1_{.j} \neq A^2_{.j}$, $|x_j^1| + |x_j^2| > 0$. Set $J = \{i;\ A^1_{.i} \neq A^2_{.i}$ and $x_i^1 x_i^2 < 0\}$ and define A, x by

$$A_{.i} = \begin{cases} (x_i^1 / (x_i^1 - x_i^2))\, A^1_{.i} + (x_i^2 / (x_i^1 - x_i^2))\, A^2_{.i} & \\ & \text{if } i \in J \\ A^1_{.i} \quad \text{if } i \notin J \text{ , } x^1_{.i} \neq 0 & \\ A^2_{.i} \quad \text{otherwise} & \end{cases}$$

$$x_i = \begin{cases} x_i^1 - x_i^2 & \text{if } i \in J \\ x_i^1 - x_i^2 & \text{if } i \notin J \text{ , } x^1_{.i} \neq 0 \\ - x_i^2 & \text{otherwise} \end{cases}$$

$(i \in N)$. If $i \in J$, then $A_{.i}$ is a convex combination of $A^1_{.i}$, $A^2_{.i}$, hence $A \in A^I$ and $Ax = 0$. But $x_j \neq 0$, showing that A is singular, a contradiction. ■

We shall prove theorems 2 and 4 of [6] together.

<u>Theorem 1.2.</u> Let A^I be regular. Then for each $y \in Y$, the system $A_{yz}x = b_y$, $T_z x \geqslant 0$ has a unique solution $x_y \in X$, which can be computed by the following finite algorithm, and we have $\text{Conv } X = \text{Conv } \{x_y;\ y \in Y\}$.

<u>Algorithm 1.1.</u> (computing x_y for a given $y \in Y$).

0. Select a $z \in Y$ (arbitrarily).
1. Solve $A_{yz}x = b_y$.
2. If $T_z x \geqslant 0$, terminate with $x_y = x$.
3. Otherwise compute $k = \min \{j;\ z_j x_j < 0\}$.
4. Set $z_k := -z_k$ and go to step 1.

<u>Proof</u>. Finiteness: We shall prove the finiteness of the sequence of $k's$ defined in step 3 by induction, showing that each k can occur there at most 2^{n-k} times $(k = n,\dots,1)$.
Case $k = n$: assume n appears at least twice in the sequence, and let z,x,z',x' correspond to its two nearest occurrences. Then $z_j x_j \geqslant 0$, $z'_j x'_j \geqslant 0$ $(j < n)$, $z_n z'_n = -1$, $z_n x_n < 0$, $z'_n x'_n < 0$, hence $z_j z'_j x_j x'_j \geqslant 0$ for each $j \in N$. But according to lemma 1.2 there is an i with $z_i z'_i = -1, x_i x'_i > 0$, hence $z_i z'_i x_i x'_i < 0$, a contradiction.
Case $k < n$: let again z,x,z',x' correspond to two successive occurrences of k , so that $z_j z'_j x_j x'_j \geqslant 0$ for each $j \leqslant k$. Then lemma 1.2 implies the existence of an $i > k$ such that $z_i z'_i = -1$. Hence between any two occurrences of k there is an occurences of some $i > k$ in the sequence; this means that k cannot occur more than $1 + (2^{n-k-1} + \dots + 2 + 1) = 2^{n-k}$ times.

Existence: Due to what we have just proved, the algorithm gives after a finite number of steps an x with $A_{yz}x = b_y$, $T_z x \geqslant 0$.

Uniqueness: Assume $A_{yz}x = A_{yz'}x' = b_y$, $T_z x \geqslant 0$, $T_{z'}x' \geqslant 0$, $x \neq x'$. Then according to lemma 1.2 there exists an i with $z_i z'_i = -1$, $x_i x'_i > 0$, hence $z_i z'_i x_i x'_i < 0$ contrary to $z_i x_i \geqslant 0$, $z'_i x'_i \geqslant 0$.

Convex hull: Since $A_{yz}x_y = A_{ye}x_y^+ - A_{yf}x_y^- = b_y$ for each $y \in Y$, it follows from theorem 1.1 that for each $A \in A^I$, $b \in b^I$, the unique solution to $Ax = b$ belongs to $\mathrm{Conv}\{x_y;\ y \in Y\}$; thus $\mathrm{Conv}\ X \subset \mathrm{Conv}\ \{x_y;\ y \in Y\}$. The converse inclusion follows from the fact that $x_y \in X$ for each $y \in Y$. ■

In [7] , we proved this theorem with the help of the P-property of regular interval matrices (theorem A in [7]). Also this assertion can be proved by simple means:

Theorem 1.3. If A^I is regular, then for each $A_1, A_2 \in A^I$, both $A_1^{-1}A_2$ and $A_1A_2^{-1}$ are P-matrices.

Proof. According to the characterization of P-matrices arrived at independently by Fiedler, Pták [3] and Gale, Nikaido [4] , $A_1^{-1}A_2$ is a P-matrix if for each $x \neq 0$ (say, $x_j \neq 0$), there is an i with $x_i(A_1^{-1}A_2x)_i > 0$. Put $y = A_1^{-1}A_2x$, then $A_1y = A_2x$ and lemma 1.2 gives that either (a) $y_ix_i > 0$ for some i , or (b) $y = x$, hence $y_jx_j > 0$; we are done. Applying this result to the transpose $(A^I)^T$, we obtain that $A_1A_2^{-1}$ is also a P-matrix. ∎

At the end of this section, we note parenthetically that the mapping constructed in theorem 1.4 in [8] is a homeomorphism of X and [f,e] ; in fact, since it is a continuous one - to - one mapping of the compact set X onto the compact set [f,e] , its inverse mapping is also continuous.

2. Duality - type theorem for interval linear systems

In [6] , we presented theorem 3, which in many cases contributes to an essential reduction in the amount of computations. Meanwhile we realized that this theorem can be given a more symmetric form, leading to an interesting duality - type result. Consider a pair of optimization problems

$$\max\left\{c^Tx;\ A^Ix = b^I\ ,\ c \in c^I\right\} \tag{2.1}$$

$$\max\left\{b^Tp;\ A^{IT}p = c^I,\ b \in b^I\right\} \tag{2.2}$$

where, as usual, " $A^Ix = b^I$ " denotes " $Ax = b$ for some $A \in A^I$, $b \in b^I$ " , similarly for " $A^{IT}p = c^I$ " with $A^{IT} = \left\{A^T\ ;\ A \in A^I\right\}$. In the sequel, we speak of feasible and optimal solutions to (2.1), (2.2) in the sense usual in mathematical programming. We have this result :

Theorem 2.1. Let A^I be regular. Then the problems (2.1), (2.2) have a pair of optimal solutions x, p satisfying

$$A_{yz}x = b_y\ \ ,\ T_zx \geqslant 0 \tag{2.3}$$

$$A^T_{yz}p = c_z\ \ ,\ T_yp \geqslant 0 \tag{2.4}$$

for some $y,z \in Y$ and the optimal values of (2.1), (2.2) are equal : $c_z^Tx = b_y^Tp$.

Comment. It follows from (2.3), (2.4) that $x = x_y$, $p = p_z$, so that we could speak of optimal solutions x_y , p_z satisfying $T_zx_y \geqslant 0$, $T_yp_z \geqslant 0$; we preferred the form (2.3), (2.4) for easier references.

Proof. Assume first that $\delta > 0$. Since both X and c^I are compact, the problem (2.1) has an optimal solution x'. Set $z = \operatorname{sgn} x'$ and consider the problem $\max\left\{c_z^Tx;\ A^Ix=b^I\right\}$. According to theorem 1.2, this problem has an optimal solution x_y, $y \in Y$; obviously, $c_z^Tx_y = c_z^Tx'$ is the optimal value of (2.1). Now let p be the solution to $A_{yz}^T p = c_z$. To prove that $T_y p \geqslant 0$, consider the value of $c_z^T x_y =$ $= p^T A_{yz} x_y = p^T b_y = p^T b_c + \sum_i p_i y_i \delta_i$. If $y_j p_j < 0$ for some j , then setting $y'_j = -y_j$ and $y'_i = y_i$ for $i \neq j$, we would have $c_z^T x_y < p^T b_{y'} = c_z^T A_{yz}^{-1} b_{y'} \leq \max \{ c_z^T x \ ; \ A^I x = b^I \} = c_z^T x_y$, a contradiction. Hence $T_y p \geqslant 0$, p is an optimal solution to (2.2), and $c_z^T x_y = b_y^T p$.

Now let $\delta_i = 0$ for some i . For each $k = 1,2,\dots$, put $\delta_k = \delta + \frac{1}{k} e$, $b_k^I = [b_c - \delta_k \ , \ b_c + \delta_k]$ and consider the pair

$$\max \{ c^T x \ ; \ A^I x = b_k^I \ , \ c \in c^I \} \qquad (2.1_k)$$

$$\max \{ b^T p \ ; \ A^{IT} p = c^I \ , \ b \in b_k^I \} . \qquad (2.2_k)$$

The first part of the proof assures the existence of $y_k, z_k \in Y$ and of optimal solutions x^k, p^k to (2.1_k), (2.2_k) such that

$$A_{y_k z_k} x^k = b_{y_k} \ , \ T_{z_k} x^k \geqslant 0 \ , \ A_{y_k z_k}^T p^k = c_{z_k} \ , \ T_{y_k} p^k \geqslant 0 .$$

Then there are $y,z \in Y$ such that $y_{k_j} = y$, $z_{k_j} = z$ for infinitely many k_j . Letting $k_j \longrightarrow \infty$ and using the compactness of solution sets, we obtain (2.3), (2.4). ∎

Setting $c^I = [e_i, e_i]$, we obtain a neccessary optimality condition in the form given in [6] : $\bar{x}_i = (x_y)_i$ for some y,z, where $A_{yz}^T p = e_i$, $T_y p \geqslant 0$, $T_z x_y \geqslant 0$; hence $(A_{yz}^{-1} T_y)_{i\bullet} \geqslant 0^T$. This condition is, generally, not sufficient :

Example 2.1. Consider the problem $\max\{x_2;\ A^I x = b^I\}$ for the well-known example by Barth, Nuding [1, p. 118]. Here the conditions (2.3), (2.4) are satisfied for $y = z = (1,1)$, $x = (4,3)$, $p = (\frac{1}{2}, 1)$, which is not an optimal solution ($\max\ x_2 = 4$).

However, (2.3) and (2.4) become sufficient optimality conditions in case that the whole solution set X lies in a single orthant. Denote $R^n_z = \{x \in R^n\ ;\ T_z x \geqslant 0\}$. Then, we have :

Theorem 2.2. Let A^I be regular, let $X \subset R^n_z$ and let (2.3), (2.4) hold. Then x,p are optimal solutions of (2.1), (2.2).

Proof. Since $X \subset R^n_z$, we may use the description of X given in [8, corollary 1.4] to bring (2.1) to an equivalent form :

$$\max\{c_z^T x\ ;\ A_{ez}x \leq \overline{b},\ -A_{fz}x \leq -\underline{b}\ ,\ -T_z x \leq 0\}\ , \qquad (2.5)$$

which is a linear programming problem whose dual problem (in the LP sense) is

$$\min\{\overline{b}^T p_1 - \underline{b}^T p_2\ ;\ p_1^T A_{ez} - p_2^T A_{fz} - p_3^T T_z = c_z^T,\ p_1, p_2, p_3 \geqslant 0\}. \qquad (2.6)$$

We shall show that x,p satisfying (2.3), (2.4) also satisfy the complementarity slackness conditions for this pair of LP problems if we put $p_1 = p^+$, $p_2 = p^-$, $p_3 = 0$. In fact, x is obviously a feasible solution to (2.5), and since $p_1^T A_{ez} - p_2^T A_{fz} = p^T A_{yz} = c_z^T$, we have that p_1, p_2, p_3 is a feasible solution to (2.6). Now if $p_{1i} > 0$ for some i, then from $T_y p \geqslant 0$, $A_{yz}x = b_y$ it follows that $y_i = 1$ and $(A_{ez}x)_i = \overline{b}_i$. Similarly, if $p_{2i} > 0$, then $(A_{fz}x)_i = \underline{b}_i$. So the complementary slackness conditions are met, hence x is an optimal solution of (2.5), thus also of (2.1); and since (2.3), (2.4) imply $c_z^T x = b_y^T p$, we also get that p is an optimal solution of (2.2). ∎

We add a simple verifiable sufficient condition for $X \subset R_z^n$. Denote $D = |A_c^{-1}|\Delta$, $d = |A_c^{-1}|\delta$, $x_c = A_c^{-1}b_c$, $C = D(E-D)^{-1}$.

Theorem 2.3. Let $\rho(D) < 1$ and let

$$(E-D)^{-1}(|x_c|+d) < 2|x_c| \qquad (2.7)$$

hold. Then $X \subset R_z^n$, where $z = \operatorname{sgn} x_c$.

Proof. Let $(A_c - \Delta_0)x = b_c + \delta_0$, $|\Delta_0| \leq \Delta$, $|\delta_0| \leq \delta$. Then from $x = x_c + A_c^{-1}\delta_0 + \sum_{j=1}^{\infty} (A_c^{-1}\Delta_0)^j (x_c + A_c^{-1}\delta_0)$ we have $|x - x_c| \leq d + C(|x_c| + d) = (E-D)^{-1}(|x_c|+d) - |x_c| < |x_c|$, hence x and x_c lie in the same orthant R_z^n . ∎

3. LCP procedure for computing $\underline{x}_i$, $\overline{x}_i$ directly

In this section, we give a certain kind of solution of problem 3, set in [8, p. 55] ; namely, we present a procedure for computing $\underline{x}_i$, which avoids at all computations of the x_y's . Since $\overline{x}_i = \max_{x \in X} x_i = -\min_{x \in X}(-x_i)$, we shall deal only with computations of the $\underline{x}_i'$s .

It follows from [8, corollary 1.4] that $X = \{x;\ \underline{A}x^+ - \overline{A}x^- \leq \overline{b} ,\ -\overline{A}x^+ + \underline{A}x^- \leq -\underline{b}\}$. Hence for $\underline{x}_i = \min_{x \in X} x_i$ $(i \in N)$ we have

$$\begin{aligned} \underline{x}_i = \min \Big\{ e_i^T(x_1 - x_2);\quad & \underline{A}x_1 - \overline{A}x_2 \leq \overline{b} \\ & -\overline{A}x_1 + \underline{A}x_2 \leq -\underline{b} \\ & x_1^T x_2 = 0,\ x_1 \geq 0,\ x_2 \geq 0 \Big\}, \end{aligned} \tag{3.1}$$

which is a nonlinear programming problem. In order to enforce x_1, x_2 to be complementary, we may include the term $x_1^T x_2$ into the objective function with a sufficiently big positive penalty parameter M, thus obtaining a quadratic programming problem :

$$\begin{aligned} \underline{x}_i = \min \Big\{ e_i^T(x_1 - x_2) + Mx_1^T x_2\ ;\quad & \underline{A}x_1 - \overline{A}x_2 \leq \overline{b} \\ & -\overline{A}x_1 + \underline{A}x_2 \leq -\underline{b} \\ & x_1 \geq 0\ ,\ x_2 \geq 0 \Big\}. \end{aligned} \tag{3.2}$$

Denoting the Lagrange multipliers by p_1, p_2 , the Kuhn-Tucker optimality conditions for the problem (3.2) are

$$\begin{array}{rl} Mx_2 + \underline{A}^T p_1 - \overline{A}^T p_2 + e_i & \geq 0 \\ Mx_1 - \overline{A}^T p_1 + \underline{A}^T p_2 - e_i & \geq 0 \\ -\underline{A}x_1 + \overline{A}x_2 + \overline{b} & \geq 0 \\ \overline{A}x_1 - \underline{A}x_2 - \underline{b} & \geq 0 \\ x_1^T(Mx_2 + \underline{A}^T p_1 - \overline{A}^T p_2 + e_i) & = 0 \\ x_2^T(Mx_1 - \overline{A}^T p_1 + \underline{A}^T p_2 - e_i) & = 0 \\ p_1^T(-\underline{A}x_1 + \overline{A}x_2 + \overline{b}) & = 0 \\ p_2^T(\overline{A}x_1 - \underline{A}x_2 - \underline{b}) & = 0 \end{array} \tag{3.3}$$

where $x_1, x_2, p_1, p_2 \geqslant 0$ and $x_1^T x_2 = 0$. Hence putting $x = (x_1^T, x_2^T, p_1^T, p_2^T)^T$ and introducing

$$A = \begin{pmatrix} 0 & ME & \underline{A}^T & -\overline{A}^T \\ ME & 0 & -\overline{A}^T & \underline{A}^T \\ -\underline{A} & \overline{A} & 0 & 0 \\ \overline{A} & -\underline{A} & 0 & 0 \end{pmatrix} \qquad q = \begin{pmatrix} e_i \\ -e_i \\ \overline{b} \\ -\underline{b} \end{pmatrix}$$

we may rewrite (3.3) as a linear complementarity problem (LCP)

$$\begin{aligned} Ax + q &\geqslant 0 \\ x^T(Ax + q) &= 0 \\ x &\geqslant 0 \quad . \end{aligned} \qquad (3.4)$$

This problem can be solved by Lemke's complementary pivot algorithm (see [5, sect. 16.6] for its description). Although the objective function in (3.2) is nonconvex, so that (3.3) are necessary, but generally not sufficient optimality conditions, our experience gathered so far on a limited number of examples showed that the algorithm, when performable, always found the optimal solution (even at the example 2.1, where some other methods for solving (3.1) we tested failed due to jamming). The major setback with the problem (3.4) is the fact that A is of size $4n \times 4n$, which slows down the computations considerably. According to our experience, even moderate values of the penalty parameter M suffice to enforce $x_1^T x_2 = 0$ ($M \sim 10^1 \div 10^2$). If $x_1^T x_2 > 0$, the procedure must be repeated with an increased value of M (e.g. $M := 10M$).

When solving (3.4) using Lemke's algorithm, a phenomenon called ray termination may occur (no entry in the column being introduced into the basis is positive, so that pivot cannot be found), in which case the algorithm fails.
We shall show that ray termination cannot occur for a sufficiently large class of interval matrices having in each row at least one nondegenerate interval coefficient :

Theorem 3.1. Let A^I be a regular matrix satisfying $\Delta e > 0$. Then ray termination cannot occur for any $i \in N$.

Proof. Lemke's algoritm generates solutions to the system $z = Ax + x_0 e + q$, $z^T x = 0$, $z \geqslant 0$, $x \geqslant 0$, $x_0 \geqslant 0$ until $x_0 = 0$. Assume for contrary that ray termination occurs at some step; then from the current tableau we may construct (see [5] , p. 504) z, x, x_0 such that $z = Ax + x_0 e$, $z^T x = 0$, $(z,x,x_0) \geqslant 0$, $(z,x,x_0) \neq 0$. Then from $0 = x^T z = x^T Ax + x_0(x^T e) = 2Mx_1^T x_2 + x_0(x^T e)$ it follows due to the nonnegativity of both the right-hand side terms that $x_1^T x_2 = x_0 = 0$ (since $x = 0$ would imply cycling). Hence we have $Ax = z \geqslant 0$, $x_1^T x_2 = 0$, $x \neq 0$. Assume first that $x_1 = x_2 = 0$, so that $p_1 \neq 0$ or $p_2 \neq 0$. Then from $\underline{A}^T p_1 - \overline{A}^T p_2 \geqslant 0$, $-\overline{A}^T p_1 + \underline{A}^T p_2 \geqslant 0$ we obtain $\underline{A}^T p_1 - \overline{A}^T p_2 = \overline{A}^T p_1 - \underline{A}^T p_2$, hence $\Delta^T(p_1 + p_2) = 0$. Since $\Delta^T \geqslant 0$, $p_1 + p_2 \geqslant 0$ and $(p_1 + p_2)_i > 0$ for some i , we have $\Delta_{i.} = 0^T$ contrary to the assumption $\Delta e > 0$. Hence $x_1 \neq 0$ or $x_2 \neq 0$. Then from $Ax \geqslant 0$ we have $-\underline{A}x_1 + \overline{A}x_2 \geqslant 0$, $\overline{A}x_1 - \underline{A}x_2 \geqslant 0$ which in the light of corollary 1.4 in [8] means that $A(x_1 - x_2) = 0$ for some $A \in A^I$. Since $x_1 - x_2 \neq 0$, it implies singularity of A^I, a contradiction. Hence ray termination cannot occur. ■

Example 3.1 (see [8], system (3.1) for n = 3). Ray termination occurs when computing $\underline{x}_2$ for the system

$$\begin{aligned} x_1 + [-2,2]\, x_2 \qquad\qquad &= -1 \\ x_2 \quad + [-2,2]x_3 &= -1 \\ x_3 &= -1 \ , \end{aligned}$$

where the condition $\Delta e > 0$ is not met in the third row.

Let x, p satisfy (2.3), (2.4) with $c^I = [-e_i, -e_i]$. Then it can be easily verified that $x_1 = x^+$, $x_2 = x^-$, $p_1 = p^+$, $p_2 = p^-$ satisfy (3.3) for arbitrary $M \geqslant 0$. Hence (2.3), (2.4) are a special case of the Kuhn-Tucker conditions.

4. Regularity conditions

In our earlier papers we gave some necessary and sufficient regularity conditions (theorems 1,6 and 8 in [6] , theorems 6.3, 6.6 in [8]). Several others are added here.

Notation : $\widetilde{D}_{yz} = T_y \Delta T_z A_c^{-1}$ $(y,z \in Y)$, $Y_1 = \{ t \in R^n \ ; \ |t_j| \leq 1$ for some $j \in N$, $|t_i| = 1$ for each $i \neq j\}$, $D_y = A_c^{-1} T_y \Delta$ $(y \in Y)$.

Theorem 4.1. The following assertions are mutually equivalent :

(i) A^I is regular

(ii) for each $y \in Y$, $A_{ye}x^1 - A_{yf}x^2 = y$ has a solution $x^1 \geqslant 0$, $x^2 \geqslant 0$

(iii) for each $y \in Y$, $A_{yz}x = y$, $T_z x \geqslant 0$ has a solution

(iv) for each $y \in Y$ there exists an $x > 0$ such that $A_{ye}^{-1} A_{yf} x > 0$

(v) for each $y \in Y$ there exists a q such that $A_{ye}^{-1} q > 0$, $A_{yf}^{-1} q > 0$

(vi) for each $y \in Y$ there exists a p such that $|D_y p| < p$

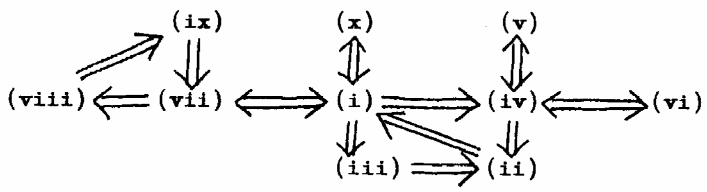
(vii) for each $y, z \in Y$, $i \in N$ there holds $(A_c A_{yz}^{-1})_{ii} > \frac{1}{2}$

(viii) for each $A \in A^I$, $i \in N$ there holds $(A_c A^{-1})_{ii} > \frac{1}{2}$

(ix) for each $y, z \in Y$, $i \in N$ there holds $((E - \tilde{D}_{yz})^{-1})_{ii} > \frac{1}{2}$

(x) for each $t \in Y_1$, $z \in Y$ there holds $\det A_{tz} \neq 0$.

Proof scheme :



(i) $\Rightarrow$ (iii) : Follows from theorem 1.2. (iii) $\Rightarrow$ (ii): Putting $x^1 = x^+$, $x^2 = x^-$, we obtain $A_{ye}x^1 - A_{yf}x^2 = A_{yz}x = y$. (ii) $\Rightarrow$ (i) : Let $A \in A^I$, $k \in N$, $J \subset N$ and let y be such that $y_j = -1$ $(j \in J)$, $y_j = 1$ $(j \notin J)$. Then for $x_J = x_y^1 - x_y^2$ we have $(Ax_J)_j \leq (A_{ye}x_y^1 - A_{yf}x_y^2)_j = y_j = -1 \leq (e_k)_j$ for $j \in J$ and similarly $(Ax_J)_j \geq 1 \geq (e_k)_j$ for $j \notin J$. Hence $Ax = e_k$ has a solution due to lemma 1.1. Since A and k were arbitrary, A^I is regular.

(i) $\Rightarrow$ (iv) : Since $A_{ye}^{-1}A_{yf}$ is a P-matrix, the existence of such an x follows from theorem by Gale , Nikaido [4,p.83]. (iv) $\Rightarrow$ (ii) : Since $A_{ye}^{-1}A_{yf}x > 0$, there is a positive number M such that $A_{ye}^{-1}A_{yf}(Mx) + A_{ye}^{-1}y > 0$. Now for $x^1 = A_{ye}^{-1}A_{yf}(Mx) + A_{ye}^{-1}y$, $x^2 = Mx$ we have $A_{ye}x^1 - A_{yf}x^2 = y$, $x^1 \geq 0$, $x^2 \geq 0$.

<u>(iv) $\Longleftrightarrow$ (v)</u> : Put $q = A_{yf}x$ and vice versa. <u>(iv) $\Longleftrightarrow$ (vi)</u>: Since $A_{ye}^{-1}A_{yf} = (E-D_y)^{-1}(E+D_y) = E + 2(E-D_y)^{-1}D_y = 2(E-D_y)^{-1}-E$, we obtain the equivalence by setting $p = (E-D_y)^{-1}x$ and vice versa.

<u>(i) $\Longrightarrow$ (vii)</u> : Let $y',y,z \in Y$ and let y',y differ just in the i-th entry. Then, since $A_{y'z} = (E+2y_iT_{e_i}\Delta T_zA_{yz}^{-1})A_{yz}$ and $\det(E+2y_iT_{e_i}\Delta T_zA_{yz}^{-1}) = 1+2(T_y\Delta T_zA_{yz}^{-1})_{ii} = 2(A_cA_{yz}^{-1})_{ii} - 1$, we have

$$\det A_{y'z} = (2(A_cA_{yz}^{-1})_{ii} - 1) \det A_{yz} . \qquad (4.1)$$

Since both $\det A_{y'z}$ and $\det A_{yz}$ must be of the same sign, we get $2(A_cA_{yz}^{-1})_{ii}-1 > 0$. <u>(vii) $\Longrightarrow$ (i)</u> by contradiction : Assume A^I is singular, then $\det A_{y_1z} \det A_{y_2z} \leqslant 0$ for some $y_1,y_2,z \in Y$ (Baumann [2], proof of theorem 1). Consider a path in the unit cube in R^n leading from y_1 to y_2 ; since $\det A_{y_1z} \det A_{y_2z} \leqslant 0$, there exists a pair of neighbouring vertices y',y (differing in the i-th entry only) such that $\det A_{y'z} \det A_{yz} \leqslant 0$. Then (4.1) implies $2(A_cA_{yz}^{-1})_{ii} - 1 \leqslant 0$.

<u>(vii) $\Longrightarrow$ (viii)</u> : Since for each $A \in A^I$ we have $A^{-1} = \sum_{y,z\in Y} A_{yz}^{-1}L_{yz}$ for some nonnegative diagonal matrices L_{yz} satisfying $\sum_{y,z\in Y} L_{yz} = E$ ([8] , theorem 7.1) , for any $i \in N$ we have $(A_cA^{-1})_{ii} = \sum_{y,z\in Y}(A_cA_{yz}^{-1})_{ii}(L_{yz})_{ii} > \frac{1}{2}$.
<u>(viii) $\Longrightarrow$ (ix)</u> : From (viii) follows (vii), hence $((E-\widetilde{D}_{yz})^{-1})_{ii} = (A_cA_{yz}^{-1})_{ii} > \frac{1}{2}$. <u>(ix) $\Longrightarrow$ (vii)</u> : again from $(A_cA_{yz}^{-1})_{ii} = ((E-\widetilde{D}_{yz})^{-1})_{ii}$. <u>(i) $\Longrightarrow$ (x)</u> is obvious. <u>(x) $\Longrightarrow$ (i)</u>: Assume for contrary that A^I is singular, then, as in the proof of

(vii) $\Longrightarrow$(i) , we may argue that $\det A_{y'z} \det A_{yz} \leq 0$ for some $y, y' \in Y$ differing just in the i-th entry. For $\tau \in [0,1]$, define a real function $f(\tau) = \det A_{y+\tau(y'-y),z}$. Since f is linear and $f(0)\, f(1) \leq 0$, there is a $\tau_0 \in [0,1]$ with $f(\tau_0) = 0$. Then for $t = y + \tau_0(y'-y)$ we have $t \in Y_1$ and $\det A_{tz} = 0$. ■

Another set of regularity conditions can be obtained when applying theorem 4.1 to the transpose $(A^I)^T$. E.g., this leads to conditions

(vii*) for each $y,z \in Y$, $i \in N$ there holds $(A_{yz}^{-1} A_c)_{ii} > \frac{1}{2}$,

(x*) for each $t \in Y_1$, $z \in Y$ there holds $\det A_{zt} = 0$,

etc. From (viii) it follows that A^I is singular if $(A_c A^{-1})_{ii} \leq \frac{1}{2}$ for some $A \in A^I$, $i \in N$ (although A need not be singular). In (ii), the number of equations is reduced in comparison with assertion (iii) of theorem 6.6 in [8] from $n \cdot 2^n$ to 2^n . Assertion (iv) shows that a necessary condition for P-matrices becomes also sufficient in this context if it is valid for each $y \in Y$. It is better to read (x) negated : A^I is singular iff A_{tz} is singular for some $t \in Y_1$, $z \in Y$. Generally we cannot assert $t \in Y$, as the interval matrix $A^I = [-E,E]$ shows, which is obviously singular ($0 \in A^I$) but $\det A_{yz} \neq 0$ for each $y,z \in Y$. An application of (x) is given in the next theorem. Recall that $\rho_0(A)$ denotes the maximum absolute value of real eigenvalues ($\rho_0(A) = 0$ if no real eigenvalue exists) and denote $\rho_0(A^I) = \max \{ \rho_0(A) \, ; A \in A^I \}$.

<u>Theorem 4.2.</u> Let A^I be a square interval matrix. Then

$$\rho_0(A^I) = \max_{y,z\in Y} \rho_0(A_{yz}) .$$

<u>Proof.</u> The assertion obviously holds if $\rho_0(A^I) = 0$. Let $\lambda_0 = \rho_0(A)$ for some A, $\lambda_0 \neq 0$. Then $\det(A-\lambda_0 E)=0$, hence the interval matrix $[A_c - \lambda_0 E - \Delta , A_c - \lambda_0 E + \Delta]$ is singular, therefore according to (x), $\det(A_{tz} - \lambda_0 E) = 0$ for some $t \in Y_1$, $z \in Y$. If $t \in Y$, we are done; thus assume that $|t_i| < 1$ for some i . Since $\det(A_{tz} - \lambda_0 E)$ is linear in t_i , defining $y_i = -1$, $y_i' = 1$, $y_j = y_j' = t_j$ $(j \neq i)$, we have $y \in Y$, $y' \in Y$ and $\det(A_{yz}-\lambda_0 E)\det(A_{y'z}-\lambda_0 E) \leq 0$. Assume that the left-hand side is negative. Since the polynomial $p(\lambda) = \det(A_{yz} - \lambda E)\det(A_{y'z} - \lambda E)$ is of even degree, it holds $p(\lambda) \to \infty$ for $\lambda \to \infty$ as well as $\lambda \to -\infty$. If $\lambda_0 > 0$, taking $\lambda \to \infty$, we obtain that $p(\lambda) = 0$ for some $\lambda > \lambda_0$; if $\lambda_0 < 0$, taking $\lambda \to -\infty$, we have $p(\lambda) = 0$ for some $\lambda < \lambda_0$. In both the cases, $|\lambda_0| < |\lambda|$, where λ is a real eigenvalue of some A_{yz} . ■

<u>5. Descent method for testing singularity</u>

We present here a simple method for testing singularity of interval matrices. This method is, however, not universal since it may fail without stating singularity or regularity.

Let $A \in A^I$ and consider some perturbation of coefficients in the j-th row; more precisely, consider $A + D_j$, where all rows of D_j except the j-th are zero and $(D_j)_{jk} = d_{jk}$, $k \in N$. Since $A + D_j = (E+D_jA^{-1})A$ and $\det(E+D_jA^{-1}) = 1 + (D_jA^{-1})_{jj}$, we have

$$\det(A + D_j) = (1 + (D_jA^{-1})_{jj})\det A. \qquad (5.1)$$

Set $\alpha = 1 + (D_jA^{-1})_{jj}$ for a while and consider three cases:

(a) if $\alpha \leq 0$, then A^I is obviously singular;

(b) if $0 < \alpha < 1$, then $|\det(A+D_j)| < |\det A|$;

(c) if $\alpha \geq 1$, then $|\det(A+D_j)| \geq |\det A|$.

Hence an obvious idea arises to organize the method in such a way as to decrease $|\det A|$ according to (b) in each step until (a) occurs or (c) holds for any j. Since $\alpha = 1 + \sum_k d_{jk}A^{-1}_{kj}$, we must look for d_{jk} such that $d_{jk}A^{-1}_{kj} < 0$ and, since the expression is linear in d_{jk}, to take d_{jk} as large as possible. If $A_{jk} = \underline{A}_{jk}$, we take $d_{jk} = \overline{A}_{jk} - \underline{A}_{jk}$, if $A_{jk} = \overline{A}_{jk}$, we take $d_{jk} = \underline{A}_{jk} - \overline{A}_{jk}$, in both the cases $d_{jk} = 2(A_c - A)_{jk}$. We can summarize : during the algorithm, we look for j,k satisfying

$$(A_c - A)_{jk}\, A^{-1}_{kj} < 0 \qquad (5.2)$$

until

$$(A_c - A)_{jk}\, A^{-1}_{kj} \geq 0 \quad \text{for each } j,k \qquad (5.3)$$

holds. If (5.2) occurs for some j,k, we set $A_{jk} := A_{jk} + 2(A_c-A)_{jk} = (2A_c-A)_{jk}$ for each k satisfying (5.2) (we do it only for elements of the j-th row), thus obtaining a matrix with a lower absolute value of the determinant (case (b)). If the sum over all k's satisfying (5.2) is less than $-\frac{1}{2}$, then the above case (a) occurs and A^I is singular.

Hence the full description of the algorithm looks like this :

Algorithm 5.1 (testing singularity of A^I).

0. Start with an $A \in A^I$ satisfying $|A-A_c| = \Delta$ (e.g. $A = \underline{A}$ or $A = \overline{A}$).
1. Compute A^{-1}.
2. If $(A_c - A)_{jk}A^{-1}_{kj} \geqslant 0$ for each $j,k \in N$, terminate. The algorithm fails.
3. Otherwise find the minimum index j for which the set $K_j = \{k;\ (A_c - A)_{jk}A^{-1}_{kj} < 0\}$ is nonempty.
4. If $\sum_{K_j} (A_c - A)_{jk}A^{-1}_{kj} \leqslant -\frac{1}{2}$, terminate. A^I is singular.
5. Otherwise set $A_{jk} := (2A_c - A)_{jk}$ for each $k \in K_j$ and go to step 1.

The algorithm is finite. In fact, since each matrix A appearing in the course of computations satisfies $A_{jk} \in \{\underline{A}_{jk}, \overline{A}_{jk}\}$ for each $j,k \in N$, the algorithm goes through a finite set of matrices and since $|\det A|$ decreases at each step, no matrix can appear twice. When returning from step 5 to step 1, the algorithm requires to invert a matrix differing only in the j-th row from the previous one. This can be done on the base of lemma 1 in [6] using only one pivot operation. In case of terminating in step 2, we may expect in many practical cases A^I to be regular (theorem 5.1 below supports in some sense this statement), but one must be careful because it is not always so. The following nice counterexample was constructed by Dr. M. Baumann:

Example 5.1. The interval matrix

$$A^I = \begin{pmatrix} [0,2] & [2,4] \\ [1,3] & [0,2] \end{pmatrix}$$

is singular, but (5.3) holds for $A = \underline{A}$, hence the algorithm, when started from this A , fails immediately.

Nevertheless, we have this partial result:

Theorem 5.1. Let $A \in A^I$ satisfy (5.3) and let $|A^{-1}| > 0$. Then A is a local minimum point of $|\det|$ over A^I .

Proof. Since $|A^{-1}| > 0$, there exists an $\varepsilon > 0$ such that for each $B \in S = \{ A' \in A^I;\ \|A' - A\| < \varepsilon \}$ there holds $|B^{-1}| > 0$, so that $B^{-1}_{jk} A^{-1}_{jk} > 0$ $(j,k \in N)$ and $\det B \det A > 0$. For $j = 1,\dots,n$, define matrices B_j, C_j in this way $(i \in N)$:

$$(B_j)_{i.} = \begin{cases} A_{i.} & \text{if } i < j \\ B_{i.} & \text{if } i \geqslant j \end{cases}, \quad (C_j)_{i.} = \begin{cases} A_{i.} & \text{if } i < j \\ (B-A)_{i.} & \text{if } i = j \\ B_{i.} & \text{if } i > j \end{cases}.$$

Then $B_j \in S$ $(j \in N)$ and we have $\det B - \det A = \sum_{j=1}^{n} \det C_j =$
$= \sum_{j=1}^{n} \det B_j \sum_{k=1}^{n} (B-A)_{jk} (B_j^{-1})_{kj}$. Since $(B-A)_{jk} =$
$= \delta_{jk} (A_c - A)_{jk}$ for some $\delta_{jk} \geqslant 0$ and $(B_j^{-1})_{kj} \cdot A^{-1}_{kj} > 0$,
we have that $\beta_j = \sum_{k=1}^{n} (B-A)_{jk} (B_j^{-1})_{kj} \geqslant 0$ due to (5.3).

Now with σ = sgn det A we obtain

$$|\det B| - |\det A| = \sigma(\det B - \det A) = \sum_{j=1}^{m} \beta_j \sigma \det B_j \geq 0 ,$$

hence $|\det A| \leq |\det B|$. Since $B \in S$ was arbitrary, this means that A is a local minimum point of the absolute value of the determinant. ■

Recall ([6]) that A^I is called inverse-stable if $|A^{-1}| > 0$ for each $A \in A^I$.

Theorem 5.2. Let A^I be inverse-stable. Then $|\det|$ achieves its unique global minimum over A^I at the matrix A_0 given by

$$(A_0)_{jk} = \begin{cases} \underline{A}_{jk} & \text{if} \quad (A_c^{-1})_{kj} > 0 \\ \overline{A}_{jk} & \text{if} \quad (A_c^{-1})_{kj} < 0 \end{cases} ,$$

which can be found by algorithm 5.1 in at most n iterations (ending with failure).

Proof. Both the expression for A_0 and its uniqueness follow from (5.3), which must be satisfied at the global minimum point. Due to the inverse-stability, the algorithm never returns to the same row index j , hence it takes at most n steps. ■

Before arriving at the idea of algorithm 5.1, we tested (also with rather good results) another algorithm based on negated assertions (vii), (vii*) from section 4. In case of detecting singularity, it also produces a singular matrix A_{yz} with either $y \in Y_1$, $z \in Y$ or $y \in Y$, $z \in Y_1$ (see (x),(x*)).

We present it here without further comments; example 5.1 again serves as an counterexample.

Algorithm 5.2 (testing singularity of A^I).

0. Select $y,z \in Y$ (e.g. $y = z = e$).
1. Invert A_{yz} and compute $p = \min_k (A_c A_{yz}^{-1})_{kk} = (A_c A_{yz}^{-1})_{ii}$, $q = \min_k (A_{yz}^{-1} A_c)_{kk} = (A_{yz}^{-1} A_o)_{jj}$.
2. If $p \geqslant 1$ and $q \geqslant 1$, terminate. The algorithm fails.
3. If $p \leqslant \frac{1}{2}$, set $y_i := (y_i p)/(p-1)$ and terminate. A^I is singular: $\det A_{yz} = 0$, $y \in Y_1$, $z \in Y$.
4. If $q \leqslant \frac{1}{2}$, set $z_j := (z_j q)/(q-1)$ and terminate. A^I is singular: $\det A_{yz} = 0$, $y \in Y$, $z \in Y_1$.
5. Otherwise, if $p \leqslant q$, set $y_i := -y_i$ and go to step 1; if $p > q$, set $z_j := -z_j$ and go to step 1.

Added after finishing the MS. It seems reasonable to start algorithm 5.1 in step 0 with matrix A_0 defined as in theorem 5.2 (with e.g. $(A_0)_{jk} = \underline{A}_{jk}$ if $(A_c^{-1})_{kj} = 0$). If $\det A_c \det A_0 \leqslant 0$, then singularity is detected already in step 0 (this is the case of example 5.1 now). Global minimum of $|\det|$ for inverse-stable matrices is then always found in only one iteration.

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Author´s address: J. Rohn, Faculty of Mathematics and Physics, Charles University, Malostranské nám. 25, 11800 Prague, Czechoslovakia