Miscellaneous Results on Linear Interval Systems

by

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This paper consists of results that were not included into author's previous papers /5/ - /8/ for various reasons (other proofs, obvious consequences, little applicability etc.). An attempt to publish them all together gave birth to this incoherent paper. Notations used are the same as in /5/ - /8/.

1. On theorems by Oettli, Prager and Gerlach

The following theorem due to Oettli and Prager /4/ describes the set $X = \{x;\ Ax = b\ ,\ A \in A^I\ ,\ b \in b^I\}$:

Theorem 1.1. We have $X = \{x;\ |A_c x - b_c| \leq \Delta|x| + \delta\}$.

Proof. Using $A_c = \frac{1}{2}(\underline{A} + \overline{A})$, $b_c = \frac{1}{2}(\underline{b} + \overline{b})$, we may rewrite $X = \{x;\ \underline{A}x^+ - \overline{A}x^- \leq \overline{b}\ ,\ \overline{A}x^+ - \underline{A}x^- \geq \underline{b}\}$, which is the form we shall prove. If $Ax = b$ for some $A \in A^I$, $b \in b^I$, then $\underline{A}x^+ - \overline{A}x^- \leq$ $\leq A(x^+ - x^-) = b \leq \overline{b}$, $\overline{A}x^+ - \underline{A}x^- \geq A(x^+ - x^-) \geq \underline{b}$. Conversely, let the two inequalities hold; for each i , define $f_i : R^{n+1} \to R^1$ by

$$f_i(\alpha_1,\dots,\alpha_{n+1}) = \sum_{j=1}^{n} \left[(\underline{A}_{ij} + \alpha_j(\overline{A}_{ij} - \underline{A}_{ij}))x_j^+ - \right.$$
$$\left. - (\overline{A}_{ij} + \alpha_j(\underline{A}_{ij} - \overline{A}_{ij}))x_j^-)\right] -$$
$$- (\overline{b}_i + \alpha_{n+1}(\underline{b}_i - \overline{b}_i))\ ,$$

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then $f_i(0,\dots,0) \leq 0$, $f_i(1,\dots,1) \geq 0$, hence there is a j such that $f_i(1,\dots,1,\overset{j}{0},\dots,0) \leq 0$ and $f_i(1,\dots,1,\overset{j}{1},\dots,0) \geq 0$. Thus there is a $\tau \in [0,1]$ such that $f_i(1,\dots,1,\overset{j}{\tau},0,\dots,0) = 0$. Now for $t = (1,\dots,1,\tau,0,\dots,0)$, defining $A_{ij} = \underline{A}_{ij} + t_j(\overline{A}_{ij} - \underline{A}_{ij})$ if $x_j^+ > 0$ and $A_{ij} = \overline{A}_{ij} + t_j(\underline{A}_{ij} - \overline{A}_{ij})$ otherwise, and $b_i = \overline{b}_i + t_{n+1}(\underline{b}_i - \overline{b}_i)$, we have $(Ax)_i = b_i$, hence $x \in X$.■

We have simultaneously proved this result :

Theorem 1.2. Let $x \in X$. Then $Ax = b$ for some $A \in A^I$, $b \in b^I$ such that for each i we have $A_{ij} \in \{\underline{A}_{ij}, \overline{A}_{ij}\}$ $(1 \leq j \leq n)$, $b_i \in \{\underline{b}_i, \overline{b}_i\}$ for all but at most one entry.

For given A^I and $x \in R^n$ denote by $A^I \odot x$ the set $\{Ax; A \in A^I\}$. The Oettli-Prager theorem is equivalent to this result (in the sense that each of them can be proved from the other) :

Theorem 1.3. We have $A^I \odot x = [A_c x - \Delta|x| , A_c x + \Delta|x|]$.

Proof. If $y = Ax$ for some $A \in A^I$, then $|y - A_c x| \leq \Delta|x|$, hence $y \in [A_c x - \Delta|x| , A_c x + \Delta|x|]$. Conversely, if this is true, then $|A_c x - y| \leq \Delta|x|$, hence $Ax = y$ for some $A \in A^I$ due to theorem 1.1.■

Theorem 1.1 can be, in turn, proved from theorem 1.3. In fact,

$$X = \{x;\ Ax = b\ ,\ A \in A^I\ ,\ b \in b^I\} = \{x;\ A^I \odot x \cap b^I \neq \emptyset\} =$$
$$= \{x;\ A_c x - \Delta|x| \leq \overline{b}\ ,\ A_c x + \Delta|x| \geq \underline{b}\} = \{x;\ |A_c x - b_c| \leq \Delta|x| + \delta\} .$$

Gerlach /2/ was the first to investigate the set $X_1 = \{x;\ Ax \leq b\ ,\ A \in A^I,\ b \in b^I\}$. His result can be also proved from theorem 1.3 :

Theorem 1.4 (Gerlach). We have $X_1 = \{x;\ A_c x - b_c \le \Delta|x| + \delta\}$.

Proof. If $Ax \le b$ for some $A \in A^I$, $b \in b^I$, then, since $Ax \in A^I \odot x$, we have $A_c x - \Delta|x| \le Ax \le b \le b_c + \delta$. Conversely, if $A_c x - b_c \le \Delta|x| + \delta$, then setting $z = \operatorname{sgn} x$, we obtain $A_{ez} x \le \bar{b}$, where $A_{ez} \in A^I$, $\bar{b} \in b^I$. ∎

Denote $X_2 = \{x;\ Ax \ge b,\ A \in A^I,\ b \in b^I\}$. In a similar way we may prove that $X_2 = \{x;\ A_c x - b_c \ge -\Delta|x| - \delta\}$.

Theorem 1.5. We have $X = X_1 \cap X_2$.

Proof. Obviously, $X = \{x;\ |A_c x - b_c| \le \Delta|x| + \delta\} = \{x;\ A_c x - b_c \le \Delta|x| + \delta\} \cap \{x;\ A_c x - b_c \ge -\Delta|x| - \delta\} = X_1 \cap X_2$. ∎

This result is not quite trivial since it shows that if $A_1 x \le b_1$, $A_2 x \ge b_2$ for some $A_1, A_2 \in A^I$, $b_1, b_2 \in b^I$, then also $A_3 x = b_3$ for some $A_3 \in A^I$, $b_3 \in b^I$.

When dealing with interval linear systems, one could be tempted to introduce solutions satisfying $A^I \odot x = b^I$. Theorem 1.3 shows that then $A_c x - \Delta|x| = \underline{b}$, $A_c x + \Delta|x| = \bar{b}$, implying $x = A_c^{-1} b_c$ and $\Delta|A_c^{-1} b_c| = \delta$. Hence unless the last condition is met , a solution to $A^I \odot x = b^I$ does not exist .

2. Estimation of the interval solution

In this section we give a (practically hardly applicable) method for constructing an estimation of the exact interval solution $[\underline{x}, \bar{x}]$ of an interval linear system $A^I x = b^I$. As before, we use the notation $D_y = A_c^{-1} T_y \Delta$, $d_y = A_c^{-1} b_y$ $(y \in Y)$.

<u>Theorem 2.1.</u> Let A^I be regular and let for each $y \in Y$, q_y be a solution to the system of linear inequalities

$$\begin{aligned} (E - D_y)q &\geqslant d_y \\ (E + D_y)q &\geqslant -d_y \\ q &\geqslant 0 \end{aligned} \qquad (2.1)$$

(whose solution set is nonempty). Then, we have

$$\begin{aligned} \underline{x} &\geqslant \min\left\{D_y q_y + d_y \; ; \; y \in Y\right\} \\ \overline{x} &\leqslant \max\left\{D_y q_y + d_y \; ; \; y \in Y\right\} . \end{aligned} \qquad (2.2)$$

<u>Proof.</u> First observe that (2.1) can be equivalently rewritten as

$$\begin{aligned} -q \leqslant D_y q + d_y &\leqslant q \\ q &\geqslant 0 . \end{aligned} \qquad (2.3)$$

Since A^I is regular, the equation $x = D_y|x| + d_y$ has a solution x_y (/5/). Hence $q = |x_y|$ satisfies $|D_y q + d_y| = q$, so that it solves both (2.3) and (2.1); hence the solution set of (2.1) is nonempty. Let q_y be an arbitrary solution to (2.1) $(y \in Y)$. Define $p_y = D_y q_y + d_y$, $x_y^1 = \frac{1}{2}(q_y + p_y)$, $x_y^2 = \frac{1}{2}(q_y - p_y)$, then $|p_y| \leq q_y$, hence $x_y^1 \geqslant 0$, $x_y^2 \geqslant 0$, and from $x_y^1 - x_y^2 = D_y q_y + d_y = D_y(x_y^1 + x_y^2) + d_y$ it follows $A_{ye} x_y^1 - A_{yf} x_y^2 = b_y$. Hence from theorem 0 in /5/ we obtain $X \subset \mathrm{Conv}\left\{x_y^1 - x_y^2 \; ; \; y \in Y\right\} = \mathrm{Conv}\left\{D_y q_y + d_y ; y \in Y\right\}$, which justifies the estimations (2.2).■

3. Other proofs I : P-property

In /6/ we proved and in /8/ reproved the following theorem. In view of its importance for deriving other results, we give here a still another proof.

Theorem 3.1. Let A^I be regular. Then for each $A_1, A_2 \in A^I$, both $A_1 A_2^{-1}$ and $A_1^{-1} A_2$ are P-matrices.

Proof. We shall prove only the first part of the assertion since the second follows from the first one applied to the transpose $(A^I)^T$. Thus assume that $A_1 A_2^{-1}$ is not a P-matrix for some $A_1, A_2 \in A^I$. Then according to Gale and Nikaido /1/ there exists $x \neq 0$ such that $x_i (A_1 A_2^{-1} x)_i \leq 0$ for each i. Define $A = A_1 + T_t (A_2 - A_1)$, where $t_i = 1$ if $x_i = 0$ and t_i is (any) root of the real function

$$f_i(t) = x_i (A_1 + t(A_2 - A_1))_{i.} A_2^{-1} x$$

in $[0,1]$ if $x_i \neq 0$ (such a root exists because $f_i(0) = x_i (A_1 A_2^{-1} x)_i \leq 0$, $f_i(1) = x_i^2 \geq 0$). Now we have $A \in A^I$ and $A A_2^{-1} x = 0$, contradicting regularity. ∎

4. Other proofs II : a "schrankentreu" result

The "schrankentreu" (after Nuding /3/) result quoted below was proved in /7/ with the help of some properties of nonnegatively invertible matrices. The proof given here is more elementary.

Theorem 4.1. Let A^I be nonnegatively invertible and let $\overline{A}^{-1}\underline{b} \geq 0$. Then for the exact interval solution $[\underline{x},\overline{x}]$ of $A^I x = b^I$ we have $\underline{x} = \overline{A}^{-1}\underline{b}$, $\overline{x} = \underline{A}^{-1}\overline{b}$.

Proof. Let $Ax = b$ for some $A \in A^I$, $b \in b^I$. Then in view of the nonnegative invertibility we have $x = A^{-1}b \geq A^{-1}\underline{b} = \overline{A}^{-1}\underline{b} + A^{-1}(\overline{A}-A)\overline{A}^{-1}\underline{b} \geq \overline{A}^{-1}\underline{b}$, so that $\underline{x} = \overline{A}^{-1}\underline{b}$. Further, since $x \geq 0$, we get $\underline{A}x \leq Ax \leq \overline{b}$, implying $x \leq \underline{A}^{-1}\overline{b} = \overline{x}$. ∎

5. Other proofs III : Farkas-type theorem

The classical Farkas theorem states that $Ax = b$ has a nonnegative solution iff for each y, $A^T y \geq 0$ implies $b^T y \geq 0$. In /7/ we showed that this result can be extended to interval linear systems (with A^I of arbitrary size $m \times n$) in this way :

Theorem 5.1. $A^I x = b^I$ has a nonnegative solution if and only if

$$(\forall y)(A^T y \geq 0 \text{ for each } A \in A^I \Rightarrow b^T y \geq 0 \text{ for some } b \in b^I) \qquad (5.1)$$

holds.

We give here a direct proof of this theorem.

Proof. "Only if" : If $A_o x = b_o$, $x \geq 0$, $A_o \in A^I$, $b_o \in b^I$, then if $A^T y \geq 0$ for each $A \in A^I$, then also $A_o^T y \geq 0$, hence $b_o^T y \geq 0$ according to the Farkas theorem. "If" : Assume that (5.1) holds. To conclude the proof, it suffices in view of corollary 1.2 in /7/ to show that $\underline{A}x \leq \overline{b}$, $-\overline{A}x \leq -\underline{b}$ has a nonnegative solution, which, due to Farkas theorem, is reduced to the proof of the implication

$\underline{A}^T p_1 - \overline{A}^T p_2 \geq 0 \Rightarrow \overline{b}^T p_2 - \underline{b}^T p_2 \geq 0$ for each $p_1, p_2 \geq 0$. Thus let $\underline{A}^T p_1 - \overline{A}^T p_2 \geq 0$ for some $p_1, p_2 \geq 0$. Then for each $A \in A^I$ we have $A^T(p_1 - p_2) \geq \underline{A}^T p_1 - \overline{A}^T p_2 \geq 0$, hence (5.1) assures the existence of a $b_1 \in b^I$ such that $b_1^T(p_1 - p_2) \geq 0$. But $\overline{b}^T p_1 - \underline{b}^T p_1 \geq b_1^T(p_1 - p_2) \geq 0$, and we are done. ∎

6. Determinants and singularity

In this section, we first derive some determinant theorem and then we apply it to singular interval matrices to show that such a matrix always contains a real singular matrix in some "normal form".

Let $A^I = [\underline{A}, \overline{A}]$ be a square interval matrix. For each $A \in A^I$ denote by $h(A)$ the number of pairs (i,j) for which $A_{ij} \in (\underline{A}_{ij}, \overline{A}_{ij})$ (so that for the pairs not counted we have either $A_{ij} = \underline{A}_{ij}$ or $A_{ij} = \overline{A}_{ij}$). We have this result.

Theorem 6.1. Let A^I be a square interval matrix. Then for each $A \in A^I$ there exists an $A' \in A^I$ such that $h(A') \leq 1$ and $\det A' = \det A$.

Proof. Given an $A \in A^I$, let $h = \min\{h(B);\ B \in A^I,\ \det B = \det A\}$, then $h = h(A')$ for some $A' \in A^I$, $\det A' = \det A$. Assume for contrary that $h(A') \geq 2$, so that there are (p,q) , (r,s) , $(p,q) \neq (r,s)$ such that $a'_{pq} \in (\underline{a}_{pq}, \overline{a}_{pq})$, $a'_{rs} \in (\underline{a}_{rs}, \overline{a}_{rs})$. Then we can choose η , ν such that the matrix A'' formed from A' by replacing a'_{pq}, a'_{rs} by $a'_{pq} + \eta$, $a'_{rs} + \nu$, respectively will satisfy $\det A'' = \det A'$, $A'' \in A^I$ and either

$a'_{pq} + \eta \in \{\underline{a}_{pq}, \bar{a}_{pq}\}$, or $a'_{rs} + \nu \in \{\underline{a}_{rs}, \bar{a}_{rs}\}$. Then $\det A'' = \det A$ and $h(A'') < h$, which is a contradiction. ∎

The next theorem shows that a singular interval matrix contains a singular (real) matrix of a simple form. We shall give two proofs to this result.

Theorem 6.2. If A^I is a singular interval matrix, then there exists a singular matrix $A \in A^I$ with $h(A) \leq 1$.

Proof I. If A^I is singular, then it contains a singular matrix A_o ; hence theorem 6.1 assures the existence of a matrix $A \in A^I$ such that $\det A = \det A_o = 0$ and $h(A) \leq 1$. ∎

Proof II. If A^I is singular, then there exists a singular matrix A_{tz} such that $z \in Y$, $|t| \leq e$ and $|t_i| = 1$ for all but at most one i (/8/ , theorem 4.1, (x)). If $|t| = e$, then $h(A_{tz}) = 0$ and we are done. Thus assume that $|t_r| < 1$ for some (unique) r and define matrices A_j $(j = 0,\dots,n)$ as follows: if $i \neq r$, put $(A_j)_{ik} = (A_{tz})_{ik}$ for each k ; for $i = r$ define $(A_j)_{rk} = \underline{A}_{rk}$ for $k \leq j$ and $(A_j)_{rk} = \bar{A}_{rk}$ for $k > j$. Since $\det A_{tz}$ is linear in t_r , we must have $\det A_o \det A_n \leq 0$. Then there exists a j , $j \in \{0,\dots,n-1\}$, such that $\det A_j \det A_{j+1} \leq 0$, for $\det A_i \det A_{i+1} > 0$ for each $i \in \{0,\dots,n-1\}$ would imply $\det A_o \det A_n > 0$. Since A_j and A_{j+1} differ only in the $(r,j+1)$-th coefficient, assigning it a proper value from $[\underline{a}_{r,j+1}, \bar{a}_{r,j+1}]$ will yield a singular matrix A with $h(A) \leq 1$. ∎

The second proof is constructive. A singular matrix of the form A_{tz} (or A_{zt} , in which case we argue similarly) can be constructed by algorithm 5.2 in /8/.

7. Transformation of the inverse

Assume we know the inverse A^{-1} to a square matrix A and we would like to compute $(A + D_j)^{-1}$, where D_j is a square matrix whose all rows are zero except the j-th one which is equal to d^T (we have encountered such a situation several times in our previously published algorithms). We shall show here that this may be done by only one Gaussian pivoting using the following procedure (e_j is the j-th coordinate vector):

1. Compute $a = d^T A^{-1} + e_j^T$. If $a_j = 0$, stop! $A + D_j$ is singular.
2. Form an $(n+1) \times n$ tableau $\frac{a}{A^{-1}}$
3. Perform a Gaussian elimination on columns of the tableau with pivot a_j so that a become e_j^T.
4. Then the square lower part of the tableau is equal to $(A + D_j)^{-1}$.

To justify the last assertion, we observe that the k-th column of the square lower part is for $k \neq j$ equal to

$$(A^{-1})_{.k} - \frac{a_k}{a_j} (A^{-1})_{.j} = (A^{-1} - \frac{1}{a_j} A^{-1} D_j A^{-1})_{.k}$$

and for $k = j$ to

$$\frac{1}{a_j} (A^{-1})_{.j} = (A^{-1} - \frac{1}{a_j} A^{-1} D_j A^{-1})_{.j} ,$$

where $a_j = d^T(A^{-1})_{\cdot j} + 1 = (D_jA^{-1})_{jj} + 1$. Hence the square lower part of the tableau is equal to $(A+D_j)^{-1}$ due to the lemma in /5/. If $a_j = 0$, then $\det(A + D_j) = \det(E + D_jA^{-1})\det A = a_j \det A = 0$.

Now assume we know the solution to a system $Ax = b$ and we are looking for the solution x' to $(A+D_j)x' = b + \varepsilon e_j$ (changes in j-th row only!). Using the above expression for $(A + D_j)^{-1}$, we arrive at the formula $x' = x + \frac{\varepsilon - d^Tx}{a_j}(A^{-1})_{\cdot j}$. Hence x' may be found by the same procedure as above when working with an $(n+1)x(n+1)$ tableau with a_i $(i \leq n)$ as before, $a_{n+1} = d^Tx - \varepsilon$ and with the lower part of the form $A^{-1} | x$.

8. Descent method for computing $\underline{x}_i$

Employing the idea developed in /8/, section 5 we may construct a descent algorithm for computing $\underline{x}_i$; as shown below, it works only under some restrictions (modifications for $\overline{x}_i$ are obvious) :

<u>Algorithm 8.1.</u> (computing $\underline{x}_i$ for a given i).

0. Select $A \in A^I$, $b \in b^I$ so that $|A-A_c| = \Delta$, $|b-b_c| = \delta$.
1. Solve $Ax = b$.
2. If $A^{-1}_{ij}(A-A_c)_{jk}x_k \geq 0$ for each j,k and $A^{-1}_{ij}(b_c-b)_j \geq 0$ for each j , set $\underline{x}_i = x_i$ and stop.
3. Otherwise find the minimum j for which either $A^{-1}_{ij}(b_c-b)_j < 0$ or $A^{-1}_{ij}(A-A_c)_{jk}x_k < 0$ for some k .
4. For this j set $b_j := (2b_c-b)_j$ if $A^{-1}_{ij}(b_c-b)_j < 0$ and $A_{jk} := (2A_c-A)_{jk}$ if $A^{-1}_{ij}(A-A_c)_{jk}x_k < 0$.
5. Go to step 1 .

Since in step 4 changes are made only in one row j, solving the new system when returning from step 5 to 1 may be performed by the method of section 7. The algorithm is finite since each A, b satisfy $|A - A_c| = \Delta$, $|b - b_c| = \delta$ and x_i strictly decreases at each step.

Theorem 8.1. Let A^I be inverse-stable and let the solution set X lie in the interior of a single orthant. Then $\underset{\sim}{x}_i = \underline{x}_i$.

Proof. After a finite number of steps the algorithm stops with $A^{-1}_{ij}(A-A_c)_{jk}x_k \geq 0 \quad \forall j,k$, $A^{-1}_{ij}(b_c-b)_j \geq 0 \quad \forall j$. Set $y^T = -\mathrm{sgn}(A^{-1})_{i.}$, $z = \mathrm{sgn}\, x$, then due to the assumptions we have $y_j(A-A_c)_{jk}z_k \leq 0$ $\forall j,k$, $y_j(b_c-b)_j \leq 0$ $\forall j$, hence $A = A_{yz}$, $b = b_y$. Thus x satisfies $A_{yz}x = b_y$, $T_z x \geq 0$ and $(A^{-1}_{yz})_{ij}y_j \leq 0$ $\forall j$, hence the assumptions of theorem 2.2 in /8/ (for problem min ... instead of max ...) are met, so that x is optimal and $x_i = \underset{\sim}{x}_i = \underline{x}_i$. ∎

The assumptions of theorem 8.1 can be replaced by more verifiable assumptions

$$C\,|A_c^{-1}| < |A_c^{-1}|$$

$$(E+C)(|x_c| + d_o) < 2\,|x_c|$$

where $C = D(E-D)^{-1}$, $x_c = A_c^{-1}b_c$, $d_o = |A_c^{-1}|\delta$ and $D = |A_c^{-1}|\Delta$ is supposed to satisfy $\rho(D) < 1$ (see /5/, /8/). Hence the algorithm works if $|A_c^{-1}| > 0$, $|x_c| > 0$ and A^I, b^I are sufficiently narrow.

9. An interval linear programming algorithm

An interval linear programming problem (A^I of size $n \times m$)

$$\max \left\{ c^T x;\ Ax = b,\ A \in A^I,\ b \in b^I,\ x \geq 0 \right\}$$

is, as well-known /7/, equivalent to a LP problem

$$\max \left\{ c^T x;\ \underline{A}x \leq \overline{b},\ -\overline{A}x \leq -\underline{b},\ x \geq 0 \right\} \qquad (9.1)$$

which may be solved directly by the simplex method. However, the number of rows is doubled in (9.1) and therefore an algorithm working with an original size tableau, presented below, may be of some interest.

The idea behind the algorithm consists in solving problems of the form

$$\max \left\{ c^T x;\ A_{ye} x = b_y,\ x \geq 0 \right\} \qquad (9.2)$$

($y \in Y$) and is supported by the following criterion :

Theorem 9.1. Let x^* be an optimal solution of (9.2) and let (9.2) have a dual optimal solution p satisfying $T_y p \geq 0$. Then x^* is an optimal solution of (9.1).

Proof. We shall show that x^* and p^+ , p^- satisfy the complementary slackness conditions for (9.1) and its dual problem $\min\left\{ \overline{b}^T p_1 - \underline{b}^T p_2;\ \underline{A}^T p_1 - \overline{A}^T p_2 \geq c\ ,\ p_1\ ,\ p_2 \geq 0 \right\}$. Since $\underline{A}^T p^+ - \overline{A}^T p^- = A_{ye}^T p \geq c$, we have that p^+, p^- is feasible for the dual problem. Next, if $x_i > 0$, then $(\underline{A}^T p^+ - \overline{A}^T p^-)_i = c_i$; if $p_j^+ > 0$, then $y_j = 1$, hence $(\underline{A}x)_j = \overline{b}_j$, similarly for $p_j^- > 0$. Hence the complementary slackness conditions are met. ∎

Hence if $y_j p_j < 0$ for some j, we set $y_j := -y_j$ and return to solving a new problem (9.2). It is however more preferrable not to solve each problem till its end, but to change the system (9.2) as soon as the condition $T_y p \geqslant 0$ is violated. In view of the fact that each change affects one row only, we may use the procedure of section 7 to perform the computations in a tableau of the form

	a	
A_{yB}^{-1}	$A_{yB}^{-1} A_{ye}$	x
p	c^*	h

(9.3)

where a is an additional pivoting row (see below) and the rest is a usual simplex tableau for (9.2) (where A_{yB} is the current basis matrix for (9.2) with basis B).

Algorithm 9.1 (solving 9.1)).

0. Select an $y \in Y$ and apply phase I of the simplex method to obtain a feasible solution. Form (9.3), $a := 0$.
1. If $T_y p \geqslant 0$, go to step 6.
2. Otherwise select a j with $y_j p_j < 0$ and set $a := e_j^T + 2y_j [(\Delta_B)_{j.} (A_{yB}^{-1}, A_{yB}^{-1} A_{ye}, x) + (0^T, -\Delta_{j.}, \delta_j)]$, $y_j := -y_j$.
3. If $a_j \leq 0$, stop! (9.1) is unbounded.
4. Perform Gaussian elimination with pivot a_j and pivot row a.
5. Go to step 1.

6. If $c^* \geq 0$, stop! Optimal solution.
7. Otherwise find minimum j with $c_j^* < 0$ and perform a usual simplex step to introduce the respective column into basis.
8. Go to step 1.

<u>Theorem 9.2.</u> Assume that each nonnegative solution x of $Ax=b$ $(A \in A^I,\ b \in b^I)$ is nondegenerate (i.e. has at least n positive entries) and that $\delta > 0$. Then algorithm 9.1 gives in a finite number of steps an optimal solution to (9.1).

<u>Proof.</u> In the light of section 7 we can check that the transformation described in step 2 updates the tableau (9.3) to the form corresponding to the change of y_j to $-y_j$. Denoting the new value of x by x' , we have

$$c_B^T x' = c_B^T x - \frac{2(\Delta_B x + \delta)_j}{a_j} y_j p_j ,$$

hence the objective increases at each step, implying finiteness. Nondegeneracy assures $x > 0$ at each step; if $a_j \leq 0$, the objective is unbounded. When stopping occurs in step 6, the current optimal solution of (9.2) satisfies $T_y p \geq 0$, thus it is an optimal solution to (9.1) due to theorem 9.1. ■

This algorithm and convergence theorem were mentioned in /7/, p. 52.

References

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