A NOTE ON SOLVING EQUATIONS OF TYPE $A^I x^I = b^I$

by

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Let $A^I = [A_c - \Delta , A_c + \Delta]$ be a regular $n \times n$ interval matrix and let $b^I = [\underline{b}, \overline{b}]$ be an interval n-vector. In this note we show that the problem of finding an interval n-vector x^I such that $A^I x^I = b^I$ (where the left-hand multiplication is performed in interval arithmetic) can be rather easily solved if we impose an additional restriction on the concept of solution.

Definition. An interval n-vector x^I is called a strong solution if $A^I x^I = b^I$ and, moreover, if there exist $x_1, x_2 \in x^I$ such that $A_1 x_1 = \underline{b}$, $A_2 x_2 = \overline{b}$ for some $A_1, A_2 \in A^I$.

We shall show that the problem of finding a strong solution or verifying that no such solution exists can be solved by the following simple algorithm:

Algorithm.

0. Solve the equations $A_c x_1 - \Delta |x_1| = \underline{b}$, $A_c x_2 + \Delta |x_2| = \overline{b}$.
1. Construct $\tilde{x}^I = [\underset{\sim}{x}, \tilde{x}]$, where $\underset{\sim}{x}_j = \min\{(x_1)_j, (x_2)_j\}$, $\tilde{x}_j = \max\{(x_1)_j , (x_2)_j\}$, $j = 1, \dots, n$.
2. If $A^I \tilde{x}^I = b^I$, stop! $\tilde{x}$ is a strong solution.
3. Otherwise stop! No strong solution exists.

Since A^I is regular, each of the two equations described in step 0 has a unique solution, as proved in [4] . Since $|x_1| = T_z x_1$ for some diagonal matrix T_z satisfying $|T_z| = E$, we have $(A_c - \Delta T_z) x_1 = b$, where $A_c - \Delta T_z \in A^I$;

AMS Subject Classification: 65G10

similarly for x_2. Hence if $A^I\tilde{x}^I = b^I$, then $\tilde{x}^I$ is a strong solution (since $x_1, x_2 \in \tilde{x}^I$). To justify step 3, we prove this result:

Theorem. Let A^I be regular and let $A^I x^I = b^I$ have a strong solution. Then $\tilde{x}^I$ is also a strong solution.

Proof. Let x^I be a strong solution. Then $A_1 x_1^* = \underline{b}$, $A_2 x_2^* = \overline{b}$ for some $x_1^*, x_2^* \in x^I$, $A_1, A_2 \in A^I$. Due to the Oettli-Prager theorem, we have $\{Ax_1^* ; A \in A^I\} =$ $= [A_c x_1^* - \Delta |x_1^*| , A_c x_1^* + \Delta |x_1^*|]$; then $A^I x^I = b^I$ implies $A_c x_1^* - \Delta |x_1^*| = \underline{b}$ and the above-mentioned uniqueness of solution gives $x_1^* = x_1$. In a similar way we obtain $x_2^* = x_2$; hence $\tilde{x}^I \subset x^I$. Now we have $b^I \subset A^I\tilde{x}^I \subset A^I x^I = b^I$, $\underline{b} = A_1 x_1$, $\overline{b} = A_2 x_2$, hence $\tilde{x}^I$ is a strong solution.

We shall briefly sum up some methods for solving the equation $A_c x_1 - \Delta |x_1| = \underline{b}$ (similarly for $A_c x_2 + \Delta |x_2| = \overline{b}$). As described in [3], we have these options:

(a) to solve the linear complementarity problem
$$x_1^+ = (A_c - \Delta)^{-1}(A_c + \Delta)x_1^- + (A_c - \Delta)^{-1}\underline{b} \quad ,$$

(b) to solve the system $(A_c - \Delta T_z)x = \underline{b}$ until $T_z x \geq 0$; if $T_z x$ is not nonnegative in the current step, we set $z_k := -z_k$, where $k = \min\{ j;\ z_j x_j < 0 \}$ and return (T_z is a diagonal matrix with diagonal elements $z_1, \dots, z_n$) ,

(c) to solve the fixed-point equation $x_1 = A_c^{-1}\Delta |x_1| +$ $+ A_c^{-1}\underline{b}$ by Banach iterations $(x^{m+1} = A_c^{-1}\Delta |x^m| +$

$+ A_c^{-1} \underline{b}$, $x^0 = A_c^{-1}\underline{b})$; we have $x^m \longrightarrow x_1$ provided $\rho(A_c^{-1}\Delta) < 1$.

Example 1 (Hansen [2]) . The system

$$[2,3]\, x_1 + [0,1]\, x_2 = [0, 120]$$
$$[1,2]\, x_1 + [2,3]\, x_2 = [60, 240]$$

has a unique strong solution $\tilde{x}^I = [\underset{\sim}{x}, \tilde{x}]$, where $\underset{\sim}{x} = (0, 17.1429)^T$, $\tilde{x} = (30, 68.5714)^T$.

Example 2 (Barth, Nuding [1]). The system

$$[2,4]\, x_1 + [-2,1]\, x_2 = [-2,2]$$
$$[-1,2]\, x_1 + [2,4]\, x_2 = [-2,2]$$

has no strong solution.

References

[1] W.Barth, E.Nuding, Optimale Lösung von Intervallgleichungssystemen, Computing 12 (1974), 117-125

[2] E.Hansen, On Linear Algebraic Equations with Interval Coefficients, in: Topics in Interval Analysis (E.Hansen, Ed.), Clarendon Press, Oxford 1969

[3] J.Rohn, Solving Interval Linear Systems, Freiburger Intervall-Berichte 84/7, 1-14

[4] J.Rohn, Proofs to "Solving Interval Linear Systems", Freiburger Intervall-Berichte 84/7, 17-30

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