TESTING REGULARITY OF INTERVAL MATRICES

by

J. Rohn, Prague

An $n \times n$ interval matrix $A^I = [A_c - \Delta, A_c + \Delta]$ is called regular if $\det A \neq 0$ for each $A \in A^I$, otherwise it is said to be singular. Testing regularity may be a difficult task in the general case. In fact, all the necessary and sufficient regularity conditions arrived at by the author ([3],[5],[6]) are of the form "A^I is regular if and only if $\forall y \in Y \ldots$", where $Y = \{y \in R^n ; |y_j| = 1 \ \forall j\}$, so that in case of regularity they require performing 2^n computations of some sort (testing singularity is generally easier because it suffices to find some $y \in Y$ for which the respective condition is violated). Therefore it is reasonable to try some simpler sufficient regularity (singularity) conditions first, resorting to necessary and sufficient conditions only if the previous tests fail. According to our computational experience, we recommend the following sequence of tests:

0. Compute $D = |A_c^{-1}|\Delta$.
1. If $D_{jj} \geq 1$ for some j, then A^I is singular.
2. If $\rho(D) < 1$, then A^I is regular.
3. Otherwise use algorithm 5.1 in [6] for testing singularity. In case of failure proceed to step 4.

AMS Subject Classification: 65G10

4. Solve the system $A_{yz}x = y$, $T_z x \geqslant 0$ for different $y \in Y$ using the sign - accord algorithm [3,p.6] . If cycling occurs for some $y \in Y$, then A^I is singular.

5. Otherwise (i.e. if $A_{yz}x = y$, $T_z x \geqslant 0$ has a solution for each $y \in Y$), A^I is regular.

For a justification of step 1, see [3, p.8] and [4,p, 26] . Step 2 is the well-known Beeck's spectral radius criterion [2] . Algorithm 5.1 [6, p.112] will perform better if j is always chosen so that $K_j \neq 0$ and $\sum_{k,j} (A_c - A)_{jk} A^{-1}_{kj}$ is minimal. Steps 4 and 5 are based on the assertion (iii) of theorem 4.1 in [6] ; as before, we use the notations $T_z = \operatorname{diag}\{z_1, \dots, z_n\}$, $A_{yz} = A_c - T_y \Delta T_z$. We recommend to start the sign - accord algorithm with $z = \operatorname{sgn}(A_c^{-1}y)$, as mentioned in [5, p.41] , since this reduces considerably the number of systems to be solved. According to theorem 4 in [3, p.7] , in case of regularity the sign - accord algorithm finds for each $y \in Y$ the (unique) solution to $A_{yz}x = y$, $T_z x \geqslant 0$ in a finite number of steps. If A^I is singular, then the latter system does not have a solution for some $y \in Y$, hence the sign - accord algorithm either cycles (returns to the same z and x after several steps), or finds a singular matrix A_{yz} .

We add five examples. For each i , the i-th example terminates in the i-th step (i=1,...,5).

Example 1.

$$\begin{pmatrix} [1, 2] & [3, 4] \\ [2, 5] & [7, 8] \end{pmatrix}$$

We have $\max_j D_{jj} = 9$; singular matrix.

Example 2 (Barth, Nuding [1, p.118]).

$$\begin{pmatrix} [2, 4] & [-2, 1] \\ [-1, 2] & [2, 4] \end{pmatrix}$$

Here, $\max_j D_{jj} = 0.405$ and $\rho(D) = 0.946 < 1$; regular matrix.

Example 3.

$$\begin{pmatrix} [1, 2] & [3, 4] \\ [5, 6] & [7, 8] \end{pmatrix}$$

Since $\max_j D_{jj} = 0.687$ and $\rho(D) = 1.125$, we use the descent algorithm (step 3) which after two iterations produces a singular matrix

$$\begin{pmatrix} 2 & 3 \\ 5 & 7.5 \end{pmatrix} \quad .$$

Example 4 (Baumann [6, p.113]).

$$\begin{pmatrix} [0, 2] & [2, 4] \\ [1, 3] & [0, 2] \end{pmatrix}$$

Here, $\max_j D_{jj} = 0.8$, $\rho(D) = 1.4$ and the descent algorithm fails. Using the sign - accord algorithm for $y = (-1,1)$, we obtain this sequence of z's and x's :

z	x
(1, -1)	(-2 , 1.5)
(-1, -1)	(0.667 , -0.5)
(1, -1)	(-2 , 1.5),

hence the algorithm cycles; singular matrix.

<u>Example 5</u> (Baumann [7]).

$$\begin{pmatrix} [\ 0.25\ ,\ 1.75] & [0.25\ ,\ 1.75] \\ [-1.75\ ,\ -0.25] & [0.25\ ,\ 1.75] \end{pmatrix}$$

For this example, $\max_j D_{jj} = 0.75$, $\rho(D) = 1.5$ and the descent algorithm fails. However, for each $y \in Y$ the sign - accord algorithm finds a solution x_y to $A_{yz}x = y$, $T_z x \geq 0$:

y	x_y
(1, 1)	(0, 4)
(1,-1)	(4, 0)
(-1, 1)	(-4, 0)
(-1,-1)	(0, -4) .

Regular matrix.

References

[1] W.Barth, E.Nuding, Optimale Lösung von Intervallgleichungssystemen, Computing 12 (1974), 117 - 125

[2] H.Beeck, Zur Problematik der Hüllenbestimmung von Intervallgleichungssystemen, in: Interval Mathematics (K.Nickel, Ed.), Lecture Notes, Springer 1975, 150 - 159

[3] J.Rohn, Solving Interval Linear Systems, Freiburger Intervall-Berichte 84/7, 1 - 14

[4] J.Rohn, Proofs to "Solving Interval Linear Systems", Freiburger Intervall-Berichte 84/7, 17 - 30

[5] J.Rohn, Interval Linear Systems, Freiburger Intervall-Berichte 84/7, 33 - 58

[6] J.Rohn, Some Results on Interval Linear Systems, Freiburger Intervall-Berichte 85/4, 93 - 116

[7] M.Baumann, Personal communication, 1982

Author's address: J.Rohn, Faculty of Mathematics and Physics, Charles University, Malostranské nám. 25, 118 00 Prague, Czechoslovakia