A NOTE ON THE SIGN-ACCORD ALGORITHM

by

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In our papers previously published in Freiburger Intervall-Berichte [1],[2],[3] we showed that each vertex x_y of the convex hull of the solution set of an interval linear system $A^I x = b^I$ with regular interval matrix A^I can be described as a unique solution of the system

$$\begin{aligned} A_{yz}x &= b_y \\ T_z x &\geq 0 \end{aligned} \qquad (1)$$

(here, $y, z \in Y = \{\tilde{y} ; |\tilde{y}_j| = 1 \ \forall j\}$, $T_z = \operatorname{diag}\{z_1, \dots, z_n\}$, $A_{yz} = A_c - T_y \Delta T_z$, $b_y = b_c + T_y \delta$, where $A^I = [A_c - \Delta, A_c + \Delta]$, $b^I = [b_c - \delta, b_c + \delta]$) and we proposed the following finite algorithm (called the sign-accord one since it works toward reaching $z_j x_j \geq 0 \ \forall j$) for solving (1) [1, p.6], [2, p.25] :

0. Select a $z \in Y$.
1. Solve $A_{yz}x = b_y$.
2. If $T_z x \geq 0$, stop with $x_y := x$.
3. Otherwise find $k = \min\{j; z_j x_j < 0\}$.
4. Set $z_k := -z_k$ and go to step 1 .

Later [3, p.41] we recommended to specify step 0 by

0^0. Set $z = \operatorname{sgn}(A_c^{-1} b_y)$

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(where $(\operatorname{sgn} x)_i = 1$ if $x_i \geqslant 0$ and $(\operatorname{sgn} x)_i = -1$ otherwise). The idea behind it was quite simple: replacing the equation $A_{yz}x = b_y$ (with unknown z) by $A_c x' = b_y$, one may expect its solution $x' = A_c^{-1} b_y$ to lie in the same orthant as x_y provided A^I is "narrow". Our recent computational experience confirmed the impact of step 0^o upon the behavior of the algorithm, resulting in most cases in going through step 1 only once; this, in fact, was the main reason for writing this note. We shall first support our above - stated intuitive reasoning by some theoretical result and then we shall show an example of a worst-case behavior caused by an improper initialization, where the application of step 0^o leads to a drastic reduction of the number of systems to be solved.

Let $D = |A_c^{-1}|\Delta$. We have this result :

Theorem 1. Let $D|x_y| < |x_y|$ for some $y \in Y$. Then the sign-accord algorithm with step 0^o finds x_y in only one iteration.

Proof. Since $|x_y| > 0$, there exists a unique $z \in Y$ (namely, $z = \operatorname{sgn} x_y$) such that $A_{yz}x_y = b_y$, $T_z x_y \geqslant 0$ holds. Denote $x' = A_c^{-1} b_y$. Then from $A_c x_y = T_y \Delta T_z x_y + b_y = T_y \Delta |x_y| + b_y$, $A_c x' = b_y$ we obtain $A_c(x_y - x') = T_y \Delta |x_y|$, implying $|x_y - x'| \leqslant D|x_y|$. Hence x_y and x' lie in the same orthant, so that $z = \operatorname{sgn} x'$. Since the sign-accord algorithm starts in step 0^o with $z = \operatorname{sgn} x'$, the solution to $A_{yz}x = b_y$ found in step 1 is identical with x_y, so that $T_z x \geqslant 0$ in step 2 and the algorithm stops.

Since $D \to 0$ as $\Delta \to 0$, the condition $D|x_y| < |x_y|$ is satisfied if $|x_y| > 0$ and A^I is sufficiently narrow.

Now, for each $n \geq 2$ consider the interval linear system

$$A_n^I x = [-e, e] \qquad (2)$$

where $e = (1,1,\dots,1) \in R^n$ and the $n \times n$ interval matrix A_n^I is defined by

$$(A_n^I)_{ij} = \begin{cases} 1 & \text{if } i = j \\ [-2,2] & \text{if } j=i+1 \text{ and } 1 \leq i \leq n-1 \\ 0 & \text{otherwise} \end{cases}$$

(it differs only in the right-hand side term from the example (3.1) studied in [3, p.40]).

Theorem 2. Let $n \geq 2$. Then, for the interval linear system (2), we have :

(i) for each $y \in Y$, the sign-accord algorithm, when started from $z = (y_1, y_2, \dots, y_{n-1}, -y_n)$ in step 0 , solves 2^n systems to find x_y ,

(ii) for each $y \in Y$, the sign-accord algorithm, starting with step 0^o , solves only one system to find x_y .

Proof. First we find by backward substitutions that for each $y,z \in Y$ the solution of the system $A_{yz}x = b_y$ is given by

$$x_j = y_j \sum_{m=0}^{n-j} 2^m \prod_{i=j+1}^{j+m} y_i z_i \qquad (j = 1,\dots,n)$$

(where we employ the usual convention $\sum_{\emptyset} = 0, \prod_{\emptyset} = 1$). Hence

$$z_j x_j = \sum_{m=0}^{n-j} 2^m \prod_{i=j}^{j+m} y_i z_i \qquad (j = 1,\dots,n)$$

and since the last term prevails, we have

$\operatorname{sgn}(z_j x_j) = \operatorname{sgn}(\prod_{i=j}^{n} y_i z_i) = \prod_{i=j}^{n} y_i z_i$ for each $j=1,\dots,n$.
Next we prove that for each $y \in Y$, the number $p_y(z)$ of systems the sign-accord algorithm must solve to find x_y when started from vector z in step 0 is given by

$$p_y(z) = 1 + \sum_{j=1}^{n} (1 - \prod_{i=j}^{n} y_i z_i)\, 2^{j-2}. \qquad (3)$$

We shall carry out the proof by induction on $p_y(z)$. If $p_y(z) = 1$, then the sign-accord algorithm, after solving $A_{yz}\, x = b_y$, stops with $T_z x \geqslant 0$. Hence for each j we have $\prod_{i=j}^{n} y_i z_i = \operatorname{sgn}(z_j x_j) = 1$, so that the right-hand side in (3) is equal to 1. Now assume that (3) holds for each y,z with $p_y(z) \leqslant r$ and let y,z be such that $p_y(z) = r+1$. Let z' be the updated value of z after passing for the first time through step 4. Then $z'_k = -z_k$, $z'_j = z_j$ for $j \neq k$, $\prod_{i=j}^{n} y_i z_i = \operatorname{sgn}(z_j x_j) = 1$ for $j < k$, $\prod_{i=k}^{n} y_i z_i = \operatorname{sgn}(z_k x_k) = -1$, hence by the inductive assumption, $p_y(z) = 1 + p_y(z') = 2 + \sum_{j=1}^{n} (1 - \prod_{i=j}^{n} y_i z'_i)\, 2^{j-2} = \dots =$
$= 1 + \sum_{j=1}^{n} (1 - \prod_{i=j}^{n} y_i z_i)\, 2^{j-2}$ (since $\prod_{j}^{n} y_i z'_i = - \prod_{j}^{n} y_i z_i$ for $j \leqslant k$ and $\prod_{j}^{n} y_i z'_i = \prod_{j}^{n} y_i z_i$ for $j > k$), which completes the inductive proof of (3).

Now, if $z = (y_1, y_2, \dots, y_{n-1}, -y_n)$, then $\prod_j^n y_i z_i = -1$ for each j, hence $p_y(z) = 1 + \sum_{j=1}^{n} 2^{j-1} = 2^n$, which proves (i). Using step 0^o, we have $z = \mathrm{sgn}(A_c^{-1} b_y) = y$ (since $A_c = E$ and $b_y = y$), hence $\prod_j^n y_i z_i = 1$ for each j, implying $p_y(z) = 1$ in this case, which completes the proof.

Remark. The equation (3) has also another interesting consequences. E.g., for each $y \in Y$ and each k, $1 \leq k \leq 2^n$, there exists a $z \in Y$ such that $p_y(z) = k$, etc.

References

[1] J.Rohn, Solving Interval Linear Systems, Freiburger Intervall-Berichte 84/7, 1-14

[2] J.Rohn, Proofs to "Solving Interval Linear Systems", Freiburger Intervall-Berichte 84/7, 17-30

[3] J.Rohn, Interval Linear Systems, Freiburger Intervall-Berichte 84/7, 33-58

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