FORMULAE FOR EXACT BOUNDS ON SOLUTIONS OF LINEAR SYSTEMS WITH RANK ONE PERTURBATIONS

by

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Abstract. Simple formulae are given for the exact bounds on the solution of a linear interval system $A^I x = b^I$ where the radius of A^I is of the form qp^T . The results are applied to evaluating loss of significant decimals due to input data rounding, constructing new condition numbers for linear systems and matrices, investigating the expansion of bounds due to preconditioning and bounding the residuals.

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0. Introduction

For a linear interval system $A^I x = b^I$ with square $A^I = [A - \Delta , A + \Delta] = \{A'; A - \Delta \leq A' \leq A + \Delta\}$ and $b^I = [b - d, b + d] = \{b'; b - d \leq b' \leq b + d\}$, the solution set is defined as $X = \{x'; A'x' = b', A' \in A^I, b' \in b^I\}$ and the exact bounds on the solution are given by

$$\underline{x}_i = \min \{x_i'; x' \in X\}$$
$$\overline{x}_i = \max \{x_i'; x' \in X\}$$

$(i = 1,\dots,n)$. In our previous papers [6] - [8] , we described general, but complicated methods for computing $\underline{x}_i$, $\overline{x}_i$. In the present paper we show that if the radius Δ of the interval matrix A^I is of the form

$$\Delta = qp^T \tag{0}$$

for some nonnegative (column) vectors q, p (a so-called rank one perturbation), then, assuming that q, p and d are sufficiently small, the bounds $\underline{x}_i$, $\overline{x}_i$ can be computed by using simple formulae derived in section 1.1. These formulae take on an attractively simple form in the common special case

$$\Delta_{ij} = \beta = \text{const}$$
$$d_i = \gamma = \text{const}$$

(section 1.2, Eqs. (3)). In the second part of the paper, we apply the explicit results obtained in the first part to some special problems : evaluating the loss of significant decimals due to input data rounding, constructing new condition numbers for linear systems and matrices, investigating the expansion of the bounds $\underline{x}_i$, $\overline{x}_i$ due to preconditioning and deriving bounds on residuals. Since all the results are sharp, they offer a closer look onto the nature of these problems.

The possibility of expressing the bounds explicitly for problems satisfying (0) was discovered by Hansen [2] who, however, did not elaborate the results. The original impetus to this work came from my discussion with Dr. D.Hudak and Dr. G.Richter at the interval symposium in Dresden in June 1986, who drew my attention to systems satisfying (0); my thanks are due to them.

1. Formulae

In the first part of the paper, we prove the main result (section 1.1, formulae (1)) and consider some special cases (section 1.2).

1.1. Main result

Consider a linear interval system

$$[A - qp^T,\ A + qp^T]x' = [b - d,\ b + d]$$

where A is a regular $n \times n$ matrix and $b,q,p,d \in R^n$, q,p,d nonnegative. We denote

$$x = A^{-1}b$$
$$\bar{d} = |A^{-1}|\, d$$
$$\bar{q} = |A^{-1}|\, q$$
$$\bar{p}^T = p^T |A^{-1}| \quad ,$$

where, as usual, $|A^{-1}|$ stands for the matrix formed by the absolute values of the coefficients of A^{-1}. Hence $\bar{d},\bar{q},\bar{p}$ are again nonnegative. Next, for each $i \in \{1,\dots,n\}$ we introduce the numbers

$$\lambda_i = z^T T_p A^{-1} T_q y_i$$
$$\mu_i = z^T T_p A^{-1} T_d y_i \quad ,$$

where, for each $a \in R^n$, T_a denotes the diagonal matrix with diagonal vector a, z is the signature vector of x

(defined by $z_i = 1$ if $x_i \geqslant 0$ and $z_i = -1$ otherwise) and, similarly, y_i is the signature vector of the i-th row of A^{-1} (notice : the "T" in z^T denotes the transpose vector; not to be mixed with T_p). We shall introduce two assumptions :

$$\text{(i)} \quad \bar{q}\bar{p}^T + (p^T\bar{q})\,|A^{-1}| < |A^{-1}|$$

$$\text{(ii)} \quad p^T(|x| + \bar{d})\bar{q} + (1-p^T\bar{q})\bar{d} + (p^T\bar{q})|x| < |x| \ .$$

The meaning of these assumptions will become clear from the parts (b), (d) of the proof below. Here we only mention that they are satisfied if all entries of A^{-1} and x are nonzero and if q,p and d are sufficiently small.

Now we may formulate our main result :

<u>Theorem</u>. Let (i), (ii) hold. Then for each $i \in \{1,\dots,n\}$ we have

$$\begin{aligned} \underline{x}_i &= x_i - \bar{d}_i - \frac{(p^T|x| - \mu_i)\bar{q}_i}{1 + \lambda_i} \\ \bar{x}_i &= x_i + \bar{d}_i + \frac{(p^T|x| + \mu_i)\bar{q}_i}{1 - \lambda_i} \end{aligned} \qquad (1)$$

<u>Proof</u>. The proof goes through several steps.

(a) From (i) it follows $(p^T\bar{q})\,|A^{-1}| < |A^{-1}|$, hence $p^T\bar{q} < 1$. Put $\Delta = qp^T$ and $D = |A^{-1}|\Delta = \bar{q}p^T$, and let $\bar{\rho} = \rho(D)$, the spectral radius of D. Then $Dx = \bar{q}p^Tx = \bar{\rho}x$ for some $x \neq 0$. Hence either $p^Tx = 0$, implying $\bar{\rho} = 0$, or $p^T x \neq 0$, in which case premultiplying the above equation by p^T yields $\bar{\rho} = p^T \bar{q}$, in both the cases $\bar{\rho} < 1$. Hence the interval matrix $A^I = [A - qp^T, A + qp^T]$ is regular due to Beeck's criterion [1] . Moreover, for each $i \in \{1, \ldots, n\}$ we have $|\lambda_i| \leq p^T \bar{q} < 1$.

(b) For $D = \bar{q} p^T$ we have $D^2 = \bar{q}(p^T \bar{q}) p^T = (p^T \bar{q}) D$, and by induction $D^j = (p^T \bar{q})^{j-1} D$, $j \geq 1$. Hence the matrix $C = D(E-D)^{-1}$ satisfies $C = D + D^2 + \ldots = [1 + p^T \bar{q} + (p^T \bar{q})^2 + \ldots] D = \frac{1}{1 - p^T \bar{q}} D$ and rearranging the condition (i) gives

$$C |A^{-1}| = \frac{\bar{q} \bar{p}^T}{1 - p^T \bar{q}} < |A^{-1}|$$

which means (see [6, p.10] and [7, p.24]) that A^I is inverse stable, i.e. the coefficients of the inverse matrix preserve their signatures over A^I .

(c) In [6],[7],[8] it was shown that for each $y \in Y = \{y \in R^n;\ |y_j| = 1 \text{ for each } j\}$, the nonlinear equation

$$\tilde{x} = A^{-1} T_y \Delta |\tilde{x}| + A^{-1}(b + T_y d)$$

has a unique solution x_y . Let t be the signature vector of x_y , then $|x_y| = T_t x_y$ and x_y satisfies

$$x_y = x + A^{-1} T_y d + (p^T T_t x_y) A^{-1} T_y q \ .$$

Set $\alpha_y = p^T T_t x_y$, then $x_y = x + A^{-1} T_y d + \alpha_y A^{-1} T_y q$, and premultiplying this equation by $p^T T_t$ gives

$$\alpha_y = \frac{p^T T_t (x + A^{-1} T_y d)}{1 - p^T T_t A^{-1} T_y q} \quad ,$$

and finally

$$x_y = x + A^{-1} T_y d + \frac{p^T T_t (x + A^{-1} T_y d)}{1 - p^T T_t A^{-1} T_y q} A^{-1} T_y q. \qquad (*)$$

This expression, however, still contains an unknown signature vector t .

(d) From $(*)$, using the fact that $|T_t| = |T_y| =$ = unit matrix and that $|p^T T_t A^{-1} T_y q| \leq p^T \bar{q} < 1$, we obtain

$$|x_y - x| \leq \bar{d} + \frac{p^T(|x| + \bar{d})}{1 - p^T \bar{q}} \bar{q} < |x| \quad ,$$

the last inequality being a consequence of the assumption (ii). This shows that x_y and x lie in the interior of the same orthant, hence t is equal to z , the signature vector of x.

(e) From the general theory in [6],[7] it follows, in view of the inverse stability established in (b), that $\underline{x}_i = (x_{-y_i})_i$, $\bar{x}_i = (x_{y_i})_i$, where y_i is the signature vector of the i-th row of A^{-1} . Since $(A^{-1} T_{y_i} d)_i = \bar{d}_i$ and $(A^{-1} T_{y_i} q)_i = \bar{q}_i$, from $(*)$ (with $y := y_i$ and $t := z$) we directly get

$$\bar{x}_i = (x_{y_i})_i = x_i + \bar{d}_i + \frac{p^T|x| + \mu_i}{1 - \lambda_i} \bar{q}_i$$

(similarly for $\underline{x}_i$) , which proves (1). ■

To determine $\underline{x}$ and $\bar{x}$ using (1), we must solve the linear system $Ax = b$, compute the inverse matrix A^{-1} and evaluate the quantities $\bar{q}, \bar{p}, \bar{d}, \mu_i, \lambda_i$ $(i=1,\dots,n)$, hence $\underline{x}, \bar{x}$ may be found in $O(n^3)$ arithmetic operations.

Notice that if $qp^T = 0$ (i.e. the left-hand side matrix contains no interval coefficients), then the bounds are given by $\underline{x} = x - d$, $\bar{x} = x + d$; hence the fractions in (1) amount to the influence of the input data errors in the matrix coefficients.

As seen from the proof, the assumptions (i), (ii) provide for the signature stability of both the inverse matrix and the solution vector (cf. theorem 2 in [6]).

1.2. Special cases

Under additional restrictions on q, p and d , formulae (1) take on simpler forms.

Consider first the case $d = \gamma q$, where γ is a non-negative real number. Then (i) remains the same, (ii) reduces to

$$(ii_1) \qquad (p^T|x| + \gamma)\bar{q} + (p^T\bar{q})|x| < |x|$$

and, since $\mu_i \bar{q} = \lambda_i \bar{d}$ for each i , formulae (1) now have the form

$$\underline{x}_i = x_i - \frac{(p^T|x| + \gamma)\bar{q}_i}{1 + \lambda_i}$$

$$\bar{x}_i = x_i + \frac{(p^T|x| + \gamma)\bar{q}_i}{1 - \lambda_i} \,. \qquad (2)$$

Notice that $\bar{x}_i - x_i > x_i - \underline{x}_i$ iff $\lambda_i > 0$ and $\bar{x}_i - x_i < x_i - \underline{x}_i$ iff $\lambda_i < 0$.

Second, consider a more special case

$$\Delta_{ij} = \beta = \text{const}$$
$$d_i = \gamma = \text{const} \qquad (i,j=1,\dots,n)$$

that corresponds to $p = \beta e$, $q = e$, $d = \gamma e$, where $e = (1,1,\dots,1)^T$. Define

$$r = |A^{-1}|e$$
$$s^T = e^T|A^{-1}|$$
$$\nu_i = z^T A^{-1} y_i \qquad (i=1,\dots,n) \,,$$

then (i), (ii) can be reformulated as

$$(i_2) \quad \beta\,[\,rs^T + \|r\|\,|A^{-1}|\,] < |A^{-1}|$$

$$(ii_2) \quad \beta\,[\,\|x\|\,r + \|r\|\,|x|\,] + \gamma r < |x|$$

where we used the 1-norm $\|x\| = \|x\|_1 = \sum_i |x_i|$. For $\underline{x}, \bar{x}$ we then get

$$\underline{x}_i = x_i - \frac{(\beta \|x\| + \gamma) r_i}{1 + \beta \nu_i}$$

$$\bar{x}_i = x_i + \frac{(\beta \|x\| + \gamma) r_i}{1 - \beta \nu_i} \, . \tag{3}$$

Finally, in the case $\beta = \gamma$ (all the coefficients are given with a uniform absolute error), we have

$$\underline{x}_i = x_i - \frac{\beta (\|x\| + 1) r_i}{1 + \beta \nu_i}$$

$$\bar{x}_i = x_i + \frac{\beta (\|x\| + 1) r_i}{1 - \beta \nu_i} \tag{4}$$

In this case, β is specified by the assumptions

$$(i_3) \quad \beta [r s^T + \|r\| \, |A^{-1}|] < |A^{-1}|$$

$$(ii_3) \quad \beta [(\|x\| + 1) r + \|r\| \, |x|] < |x| \tag{5}$$

which are equivalent to $\beta \in [0, \beta_0)$, where β_0 can be easily computed from (5) as the minimum of ratios of the respective coefficients.

Notice that in the last case, all the quantities r, s, ν_i, β_0 appearing in (4) and (5) depend on A and x. Hence setting $\beta \to 0_+$, we may get some assertions concerning the sensitivity of a given real (not interval) linear system $Ax = b$. This will be done in the second part of the paper.

The results obtained can be also used for the evaluation of the inverse interval matrix $(A^I)^{-1} = [\underline{B}, \overline{B}]$ given by

$$\underline{B}_{ij} = \min \{(A_o^{-1})_{ij} \ ; \ A_o \in A^I\}$$
$$\overline{B}_{ij} = \max \{(A_o^{-1})_{ij} \ ; \ A_o \in A^I\} \quad (i,j=1,\dots,n).$$

We shall give the results only for the (most interesting) case $A^I = [A - \beta ee^T, \ A + \beta ee^T]$. Since the j-th columns of $\underline{B}$, $\overline{B}$ form the exact interval solution of the linear interval system $A^I x = e_j$, where e_j is the j-th column of the unit matrix, from (3) (with $\gamma = 0$ and $\| x \| = s_j$) we obtain

$$\underline{B}_{ij} = a_{ij}^{-1} - \frac{\beta r_i s_j}{1 + \beta \nu_{ij}} \tag{6}$$
$$\overline{B}_{ij} = a_{ij}^{-1} + \frac{\beta r_i s_j}{1 - \beta \nu_{ij}} \qquad (i,j=1,\dots,n)$$

where $A^{-1} = (a_{ij}^{-1})$ and $\nu_{ij} = \tilde{y}_j^T A^{-1} y_i$, $\tilde{y}_j$ being the signature vector of the j-th column of A^{-1} . In this case (ii_3) is identical with (i_3), so that β is bounded only by the condition

$$(i_3) \qquad \beta [rs^T + \| r \| \ |A^{-1}|] < |A^{-1}|.$$

2. Applications

In the second part, we give some consequences of formulae (3), (4). In the subsequent sections, we study the loss of significant decimals, propose new condition numbers, investigate preconditioned systems and give bounds for residuals.

2.1. Loss of significant decimals due to data rounding

Assume that both left- and right-hand side coefficients of a system of linear equations

$$A'x' = b'$$

have been rounded off to π decimals, giving a system

$$Ax = b .$$

Here we will handle the problem of determining the number of significant decimals in x , i.e. for each $i \in \{1,\dots,n\}$ we will be looking for the maximal integer δ_i satisfying

$$|x_i' - x_i| \leq 5 \times 10^{-(\delta_i+1)}$$

if nothing more is known of A', b' but that A, b are their rounded values up to π decimals.

Clearly, each such x' is an element of the solution set X of the system of linear interval equations

$$[A - \beta ee^T, A + \beta ee^T]x' = [b - \beta e, b + \beta e]$$

with

$$\beta = 5 \times 10^{-(\pi+1)} .$$

Hence, if β is sufficiently small to satisfy (5), we may use formulae (4) to get

$$\max_X |x_i' - x_i| = \max\{\overline{x}_i - x_i, x_i - \underline{x}_i\} = \frac{\beta(\|x\|+1)r_i}{1-\beta|\nu_i|}$$

which shows that δ_i is the maximal integer satisfying

$$\frac{\beta(\|x\|+1)r_i}{1-\beta|\nu_i|} \leq 5 \times 10^{-(\delta_i+1)} .$$

Taking $\log_{10}$ on both sides of this inequality, we obtain

$$\delta_i = \pi - \left[\log_{10}\frac{(\|x\|+1)r_i}{1-\beta|\nu_i|}\right]_0$$

where we denoted $[a]_0 = \min\{k;\ a \leq k\ ,\ k \text{ integer}\}$ (so that $[a]_0 = [a]$ if a is integer and $[a]_0 = [a]+1$ otherwise). Hence the number

$$\left[\log_{10}\frac{(\|x\|+1)r_i}{1-\beta|\nu_i|}\right]_0 = \pi - \delta_i$$

represents the loss of significant decimals due to input data rounding (although this loss may be negative since the possibility of $\pi - \delta_i < 0$ is not excluded). If $\pi \to \infty$ (equivalently, $\beta \to 0_+$), then $\pi - \delta_i$ tends to a finite value σ_i given by

$$\sigma_i = \begin{cases} [\log_{10}((\|x\|+1)r_i)]_0 + 1 & \text{if } \nu_i \neq 0 \text{ and } \log_{10}((\|x\|+1)r_i) \text{ is integer} \\ [\log_{10}((\|x\|+1)r_i)]_0 & \text{otherwise} \end{cases}$$

which shows that the loss becomes constant from some π_0 on. Notice that

$$\left[\log_{10} \frac{(\|x\|+1)r_i}{1-\beta|\nu_i|}\right]_0 \geqslant \sigma_i$$

for each $\beta > 0$ since the left-hand side function is non-decreasing in β. Thus the value of

$$\sigma = \max_i \sigma_i ,$$

computable in terms of A^{-1} and x, gives an information about the numerical stability of the system $Ax = b$. Notice that if none of the numbers $\log_{10}((\|x\|+1)r_i)$, $i=1,\ldots,n$, is integer, which is very probably the case in practical computations, the we have

$$\sigma = \left[\log_{10}((\|x\|+1)\|A^{-1}\|_\infty)\right]_0$$

(since $\max_i r_i = \max_i \sum_j |a_{ij}^{-1}| = \|A^{-1}\|_\infty$; recall that $\|x\| = \|x\|_1 = \sum_i |x_i|$).

Similarly, if a matrix A' is rounded off up to π decimals to yield a matrix A, then an analogous reasoning based on (6) gives for the number δ_{ij} of significant decimals in the ij-th coefficient of A^{-1} the formula

$$\delta_{ij} = \pi - \left[\log_{10} \frac{r_i s_j}{1-\beta|\nu_{ij}|}\right]_0$$

and again,

$\mathcal{T} - \delta_{ij} \longrightarrow \sigma_{ij}$ as $\mathcal{T} \longrightarrow \infty$, where

$$\sigma_{ij} = \begin{cases} [\log_{10}(r_i s_j)]_0 + 1 & \text{if } \nu_{ij} \neq 0 \text{ and} \\ & \log_{10}(r_i s_j) \text{ is integer} \\ [\log_{10}(r_i s_j)]_0 & \text{otherwise.} \end{cases}$$

If none of the numbers $\log_{10}(r_i s_j)$ is integer, then

$$\sigma_A = \max_{i,j} \sigma_{ij} = [\log_{10}(\|A^{-1}\|_\infty \|A^{-1}\|_1)]_0$$

(since $\max_j s_j = \max_j \sum_i |a_{ij}^{-1}| = \|A^{-1}\|_1$) .

Example. For Hilbert's 3×3 matrix ($A_{ij} = 1/(i+j-1)$ for $i,j = 1,2,3$) we have $\|A^{-1}\|_1 = \|A^{-1}\|_\infty = 408$, so that $\sigma_A = 6$. Hence a loss of about 6 decimals can be expected when inverting A on a computer (not taking into account additional errors arising during the computation).

2.2. Condition numbers

The asymptotic behaviour of relative errors, given by (4) and (6), shows a way how to define new condition numbers for linear systems and matrices.

To endorse the dependence upon β , denote the corresponding solution set by X_β . Then we have

$$\max_{x' \in X_\beta} \frac{|x_i' - x_i|}{|x_i|} = \frac{\beta(\|x\| + 1) r_i}{(1 - \beta|\nu_i|)|x_i|} ,$$

hence

$$\lim_{\beta\to 0_+} \frac{1}{\beta} \max_{x'\in X_\beta} \frac{|x_i' - x_i|}{|x_i|} = (\|x\| +1) \frac{r_i}{|x_i|}$$

Thus we may propose a new condition number for linear systems

$$c(A,b) = (\|x\| +1) \max_i \frac{r_i}{|x_i|} .$$

It shows that if β is sufficiently small, then the maximal relative error is equal to about $\beta\, c(A,b)$.

Similarly, on the base of (6) we may propose a condition number for matrices

$$c(A) = \max_{i,j} \frac{r_i s_j}{|a_{ij}^{-1}|} .$$

Here we have $c(A) \geqslant \max \left\{ \|A^{-1}\|_1 , \|A^{-1}\|_\infty \right\}$. Contrary to usual condition numbers, as $\|A\|\,\|A^{-1}\|$, here an ill behaviour of a single element may influence the condition number, which may lead to different results. Comparison on test examples (not performed by the author) is necessary to check the usefullness of these condition numbers.

2.3. Preconditioned systems

In this section we shall investigate the effect of preconditioning of the linear interval system

$$[A - \beta ee^T,\ A + \beta ee^T]x' = [b - \gamma e,\ b + \gamma e] \tag{7}$$

upon the bounds on its solution. Preconditioning, i.e. premultiplying the whole system by A^{-1} in interval arithmetic, leads to the system

$$[E - \beta re^T, E + \beta re^T]\ x' = [x - \gamma r,\ x + \gamma r] \qquad (8)$$

where E is the unit matrix and, as before, $r = |A^{-1}|e$. Preconditioning was proposed by Hansen and Smith [3] to bring the original system (7) to a form more suitable for interval Gauss elimination. Naturally, the solution set grows as a result of preconditioning, hence the exact bounds $\underset{\sim}{x}$, $\tilde{\overline{x}}$ for the solution of (8) satisfy $\underset{\sim}{x} \leq \underline{x}$, $\overline{x} \leq \tilde{\overline{x}}$. To compute $\underset{\sim}{x}$ and $\tilde{\overline{x}}$, we cannot apply directly the main theorem, since the condition (i) is violated for (8) (E contains zeros), but we may follow another way by computing the vectors $\tilde{x}_y$ for the system (8). As in the part (c) of the proof, we can derive

$$\tilde{x}_y = x + \frac{\beta e^T T_t x + \gamma}{1 - \beta e^T T_t T_y r}\ T_y r$$

where t is the signature vector of $\tilde{x}_y$. Assuming that (i_2) and (ii_2) hold for the original system (7), we have $\beta\|r\| < 1$ and

$$|\tilde{x}_y - x| \leq \frac{\beta\|x\| + \gamma}{1 - \beta\|r\|}\ r < |x| \quad ,$$

hence, as in the part (d) of the proof, we conclude that $t = z$ and thus we get

$$\tilde{x}_y = x + \frac{\beta\|x\| + \gamma}{1 - \beta e^T T_z T_y r}\ T_y r \qquad (9)$$

Here z is the signature vector of x, while $y \in Y$ was arbitrary so far. According to the general theory ([6], theorem 2), $\tilde{x}_i$ is achieved at some $(\tilde{x}_y)_i$. Hence, to make $(\tilde{x}_y)_i$

as large as possible, we must in the light of (9) take $y_i = 1$ and choose y_j, $j \neq i$, so as to maximize the value of $e^T T_z T_y r = \sum_j z_j y_j r_j$; this gives $y_j = z_j$ for $j \neq i$ and

$$\tilde{x}_i = x_i + \frac{(\beta \| x \| + \gamma) r_i}{1 - \beta \bar{\varkappa}_i}$$

where

$$\bar{\varkappa}_i = z_i r_i + \sum_{j \neq i} r_j = \| r \| - r_i + z_i r_i .$$

In a similar way we would obtain

$$\underset{\sim}{x}_i = x_i - \frac{(\beta \| x \| + \gamma) r_i}{1 - \beta \underline{\varkappa}_i}$$

with

$$\underline{\varkappa}_i = \| r \| - r_i - z_i r_i .$$

Finally, we get

$$\tilde{x}_i - \bar{x}_i = \frac{(\beta \| x \| + \gamma) r_i \beta (\bar{\varkappa}_i - \nu_i)}{(1 - \beta \bar{\varkappa}_i)(1 - \beta \nu_i)}$$

$$\underline{x}_i - \underset{\sim}{x}_i = \frac{(\beta \| x \| + \gamma) r_i \beta (\underline{\varkappa}_i + \nu_i)}{(1 - \beta \underline{\varkappa}_i)(1 + \beta \nu_i)}$$

which in the special case of $\gamma = \beta$ implies

$$\lim_{\beta \to 0_+} \frac{\tilde{x}_i - \bar{x}_i}{\beta^2} = (\| x \| + 1) r_i (\bar{\varkappa}_i - \nu_i)$$

$$\lim_{\beta \to 0_+} \frac{\underline{x}_i - \underset{\sim}{x}_i}{\beta^2} = (\| x \| + 1) r_i (\underline{\varkappa}_i + \nu_i) ,$$

confirming the well-known Miller's result in [4] that $\tilde{x} - \bar{x} = O(\beta^2)$, $\underline{x} - \underset{\sim}{x} = O(\beta^2)$. Finally, for the difference of the radii ratio from 1 we have

$$\lim_{\beta \to 0_+} \frac{1}{\beta} \left(\frac{\tilde{x}_i - \underset{\sim}{x}_i}{\bar{x}_i - \underline{x}_i} - 1 \right) = \|r\| - r_i ;$$

cf. Neumaier [5] , theorem 3.

2.4. Bounds for residuals

For each element x' of the solution set X of the linear interval system $[A - \beta ee^T, A + \beta ee^T]\, x' = [b - \gamma e, b + \gamma e]$ define its residual by

$$res(x') = Ax' - b .$$

Using again the fact that the maximal value $\overline{res}_i$ of $(res(x'))_i$ must be achieved at some x_y and that for each $y \in Y$,

$$Ax_y - b = \frac{\beta \|x\| + \gamma}{1 - \beta z^T A^{-1} y}\, y$$

holds, arguing as in the preceding section, we get

$$\overline{res}_i = \frac{\beta \|x\| + \gamma}{1 - \beta \overline{\Psi}_i}$$

$$\underline{res}_i = - \frac{\beta \|x\| + \gamma}{1 - \beta \underline{\Psi}_i}$$

where

$$\overline{\psi}_i = e^T|\bar{z}| - |\bar{z}_i| + \bar{z}_i$$

$$\underline{\psi}_i = e^T|\bar{z}| - |\bar{z}_i| - \bar{z}_i$$

$$\bar{z}^T = z^T A^{-1} \quad .$$

Especially, if $\gamma = \beta$, then

$$\lim_{\beta \to 0_+} \frac{\overline{res}_i}{\beta} = - \lim_{\beta \to 0_+} \frac{\underline{res}_i}{\beta} = \|x\| + 1 \ .$$

It is remarkable that the limit value is independent of i and is always greater or equal than 1.

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