EIGENVALUES OF A SYMMETRIC INTERVAL MATRIX

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Abstract. Exact bounds for eigenvalues of a symmetric interval matrix of the form $A^I = [A_c - rr^T, A_c + rr^T]$ (A_c symmetric, $r > 0$) are given under assumptions that all eigenvalues of A_c are mutually different, the eigenvectors of A_c have nonzero entries and r is sufficiently small in norm to preserve these properties over A^I.

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In this paper we investigate the eigenvalues of a symmetric interval matrix $A^I = [A_c - rr^T, A_c + rr^T]$, where A_c is a symmetric $n \times n$ matrix and r is a (column) vector whose all entries are positive. We shall give the results under three assumptions. First we shall assume that

(i) each $A \in A^I$ has n different real eigenvalues

$$\lambda_1(A) < \lambda_2(A) < \ldots < \lambda_n(A).$$

Then we may define the sets

$$L_i = \{ \lambda_i(A);\ A \in A^I \} \quad (i = 1,\ldots,n).$$

Second we shall assume that

(ii) $L_i \cap L_j = \emptyset$ for $i \neq j$ $(i,j = 1,\ldots,n)$

holds. Before formulating the third assumption, we first introduce, for any $a \in R^n$, the matrix T_a as the diagonal matrix with diagonal vector a, and define $Y = \{ z \in R^n;\ |z_j| = 1 \text{ for each } j \}$. We assume

(iii) for each $i \in \{1,\ldots,n\}$ there exists a $y_i \in Y$ such that each eigenvector x corresponding to an eigenvalue from L_i satisfies either $T_{y_i} x > 0$, or $T_{y_i} x < 0$.

Here the inequalities are to be understood componentwise. If we introduce the signature vector sgn x of a vector $x \in R^n$ by $(\mathrm{sgn}\, x)_i = 1$ if $x_i > 0$ and $(\mathrm{sgn}\, x)_i = -1$ otherwise, then each eigenvector corresponding to an eigenvalue from L_i satisfies $\mathrm{sgn}\, x = y_i$ or $\mathrm{sgn}\, x = -y_i$. To simplify notations, denote $T_i := T_{y_i}$. Since eigenvectors corresponding to different eigenvalues of A_c are orthogonal, we have $y_i \neq y_j$, thus also $T_i \neq T_j$, for each $i \neq j$.

In the key part of the proof of Theorem 1, we shall use the following lemma, which is of independent interest.

Lemma. Let B be a regular nxn matrix and let p,q be nonnegative vectors from R^n. Then the interval matrix $[B - qp^T, B + qp^T]$ is singular if and only if

$$z^T T_p B^{-1} T_q y \geq 1$$

holds for some $z, y \in Y$.

Proof. According to Theorem 6.3 in [2, p.44], $[B - qp^T, B + qp^T]$ is singular if and only if there exist $z, y \in Y$ such that the matrix $B^{-1}T_y q p^T T_z$ has a real eigenvalue λ with $|\lambda| \geq 1$. Then $B^{-1}T_y q p^T T_z x = (p^T T_z x)\, B^{-1} T_y q = \lambda x$ for some $x \neq 0$, where $p^T T_z x \neq 0$ due to $\lambda \neq 0$, hence premultiplying the equation by $p^T T_z$ gives $p^T T_z B^{-1} T_y q = \lambda$. Setting $z := -z$ if $\lambda < 0$, we obtain $z^T T_p B^{-1} T_q y = p^T T_z B^{-1} T_y q = |\lambda| \geq 1$. ∎

In the main theorem to follow, we give exact bounds for eigenvalues and also prove that the extremal eigenvalues are achieved at some symmetric matrices from A^I:

<u>Theorem 1.</u> Let $r > 0$ and let (i), (ii), (iii) hold. Then for each $i \in \{1,\dots,n\}$ we have

$$L_i = [\underline{\lambda}_i, \overline{\lambda}_i]$$

where

$$\begin{aligned} \underline{\lambda}_i &= \min \{ \lambda_i(A_c - D_i),\ \lambda_i(A_c + D_i) \} \\ \overline{\lambda}_i &= \max \{ \lambda_i(A_c - D_i),\ \lambda_i(A_c + D_i) \} \end{aligned} \qquad (1)$$

and

$$D_i = T_i r r^T T_i .$$

<u>Proof.</u> The proof consists of several steps. Let $i \in \{1,\dots,n\}$.

<u>(a)</u> We prove that L_i is compact. If $\lambda \in L_i$, then $\lambda = x^T A x$ for some $A \in A^I$ and x satisfying $\|x\|_2 = 1$, hence L_i is bounded. To prove that L_i is closed, let $\lambda^j \in L_i$ ($j = 1,2,\dots$) and $\lambda^j \rightarrow \lambda$. Then $A^j x^j = \lambda^j x^j$ for some $A^j \in A^I$, $\|x^j\|_2 = 1$, $T_i x^j > 0$ ($j = 1,2,\dots$) and there exists a subsequence $\{j_k\}$ such that $A^{j_k} \longrightarrow A \in A^I$, $x^{j_k} \longrightarrow x$, $\|x\|_2 = 1$, $T_i x \geqslant 0$, $Ax = \lambda x$. Since x is an eigenvector, it must be $T_i x > 0$ due to (iii), thus x corresponds to an eigenvalue from L_i; this shows that $\lambda \in L_i$, so that L_i is closed and thus also compact.

<u>(b)</u> Next we show that $\lambda_i(A_c) \in L_i^o$, the interior of L_i. Take an eigenvector x of A_c corresponding to $\lambda_i(A_c)$ and choose an $\varepsilon_o > 0$ such that $\sqrt{\varepsilon_o}|x| \leq r$ and $(\lambda_i(A_c) - \varepsilon_o\|x\|_2^2,\ \lambda_i(A_c) + \varepsilon_o\|x\|_2^2) \cap L_j = \emptyset$ for each $j \neq i$ (this is possible due to the assumption (ii) and the compactness of the L_j's established in (a)). Then for each $\varepsilon \in (-\varepsilon_o, \varepsilon_o)$ we have $A_c + \varepsilon x x^T \in A^I$ and $(A_c + \varepsilon x x^T)x = (\lambda_i(A_c) + \varepsilon\|x\|_2^2)x$, hence $\lambda_i(A_c) + \varepsilon\|x\|_2^2$ is an eigenvalue from L_i; thus $\lambda_i(A_c) \in L_i^o$.

<u>(c)</u> In view of (a), $L_i - L_i^o \neq \emptyset$. Let $\lambda \in L_i - L_i^o$. We shall prove that either $\lambda = \lambda_i(A_c - D_i)$, or $\lambda = \lambda_i(A_c + D_i)$. Since the

interval matrix $[A_c - \lambda E - rr^T, A_c - \lambda E + rr^T]$ is singular and λ is not an eigenvalue of A_c in view of (b) and (ii), the lemma above guarantees the existence of $z, y \in Y$ such that $z^T T_r (A_c - \lambda E)^{-1} T_r y \geq 1$. Assume for contrary that $z^T T_r (A_c - \lambda E)^{-1} T_r y > 1$. Then there exists an $\varepsilon_1 > 0$ such that $(\lambda - \varepsilon_1, \lambda + \varepsilon_1) \cap L_j = \emptyset$ for each $j \neq i$ and $z^T T_r (A_c - \lambda' E)^{-1} T_r y > 1$ for each $\lambda' \in (\lambda - \varepsilon_1, \lambda + \varepsilon_1)$, which, again employing the lemma, gives that $(\lambda - \varepsilon_1, \lambda + \varepsilon_1) \subset L_i$ contrary to $\lambda \notin L_i^o$. Hence

$$z^T T_r (A_c - \lambda E)^{-1} T_r y = 1$$

holds. Put $x = (A_c - \lambda E)^{-1} T_r y$ and $p = (A_c - \lambda E)^{-1} T_r z$, then $z^T T_r x = y^T T_r p = 1$ and $(A_c - T_r y z^T T_r) x = A_c x - T_r y = \lambda x$, $(A_c - T_r z y^T T_r) p = \lambda p$, hence x and p are eigenvectors corresponding to λ (since $|T_r y z^T T_r| = rr^T$, implying $A_c - T_r y z^T T_r \in A^I$; similarly $A_c - T_r z y^T T_r \in A^I$). We shall prove that $z_j x_j > 0$ for each j. In fact, assuming $z_j x_j < 0$ for some j (the possibility of $z_j x_j = 0$ is precluded by (iii)), for $z' \in Y$ given by $z_j' = -z_j$ and $z_k' = z_k$ for $k \neq j$ we would have $z'^T T_r (A_c - \lambda E)^{-1} T_r y = z'^T T_r x > z^T T_r x = 1$ contrary to $\lambda \notin L_i^o$, as before. Hence $z = \operatorname{sgn} x = \pm y_i$ and in a similar way, $y = \operatorname{sgn} p = \pm y_i$. Since, as established above, λ is an eigenvalue of $A_c - T_r y z^T T_r$, there holds either $\lambda = \lambda_i (A_c - T_r y_i y_i^T T_r) = \lambda_i (A_c - D_i)$, or $\lambda = \lambda_i (A_c + T_r y_i y_i^T T_r) = \lambda_i (A_c + D_i)$.

(d) We have proved that L_i is a compact set with nonempty interior and (at most) two boundary points. Hence $L_i = [\underline{\lambda}_i, \overline{\lambda}_i]$, where $\underline{\lambda}_i$, $\overline{\lambda}_i$ are the two boundary points, satisfying (1) in view of (c), and both $A_c - D_i$ and $A_c + D_i$ are symmetric. ■

Next we prove that each $\lambda \in L_i$ is an eigenvalue of a matrix in some special form:

<u>Theorem 2.</u> Let $r > 0$ and let (i), (ii), (iii) hold. Then for each $\lambda \in L_i$, $i \in \{1,\dots,n\}$, there exists a $t \in [-1,1]$ such that $\lambda = \lambda_i(A_c + tD_i)$.

<u>Proof.</u> The assertion obviously holds for $\lambda = \lambda_i(A_c)$ with $t = 0$. If $\lambda \in L_i$, $\lambda \neq \lambda_i(A_c)$, then $z_o^T T_r (A_c - \lambda E)^{-1} T_r y_o \geq 1$ for some $z_o, y_o \in Y$. Hence if $z, y \in Y$ satisfy

$$z^T T_r (A_c - \lambda E)^{-1} T_r y = \max \{ \bar{z}^T T_r (A_c - \lambda E)^{-1} T_r \bar{y};\ \bar{z}, \bar{y} \in Y \}$$

then for $x = (A_c - \lambda E)^{-1} T_r y$, $p = (A_c - \lambda E)^{-1} T_r z$ we obtain, in a similar way as in the part (c) of the above proof,

$$z^T T_r x = y^T T_r p \geq 1$$

$$\left(A_c - \frac{T_r y z^T T_r}{z^T T_r x}\right) x = \lambda x$$

$$\left(A_c - \frac{T_r z y^T T_r}{y^T T_r p}\right) p = \lambda p$$

and the optimality of z, y gives $z = \operatorname{sgn} x = \pm y_i$, $y = \operatorname{sgn} p = \pm y_i$ implying $\lambda = \lambda_i(A_c + tD_i)$ where $t = \pm \dfrac{1}{z^T T_r x}$, so that $t \in [-1, 1]$. ∎

Finally we show that for each $\lambda \in L_i$ $(i = 1,\dots,n)$, the set of all eigenvectors corresponding to λ

$$X_i^\lambda = \{ x;\ Ax = \lambda x,\ A \in A^I,\ x \neq 0 \}$$

can be described by a system of linear inequalities:

<u>Theorem 3.</u> Let $r > 0$ and let (i), (ii), (iii) hold. Then for each $\lambda \in L_i$ $(i = 1,\dots, n)$, the set X_i^λ is given by

$$(A_c - \lambda E - rr^T T_i)x \leq 0$$
$$(A_c - \lambda E + rr^T T_i)x \geq 0 \qquad (2)$$
$$x \neq 0.$$

Proof. If $x \neq 0$, then $x \in X_i^\lambda$ if and only if $(A-\lambda E)x = 0$ for some $A - \lambda E \in [A_c - \lambda E - rr^T, A_c - \lambda E + rr^T]$, which, in turn, is equivalent to $|(A_c - \lambda E)x| \leq rr^T|x|$ (Oettli, Prager [1]). Setting $|x| = T_i x$, we obtain (2). ∎

In the special case of $rr^T = \beta ee^T$, $e = (1,1,\dots,1)^T$, $\beta > 0$ (uniform tolerances), we have $D_i = \beta y_i y_i^T$ and the normalized eigenvectors from X_i^λ satisfying $\|x\|_1 = \sum_i |x_i| = 1$ are given simply by

$$-\beta e \leq (A_c - \lambda E)x \leq \beta e$$
$$y_i^T x = 1.$$

References

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