SENSITIVITY CHARACTERISTICS FOR THE LINEAR PROGRAMMING PROBLEM

by

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Dedicated to Prof. Dr. F. Nožička on his 70th birthday

Abstract. Sensitivity characteristics for the elements of the optimal simplex tableau, computable from this very tableau, are given.

Assume that the linear programming problem

$$\min \left\{ c^T x;\ Ax = b,\ x \geq 0 \right\} \qquad (1)$$

has an optimal solution which has been found by the standard simplex algorithm. Let B denote the square basis matrix corresponding to the basic optimal solution and let N be the nonbasic part of A, so that the final simplex tableau consists of B^{-1}, $B^{-1}N$, of the basic part of the optimal solution x_B, dual optimal solution y, optimal value h and the criterial row $\bar{c}_N$. We shall assume that all entries of B^{-1}, x_B and y are nonzero.

To introduce the sensitivity characteristics for all the quantities listed, we proceed as follows. For each $\beta > 0$, consider the family of linear programming problems of type

$$\min \left\{ c'^T x;\ A'x = b',\ x \geq 0 \right\} \quad , \qquad (2)$$

where $|A'_{ij} - A_{ij}| \leq \beta$, $|b'_i - b_i| \leq \beta$, $|c'_j - c_j| \leq \beta$ for each i,j. If β is sufficiently small to preserve the optimal basis, then for each element d of the final simplex tableau the corresponding values of this element over all problems of type (2) under given β-restrictions form some real interval, which we denote $[\underline{d}^\beta, \bar{d}^\beta]$. Let d_o be the value of this element for the original problem (1). Our definition of sensitivity characteristics is based on the fact that for each element d of the final tableau, the values of

$$\lim_{\beta \to 0_+} \frac{\bar{d}^{\beta} - d_o}{\beta}$$

and

$$\lim_{\beta \to 0_+} \frac{d_o - \underline{d}^{\beta}}{\beta}$$

are the same (although generally $\bar{d}^{\beta} - d_o \neq d_o - \underline{d}^{\beta}$); the common value, denoted by $\sigma(d)$, is called the sensitivity characteristic of the element d. Thus, if all the coefficients of the original problem (1) are subject to perturbations of at most β, then the element d of the final tableau will be shifted (in absolute value) of at most about $\beta\sigma(d)$ provided β is sufficiently small. For any submatrix M of the final simplex tableau (as B^{-1}, x_B, y etc.), we introduce $\sigma(M)$ as the submatrix of the same size consisting of $\sigma(d)$ for all elements d of M.

To formulate our main result, let us additionally introduce vectors $r = (r_i)$, $s = (s_j)$ by

$$r_i = \sum_j |B_{ij}^{-1}|$$

$$s_j = \sum_i |B_{ij}^{-1}|$$

and real numbers

$$\xi = \|x_B\|_1 + 1$$

$$\pi = \|y\|_1 + 1$$

where we have used the 1-norm, $\|x\|_1 = \sum_i |x_i|$. Further, let e denote the vector whose all entries are 1. The main result below gives the sensitivity characteristics for all elements of the final simplex tableau:

<u>Theorem.</u> Let all entries of B^{-1}, x_B and y are nonzero. Then we have:

$$\begin{aligned}
\sigma(B^{-1}) &= rs^T \\
\sigma(B^{-1}N) &= rs^T|N| + re^T \\
\sigma(x_B) &= \xi r \\
\sigma(y) &= \pi s^T \\
\sigma(h) &= \xi\pi - 1 \\
\sigma(\bar{c}_N) &= \pi(e^T|B^{-1}N| + e^T).
\end{aligned}$$

(Comment: $\sigma(y)$ and $\sigma(\bar{c}_N)$ are row vectors since so are y and $\bar{c}_N$ in the simplex tableau; if $N = (N_{ij})$, then $|N| = (|N_{ij}|)$, similarly for $|B^{-1}N|$).

The proof of the theorem is omitted here. It is based on some results on systems of linear interval equations. An interested reader is referred to [1] ; formulae for $\sigma(B^{-1})$, $\sigma(x_B)$ and $\sigma(y)$ are consequences of Eqs. (4) and (6) there, the rest can be then proved by similar reasonings.

It is worth mentioning that all the sensitivity characteristics can be computed **directly** from the final simplex tableau. Also, the greater r_i, the more sensitive x_{B_i}; similarly for s_j, y_j. Relative sensitivity characteristics can be introduced as $\sigma(d)/d_o$ for each element d.

Reference

[1] J.Rohn, Formulae for exact bounds on solutions of linear systems with rank one perturbations, to appear in Freiburger Intervall-Berichte.