# NEARNESS OF MATRICES TO SINGULARITY

by

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Abstract. A measure of nearness of real matrices to singularity is introduced and described. The proof employs a characterization of singular interval matrices of a special type.

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Let $A$ be an $n \times n$ real matrix. The number

$$d(A) = \min \left\{ \| B-A \| \; ; \; B \text{ singular} \right\} , \qquad (1)$$

where

$$\| A \| = \max_{i,j} |A_{ij}| , \qquad (2)$$

can be considered a measure of nearness of $A$ to singularity. The value of $d(A)$ was investigated by Kahan [1] for matrix norms induced by some vector norms. His result, however, cannot be applied to the norm (2) which seems to be natural in the context, since then $d(A)$ expresses the minimum deviation of coefficients which transforms $A$ to a singular matrix.

If $A$ is singular, then $d(A) = 0$ ; therefore we may restrict our attention only to nonsingular matrices. We shall give formulae for $d(A)$ and for the nearest singular matrix, based on a characterization of singular interval matrices of a special type. For $n \geqslant 1$, let $Y_n = \left\{ y \in R^n \; ; \; |y_j| = 1 \text{ for } j = 1,\dots,n \right\}$.

Theorem. Let $A$ be a nonsingular $n \times n$ matrix. Then there holds

$$d(A) = \frac{1}{r(A)} ,$$

where

$$r(A) = \max\{z^T A^{-1} y \ ; \ z,y \in Y_n\} \ . \tag{3}$$

If $\bar{z}$, $\bar{y}$ are vectors from $Y_n$ for which the maximum is achieved in (3), then

$$A_0 = A - \frac{1}{r(A)} \bar{y}\,\bar{z}^T \tag{4}$$

is a singular matrix nearest to $A$ and the vector

$$x_0 = A^{-1}\bar{y} \tag{5}$$

satisfies $A_0 x_0 = 0$ .

<u>Proof</u>. Denote $e = (1,1,\dots,1)^T \in R^n$ . Let $B$ be a singular $n \times n$ matrix. Put $\beta = \|B-A\|$ . Then $B$ belongs to the interval matrix $[A - \beta ee^T \ , \ A + \beta ee^T]$ , hence $[A - \beta ee^T \ , \ A + \beta ee^T]$ is singular and using the lemma in [2] , we get that there exist $z,y \in Y_n$ such that

$$\beta z^T A^{-1} y \geqslant 1$$

holds. Hence also $\beta r(A) \geqslant 1$ , and since $B$ was an arbitrary singular matrix, we get $d(A) \geqslant \frac{1}{r(A)}$ . On the other hand, for $A_0, x_0$ given by (4), (5), a direct computation gives $A_0 x_0 = 0$ , hence $A_0$ is singular and $\|A_0 - A\| = \frac{1}{r(A)}$ ; therefore $d(A) = \frac{1}{r(A)}$ . ■

Unfortunately, the value of $r(A)$ is not easy to compute. However, there exists a class of matrices for which $r(A)$ can be expressed explicitly :

<u>Corollary</u>. Let $A$ be a nonsingular $n \times n$ matrix for which there exist $\tilde{z},\tilde{y} \in Y_n$ such that

$$\tilde{z}_i A^{-1}_{ij} \tilde{y}_j \geqslant 0 \quad (i,j=1,\dots,n) \tag{6}$$

holds. Then $d(A) = \frac{1}{r(A)}$ , where

$$r(A) = \sum_{i,j} |A^{-1}_{ij}| \ .$$

Proof. Under the assumption, we have $\tilde{z}^T A^{-1} \tilde{y} \leq r(A) \leq$ $\leq \sum_{i,j} |A^{-1}_{ij}| = \sum_{i,j} \tilde{z}_i A^{-1}_{ij} \tilde{y}_j = \tilde{z}^T A^{-1} \tilde{y}$ , hence $r(A) = \sum_{i,j} |A^{-1}_{ij}|$ . ■

Especially, for inverse nonnegative matrices (where $A^{-1} \geq 0$, so that (6) is satisfied with $\tilde{z} = \tilde{y} = e$) we get that $r(A) = \sum_{i,j} A^{-1}_{ij}$ and the nearest singular matrix can be obtained by substracting the value of $\frac{1}{r(A)}$ from all coefficients of $A$ .

Let us now return to the general case. If the maximum in (3) is achieved at some $z, y \in Y_n$ , then, since $r(A) = z^T A^{-1} y =$ $= \sum_i z_i (A^{-1}y)_i = \sum_j (z^T A^{-1})_j y_j$ , there must hold

$$z_i (A^{-1}y)_i \geq 0 \quad \text{for} \quad i=1,\dots,n \tag{7}$$

and

$$(z^T A^{-1})_j y_j \geq 0 \quad \text{for} \quad j=1,\dots,n \ , \tag{8}$$

for otherwise the value of $z^T A^{-1} y$ could be increased. Thus, using the vector norm $\|x\|_1 = \sum_i |x_i|$ , we may also write

$$r(A) = \max \left\{ \|A^{-1}y\|_1 \ ; \ y \in Y_n \right\} .$$

If $n$ is large, then $r(A)$ cannot be computed in this way since $Y_n$ has $2^n$ elements. In this case, we propose the following algorithm which stops after **reaching a pseudooptimal** solution satisfying the necessary optimality conditions (7) and (8) :

Algorithm.

0. Select $z, y \in Y_n$ .
1. Set $z_i := -z_i$ for each i with $z_i (A^{-1}y)_i < 0$ .
2. Set $y_j := -y_j$ for each j with $(z^T A^{-1})_j y_j < 0$ .
3. If (7) holds, terminate. Otherwise go to step 1 .

The algorithm is finite since $Y_n$ is finite and the value of $z^T A^{-1} y$ is always **increased** during step 1 or 2, so that cycling cannot occur. The condition (8) is always satisfied after step 2, hence it need not be tested in step 3. If the algorithm terminates with some $z, y \in Y_n$ in step 3, then

$$d(A) \leq \frac{1}{z^T A^{-1} y}$$

and the matrix

$$A_0^* = A - \frac{yz^T}{z^T A^{-1} y}$$

is a singular matrix with $\| A_0^* - A \| = \frac{1}{z^T A^{-1} y}$ and $A_0^* x_0^* = 0$ for $x_0^* = A^{-1} y$ .

It is perhaps worth mentioning that according to (4), each square nonsingular matrix $A$ can be decomposed as $A = A_0 + A_1$ , where $A_0$ is singular and $A_1$ is of rank one. Also, $\| A^{-1} \| \geq \frac{1}{n^2 d(A)}$ for each nonsingular $n \times n$ matrix $A$ .

## References

[1] W. Kahan, Numerical Linear Algebra, Canad. Math. Bull. 9 (1966), 757-801

[2] J. Rohn, Eigenvalues of a Symmetric Interval Matrix, to appear in Freiburger Intervall-Berichte

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