RADIUS OF NONSINGULARITY

by

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Abstract. We introduce the radius of nonsingularity $d(A,\Delta)$ of a square matrix A subject to a nonnegative square matrix Δ as the minimum $\varepsilon \geq 0$ for which there exists a singular matrix A' satisfying $A - \varepsilon\Delta \leq A' \leq A + \varepsilon\Delta$. We show that even in the special case of Δ being the matrix of all units, computing $d(A,\Delta)$ is NP-hard for matrices A with rational entries; this is proved via establishing a connection of our problem to the problem of computing the maximum cut in an associated graph. As a consequence we prove that the problem of testing singularity of interval matrices is NP-complete.

1. Introduction

In many areas it is important to know whether a given square matrix A is sufficiently far from a singular matrix. Such areas include sensitivity analysis, control theory, numerical methods and interval analysis. Several different approaches to this question have been developed, see e.g. [4, 16].

Here we introduce the following measure. Let A, Δ be two $n \times n$ matrices, Δ nonnegative. We define the radius of nonsingularity $d(A,\Delta)$ as the minimum $\varepsilon \geq 0$ for which there exists a singular matrix A' satisfying $A - \varepsilon\Delta \leq A' \leq A + \varepsilon\Delta$. The concept of $d(A,\Delta)$ is motivated e.g. by the following situations:

(a) Rounding. Assume we are given a matrix A_0 some of whose entries are irrational numbers; such a situation may occur when the data are formally derived from some other real world values. Let the entries of A_0 be rounded off to p decimal places, giving the representation matrix A. Define $\Delta_{ij} = 0$ if $(A_0)_{ij} = A_{ij}$ and $\Delta_{ij} = 1$ otherwise. If $d(A,\Delta) > \frac{1}{2}10^{-p}$, then we can be sure that A_0 is nonsingular; otherwise the precision chosen is insufficient to make a decision (notice that A_0 is not used in the test; cf. [15]).

(b) Relative errors. If $\Delta = |A|$ (i.e., the matrix consisting of the absolute values of the entries of A), then $d(A,\Delta)$ yields the minimum relative error of the coefficients which brings A to a singular matrix.

(c) Singular interval matrices. An interval matrix $A^I = \{A'; A - \Delta \le A' \le A + \Delta\}$ is called singular if it contains a singular matrix. Hence, A^I is singular if and only if $d(A,\Delta) \le 1$.

We present the following results. The key result (Theorem 2.1) gives an explicit formula for $d(A,\Delta)$. In order to show that computing $d(A,\Delta)$ is NP-hard, we consider the special case of $\Delta = H$ (the matrix of all units) and we show in Theorem 2.2 that

$$d(A,H) = 1/r(A^{-1})$$

where r(B) is defined by

$$r(B) = \max\{z^tBy;\ z,y \in \{-1,1\}^n\}$$

("t" denotes transposition). Since r is a matrix norm, we first give some upper and lower bounds on it. Then, by establishing a connection of r to the max-cut in an associated graph, we show that computing r(B) is NP-hard for matrices B with rational entries. As a consequence of the above results we obtain that the problem of testing singularity of interval matrices is NP-complete.

Some notations. We work with square matrices of size $n \times n$ with real entries. We denote by Q the n-dimensional discrete cube $Q = \{-1,1\}^n = \{y \in R^n;\ |y| = e\}$, where $e = (1,1,\dots,1)^t$. For each

$y \in Q$, we denote by T_y the diagonal matrix with diagonal vector y (i.e. $(T_y)_{ii} = y_i$ and $(T_y)_{ij} = 0$ for $i \neq j$). For an arbitrary $n \times n$ matrix A we denote

$$\rho_0(A) = \max \{ |\lambda| ; Ax = \lambda x \text{ for some } x \neq 0, \lambda \text{ real} \},$$

i.e. an analogue of the spectral radius, with maximum being taken only over real eigenvalues; we set $\rho_0(A) = 0$ if no real eigenvalue exists. We use the following matrix norms: $\rho(A) = \sqrt{\rho_0(A^t A)}$ (the spectral norm) and $s(A) = \sum_{i,j} |a_{ij}|$.

2. Radius of nonsingularity

For an $n \times n$ matrix A and a nonnegative $n \times n$ matrix Δ, we introduce the radius of nonsingularity by

$$d(A,\Delta) = \min \{ \varepsilon \geq 0;\ A - \varepsilon\Delta \leq A' \leq A + \varepsilon\Delta \quad \text{for some singular } A' \}$$

Obviously, $d(A,\Delta) = 0$ if A is singular. On the other hand, it can be $d(A,\Delta) = \infty$: consider the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Delta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Here each A' with $A - \varepsilon\Delta \leq A' \leq A + \varepsilon\Delta$ satisfies $\det A' = -1$, hence $d(A,\Delta)$ is infinite.

Since the case of A singular is obvious, we shall consider A to be nonsingular in the sequel. In this case, under notations introduced in the previous section, we shall derive an explicit formula for $d(A,\Delta)$ (we employ the convention $\frac{1}{0} = \infty$) :

Theorem 2.1. Let A be nonsingular and $\Delta \geq 0$. Then we have

$$d(A,\Delta) = 1/\max \{ \rho_0(A^{-1} T_y \Delta T_z);\ y,z \in Q \} \tag{2.1}$$

Proof. First consider the case of $d(A,\Delta)$ finite. For a given $\varepsilon \geq 0$, existence of a singular matrix A' satisfying $A - \varepsilon\Delta \leq A' \leq A + \varepsilon\Delta$ is equivalent to singularity of the interval matrix $[A - \varepsilon\Delta, A + \varepsilon\Delta]$, which, according to the assertion (C3) of Theorem 5.1 in [14], is the case if and only if

$$\rho_0(A^{-1}T_y \varepsilon\Delta T_z) \geq 1$$

holds for some $y,z \in Q$, i.e. iff

$$\varepsilon \max \{\rho_0(A^{-1}T_y \Delta T_z);\ y,z \in Q\} \geq 1,$$

hence the minimum value of ε is given by (2.1).

If $d(A,\Delta) = \infty$, then by the same result in [14] we have $\varepsilon\rho_0(A^{-1}T_y\Delta T_z) < 1$ for each $y,z \in Q$ and each $\varepsilon \geq 0$, hence $\rho_0(A^{-1}T_y\Delta T_z) = 0$ for each $y,z \in Q$ and (2.1) again holds.

We are going to show that computing $d(A,\Delta)$ is NP-hard. For this purpose, from this point on we shall only consider the case $\Delta = H = ee^t$, and we shall simply write $d(A)$ instead of $d(A,H)$. We have this result:

Theorem 2.2. Let A be nonsingular. Then

$$d(A) = 1/r(A^{-1}) \tag{2.2}$$

where

$$r(A^{-1}) = \max \{z^tA^{-1}y;\ z,y \in Q\}.$$

Proof. For $\Delta = ee^t$, we have $A^{-1}T_y\Delta T_z = A^{-1}yz^t$ for each $y,z \in Q$. We shall show that

$$\rho_0(A^{-1}yz^t) = |z^tA^{-1}y|$$

in this case. This will be done if we show that $A^{-1}yz^t$ has only two real eigenvalues $\lambda_0 = 0$ and $\lambda_1 = z^tA^{-1}y$. Since yz^t is singular, $\lambda_0 = 0$ is an eigenvalue. Next, for $x = A^{-1}y$ we have $(A^{-1}yz^t)x = (A^{-1}y)(z^tA^{-1}y) = \lambda_1 x$, hence λ_1 is also an eigenvalue. Conversely, if λ is any real eigenvalue, then from $A^{-1}yz^tx = \lambda x$ we have either $z^t x = 0$, then $\lambda = \lambda_0$, or $z^t x \neq 0$, in which case pre-multiplying by z^t yields $\lambda = z^t A^{-1} y = \lambda_1$. Hence no other real eigenvalue exists, so that $\rho_0(A^{-1}yz^t) = |z^t A^{-1} y|$. Then Theorem 2.1 gives $d(A) = 1/r(A^{-1})$, where

$$r(A^{-1}) = \max\{|z^t A^{-1} y| ;\ z,y \in Q\} = \max\{z^t A^{-1} y;\ z,y \in Q\}.$$

The formula (2.2) could be also inferred from Kahan's result in [7]. The mapping

$$A \longrightarrow r(A) = \max\{z^t Ay;\ z,y \in Q\}$$

is obviously a matrix norm. Let us mention that $r(A)$ has been studied by Brown and Spencer [3] (see also [5]) in case that A is a ±1-matrix. They proved

$$\sqrt{\tfrac{2}{\pi}}\, n^{\frac{3}{2}} < \min\{r(A);\ a_{ij} = \pm 1\} < (1 + o(1))\ n^{\frac{3}{2}} \qquad (2.3)$$

(i.e., the minimum over all ±1-matrices A). We show in Theorem 2.4 that the lower bound remains valid for any matrix A with $s(A) = n^2$. Since $r(A)$ is a matrix norm, we have $c_1 N(A) \le r(A) \le c_2 N(A)$ for any other matrix norm $N(A)$, where c_1 and c_2 are some constants depending on n only. We present explicit values of such constants for the norms $\rho(A)$ and $s(A)$. Further, we show that computing the exact value of $r(A)$ can be reduced to the max-cut problem in a weighted graph, and conversely, max-cut can be reduced to computing $r(A)$. The former reduction provides us with a possibility of computing some bounds on $r(A)$ from approximative solution of max-cut, and the latter implies that computing $r(A)$ is NP-hard.

The next theorem gives a relation between the norms r and ρ .

Theorem 2.3. For every $n \times n$ matrix A we have

$$\rho(A) \le r(A) \le n\,\rho(A)$$
$$r(A) \ge \sqrt{n \lambda_{\min}(A^t A)}\,.$$

The proof is straightforward and will be omitted.

In the next theorem we compare $r(A)$ with the norm $s(A)$.

Theorem 2.4. We have

$$\sqrt{\tfrac{2}{\pi}}\, n^{-\frac{1}{2}} s(A) \le r(A) \le s(A).$$

Proof. It is well-known (see e.g. [5, proof of Theorem 15.2], or [12]) that $E[|e^t y|] \geq \sqrt{2n/\pi}$ for random $y \in Q$. Clearly, $E[|z^t y|;\ y \in Q] = E[|e^t y|;\ y \in Q]$ for any fixed $z \in Q$. Let $a = (a_1,\dots,a_n)^t$ be a nonnegative vector. Define vectors $a^{(i)} = (a_i, a_{i+1},\dots,a_n,a_1,\dots,a_{i-1})^t$, $i=1,\dots,n$, i.e. each $a^{(i)}$ is obtained from a by a cyclic rotation. Set $\alpha = \sum_i a_i$. We have

$$E[|y^t a|\,;\ y \in Q] = \frac{1}{n}\sum_{i=1}^{n} E[|y^t a^{(i)}|\,;\ y \in Q] = \frac{1}{n}E[\sum_{i=1}^{n}|y^t a^{(i)}|;\ y \in Q]$$

$$\geq \frac{1}{n}E[|\sum_{i=1}^{n} y^t a^{(i)}|] = \frac{\alpha}{n}E[|e^t y|] \geq \sqrt{\tfrac{2}{\pi}}\, n^{-1/2}\alpha .$$

Hence, for arbitrary $a \in R^n$ (not necessarily nonnegative) we have

$$E[|y^t a|\,;\ y \in Q] \geq c n^{-1/2}\alpha$$

where $c = \sqrt{\frac{2}{\pi}}$ and, with A_i denoting the i-th row of A,

$$E[\sum_{i=1}^{n}|A_i y|] = \sum_{i=1}^{n} E[|A_i y|] \geq \sum_{i=1}^{n} c n^{-1/2} \sum_{j=1}^{n} |a_{ij}| = c n^{-1/2} s(A),$$

hence there exists a $y \in Q$ such that

$$z^t A y = \sum_{i=1}^{n} |A_i y| \geq c n^{-1/2} s(A)$$

where z is the sign vector of Ay.

The proof for the upper bound is trivial.

Let us note that the original purely probabilistic proof of the lower bound of (2.3) from [3] can be modified to an algorithmic one. Thus, for a given ± 1-matrix A, one can construct in polynomial time a pair $y, z \in Q$ of vectors such that $z^t A y \geq c n^{3/2}$ where c is the above constant. An extension of this algorithm for arbitrary matrix A, as well as other approaches to approximative computing $r(A)$, will appear in [13].

In the rest of this section we will study a relation between $r(A)$ and the max-cut problem. Again, such a relation is not quite new since the max-cut problem has already been used for reformulation of quadratic optimization problems of type $x^tAx + c^tx$, see e.g. [1, 2].

Max-cut problem. Let $G = (N,E)$ be a graph and $c: E \to R^1$ a weight function on edges. The maximum cut in the graph G with respect to c is defined as

$$MC(G) = \max_{S \subset N} c(\delta S)$$

where δS is the set of edges with one endvertex is S and one in N - S, and $c(F) = \sum_{f \in F} c(f)$ for a subset $F \subset E$.

In order to reduce computing $r(A)$ to max-cut problem, we define the bipartite graph B_A of a matrix A as the weighted bipartite graph $B_A = (Y \cup Z)$ where Y and Z are two copies of $\{1,\dots,n\}$ and $E = \{ ij;\ a_{ij} \neq 0\}$. The weight of an edge ij is a_{ij}.

Theorem 2.5. We have $r(A) = 2MC(B_A) - e^tAe$.

Proof. Given $y, z \in Q$, define a set S by $S = \{i \in Y;\ y_i = 1\} \cup \{j \in Z;\ z_j = -1\}$. We have ($|\dots|$ denotes cardinality)
$y^tAz = \sum_{i,j} a_{ij}y_iz_j = \sum_{y_i = z_j} a_{ij} - \sum_{y_i \neq z_j} a_{ij} = 2\sum_{y_i = z_j} a_{ij} - \sum_{i,j} a_{ij} = 2|\delta S| - e^tAe$, and taking maximum on both sides gives the result.

Max-cut is a known NP-hard problem (see [6]) . A practical algorithm for solving it has been developed in [2] . Since it is difficult to find an exact solution, one may use a heuristic. We survey some of them

Lower bounds on max-cut.

(i) Poljak and Turzík [11] : If $G = (N,E)$ is a connected graph, then $MC(G) \geq \frac{1}{2}$ + minimum weight of a spanning tree of G.

A cut δS satisfying the above inequality can be found in $O(n^3)$ time.

(ii) Lieberherr and Specker have implicitly shown in [9] the bound

$$MC(G) \geq c(E)\frac{n}{2n-1} .$$

It is easy to obtain the above bound by a probabilistic method ([5]). The merit of [9] is a polynomial-time algorithm for it.

An upper bound on max-cut is given by Mohar and Poljak [10]:

$$MC(G) \leq \frac{n}{4}\lambda_{max}$$

where λ_{max} is the maximum eigenvalue of the matrix P given by

$$p_{ij} = \begin{cases} -c_{ij} & \text{if } ij \in E,\ i \neq j \\ 0 & \text{if } ij \notin E,\ i \neq j \\ \sum_k c_{ik} & \text{if } i = j. \end{cases}$$

We have shown that computing r(A) can be reduced to max-cut. Now we present an opposite reduction to establish that computing r(A) is NP-hard. We recall that even the cardinality version of max-cut is NP-hard([6]).

Theorem 2.6. Computing r(A) is NP-hard for a matrix A with rational entries.

Proof. Let $G = (N,E)$ be a graph. Define a matrix A by

$$a_{ij} = \begin{cases} -1 & \text{if } ij \in E,\ i \neq j \\ 0 & \text{if } ij \notin E,\ i \neq j \\ M & \text{if } i = j, \end{cases}$$

where M is sufficiently large integer ($M > 2|E|$ is sufficient). Let $r(A) = z^tAy$ for some $z,y \in Q$. It is easy to see that $z = y$ because of the choice of M. For each $y \in Q$, with $S = \{i;\ y_i = 1\}$ we have
$$y^tAy = \sum_{i,j} a_{ij}y_iy_j = \sum_{i,j}(-\tfrac{1}{2}a_{ij})[(y_i-y_j)^2 - 2] = -\tfrac{1}{2}\sum_{i,j}a_{ij}(y_i-y_j)^2 + \sum_{i,j}a_{ij} = Mn + 4|\delta S| - 2|E|,$$
hence $r(A) = Mn + 4MC(G) - 2|E|$ which shows that computing r(A) is NP-hard since computing the max-cut can be reduced to it.

In view of Theorem 2.2, this result shows that computing d(A) is NP-hard for matrices with rational entries.

Finally we formulate a consequence for interval matrices. A square interval matrix $A^I = \{A'; \underline{A} \leq A' \leq \bar{A}\}$ is called singular if it contains a singular matrix. Consider the decision problem

Instance: Square interval matrix A^I, where both $\underline{A}$ and $\bar{A}$ are rational matrices.

Question: Is A^I singular?

Theorem 2.7. The above problem is NP-complete.

Proof. The problem is in the NP-class since we can guess a singular matrix $A' \in A^I$ in case that the given interval matrix is singular and we can check the required property of A' in polynomial time. The problem is NP-hard since computing r(A) can be reduced to it.

A more detailed discussion of the problem of testing singularity of interval matrices can be found in [14] .

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