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New Condition Numbers for Matrices and Linear Systems

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Abstract — Zusammenfassung

New Condition Numbers for Matrices and Linear Systems. New condition numbers for matrices and linear systems are proposed, based on the dependence of the relative errors in the result upon the relative errors of the data.

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Neue Konditionszahlen für Matrizen und lineare Gleichungssysteme. Neue Konditionszahlen für Matrizen und lineare Gleichungssysteme werden eingeführt, die auf der Abhängigkeit der relativen Fehler des Resultats von den relativen Datenfehlern basieren.

The most frequently used condition number

$$\gamma(A) = \| A^{-1} \| \cdot \| A \|$$

relates the relative error of A^{-1}, measured in some norm, to the relative error of A, as it can be seen from the inequality

$$\frac{\|(A+\Delta A)^{-1} - A^{-1}\|}{\| A^{-1} \|} \le \frac{\gamma(A)\, r(A)}{1-\gamma(A)\, r(A)},$$

where $r(A) = \| \Delta A \| / \| A \|$ (see e.g. Deif [2, p. 18]). This condition number, however, has two disadvantages: (i) it depends on the norm used, (ii) it is a global characteristic, giving no information about the behaviour of single coefficients of A^{-1}. Therefore, it seems reasonable to study directly the relationship between the relative errors of the coefficients of A^{-1} and those of A.

So let A be a nonsingular $n \times n$ matrix and assume that the coefficients of A may vary in such a way that no relative error exceeds a positive real number α.

Then the matrices obtained in this way satisfy

$$\frac{|B_{ij} - A_{ij}|}{|A_{ij}|} \le \alpha$$

for each i, j with $A_{ij} \neq 0$, or shortly

$$|B - A| \leq \alpha |A|$$

(here $|A|$ stands for the matrix $(|A_{ij}|)$). Now, the maximum relative error in the ij-th coefficient of A^{-1} is given by

$$c_{ij}^{\alpha}(A) = \max \left\{ \frac{|B_{ij}^{-1} - A_{ij}^{-1}|}{|A_{ij}^{-1}|} ; |B - A| \leq \alpha |A| \right\}$$

(provided $A_{ij}^{-1} \neq 0$). This value depends on α; however, the number

$$c_{ij}(A) = \lim_{\alpha \to 0_+} \frac{c_{ij}^{\alpha}(A)}{\alpha}$$

is independent of α, and of obvious meaning: if α is small, then $c_{ij}^{\alpha}(A)$ is equal to about $c_{ij}(A)\,\alpha$, so that $c_{ij}(A)$ expresses a measure of the dependence of the relative error of A_{ij}^{-1} upon the relative errors in A.

We shall show that the numbers $c_{ij}(A)$ can be rather easily computed.

Theorem: *Let A be nonsingular. Then for each i, j with $A_{ij}^{-1} \neq 0$*

$$c_{ij}(A) = \frac{(|A^{-1}| \cdot |A| \cdot |A^{-1}|)_{ij}}{|A_{ij}^{-1}|}. \tag{1}$$

Proof: Let $|B - A| \leq \alpha |A|$. Then B belongs to the interval matrix $A^I = [A - \alpha |A|, A + \alpha |A|]$, and $B_{\cdot j}^{-1}$, the j-th column of B^{-1}, is a solution to the linear interval system $A^I x = e_j$, where e_j is the j-th column of the unit matrix. Then, applying Miller's asymptotic result [3], as quoted in [1], for each i, j we get

$$\max \{ |B_{ij}^{-1} - A_{ij}^{-1}| ; |B - A| \leq \alpha |A| \} = (|A^{-1}| \cdot |A| \cdot |A^{-1}|)_{ij}\,\alpha + 0(\alpha^2)$$

hence dividing by $|A_{ij}^{-1}|$ and taking the limit for $\alpha \to 0_+$, we obtain (1). ■

Corollary: *Let A be nonsingular. Then for each i, j with $A_{ij}^{-1} \neq 0$, $c_{ij}(A) \geq 1$.*

Proof: From $A^{-1} A = E$ we have $|A^{-1}| \cdot |A| \geq E$, hence

$$|A^{-1}| \cdot |A| \cdot |A^{-1}| \geq |A^{-1}|,$$

implying $c_{ij}(A) \geq 1$ for each i, j with $A_{ij}^{-1} \neq 0$ in view of the theorem. ■

So we may propose a new condition number

$$c(A) = \max \left\{ \frac{(|A^{-1}| \cdot |A| \cdot |A^{-1}|)_{ij}}{|A_{ij}^{-1}|} ; A_{ij}^{-1} \neq 0 \right\}.$$

Roughly said, the maximal relative error in the coefficients of A^{-1} is about $c(A)$ times greater than the maximal relative error in the coefficients of A, if the latter one is small enough. As a consequence of the corollary, we have that $c(A) \geq 1$. Thus the greater $c(A)$, the more ill-conditioned is the matrix A. This is analogous to the above-quoted condition number $\gamma(A)$, which also has the property $\gamma(A) \geq 1$ for each nonsingular A. The condition number $c(A)$ cannot be reduced by scaling: in fact, if D_1, D_2 are diagonal matrices with positive diagonal entries, then $c(D_1 AD_2)=c(A)$.

By a similar reasoning we may also arrive at a condition number for a linear system $Ax=b$. Let A, b be subject to relative errors of at most α, $\alpha>0$. Then the maximal relative error of the i-th component of the solution is given by

$$c_i^\alpha(A,b)=\max\left\{\frac{|x_i'-x_i|}{|x_i|};\ A'x'=b',\ |A'-A|\leq\alpha|A|,\ |b'-b|\leq\alpha|b|\right\}.$$

Let us again define

$$c_i(A,b)=\lim_{\alpha\to 0_+}\frac{c_i^\alpha(A,b)}{\alpha}.$$

Then, again employing Miller's asymptotic result, we obtain

$$c_i(A,b)=\frac{(|A^{-1}|\cdot|A|\cdot|x|+|A^{-1}|\cdot|b|)_i}{|x_i|}$$

for each i with $x_i\neq 0$ $(x=A^{-1}b)$.

Thus we may propose this condition number for a system $Ax=b$:

$$c(A,b)=\max\left\{\frac{(|A^{-1}|\cdot|A|\cdot|x|+|A^{-1}|\cdot|b|)_i}{|x_i|};\ x_i\neq 0\right\}. \tag{2}$$

Again, roughly speaking, the maximal relative error in the components of the solution is about $c(A,b)$ times greater than the maximal relative error in the entries of A and b. Since $|A^{-1}|\cdot|A|\cdot|x|\geq|x|$ and $|A^{-1}|\cdot|b|\geq|x|$, we have $c(A,b)\geq 2$. Also in this case scaling will not improve the condition number (2): $c(D_1 AD_2, D_1 b)=c(A,b)$ for any diagonal matrices D_1, D_2 with positive diagonal entries.

References

[1] Alefeld, G., Herzberger, J.: Introduction to Interval Computations. New York: Academic Press 1983.

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