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A Two-Sequence Method for Linear Interval Equations

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Abstract — Zusammenfassung

A Two-Sequence Method for Linear Interval Equations. It is shown that only two matrix sequences are to be constructed to solve a system of linear interval equations with an inverse stable, strongly regular interval matrix.

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Key words: Linear systems, inexact data, iterative solution.

Eine Zweisequenzmethode für lineare Gleichungssysteme. Es wird gezeigt, daß nur zwei Matrizensequenzen zu konstruieren sind, um ein lineares Intervallgleichungssystem mit einer invers-stabilen, streng regulären Intervallmatrix zu lösen.

It is shown here that some of the author's previous results can be reformulated in such a way that only two matrix sequences are to be constructed to compute the exact bounds of solution of a system of linear interval equations with an inverse stable, strongly regular interval matrix (see definitions below). The result has two specific features: (i) a componentwise matrix product is used, and (ii) the resulting vectors appear as diagonal vectors of certain matrices.

To introduce the problem in question, assume we are given a linear interval system $A^I x = b^I$ with an $n \times n$ interval matrix $A^I = [A - \Delta, A + \Delta]$, assumed to be regular ($\det A' \neq 0$ for each $A' \in A^I$), and a right-hand side interval vector $b^I = [b - \delta, b + \delta]$. Wanted is the exact interval solution $[\underline{x}, \bar{x}]$ where $\underline{x}, \bar{x}$ are given by

$$\underline{x}_i = \min \{x'_i; x' \in M\}$$
$$\bar{x}_i = \max \{x'_i; x' \in M\}, \qquad (i = 1, \ldots, n)$$

M being the solution set of $A^I x = b^I$,

$$M = \{x';\ A' x' = b',\ A' \in A^I,\ b' \in b^I\}.$$

An interval matrix A^I is called inverse stable if $|(A')^{-1}| > 0$ for each $A' \in A^I$, i.e. if each inverse matrix coefficient preserves its signature over A^I. For such an interval matrix, we define a signature matrix S by

$$S_{ij}=\begin{cases} 1 & \text{if } A^{-1}_{ji}>0 \\ -1 & \text{if } A^{-1}_{ji}<0 \end{cases} \quad (i,j=1,\ldots,n)$$

(notice the transposition of indices). For two $n\times n$ matrices, $X=(X_{ij})$ and $Y=(Y_{ij})$, we define their componentwise product as $X*Y=(X_{ij}Y_{ij})$ and the absolute value as $|X|=(|X_{ij}|)$. Further, let $\operatorname{diag} X$ denote the diagonal vector of X, i.e. $\operatorname{diag} X=(X_{11},X_{22},\ldots,X_{nn})^T$. Finally, let $e=(1,1,\ldots,1)^T\in R^n$ and denote

$$B=b\,e^T$$

and

$$D=\delta\,e^T$$

(b, δ are column vectors). Our method for computing $\underline{x}$ and $\bar{x}$ is based on the following theoretical result.

Theorem 1: *Let A^I be regular and inverse stable. Then the matrix equations*

$$A X=B-S*(\Delta|X|+D) \tag{1}$$

$$A X=B+S*(\Delta|X|+D) \tag{2}$$

have unique matrix solutions $\underline{X}$ and $\bar{X}$, respectively, and

$$\underline{x}=\operatorname{diag}\underline{X}$$

$$\bar{x}=\operatorname{diag}\bar{X}.$$

Proof: For each $i\in\{1,\ldots,n\}$, denote by $y(i)$ the signature vector of the i-th row of A^{-1}, and let $T_{y(i)}$ be the diagonal matrix with diagonal vector $y(i)$ (i.e. $y(i)=\operatorname{diag} T_{y(i)}$). In [5], Theorem 1.2, it was proved that under the regularity assumption, the equations

$$Ax=b-T_{y(i)}(\Delta|x|+\delta)$$

$$Ax=b+T_{y(i)}(\Delta|x|+\delta)$$

have unique (vector) solutions, denoted there by $x_{-y(i)}$ and $x_{y(i)}$, respectively; moreover, $\underline{x}_i=(x_{-y(i)})_i$, $\bar{x}_i=(x_{y(i)})_i$ hold if A^I is inverse stable ([3], Theorem 3; proved in [4], p. 23). Hence if we define matrices $\underline{X}$, $\bar{X}$ by $\underline{X}=(x_{-y(1)},\ldots,x_{-y(n)})$ and $\bar{X}=(x_{y(1)},\ldots,x_{y(n)})$, then $\underline{X}$, $\bar{X}$ solve uniquely (1), (2) and there holds $\underline{x}=\operatorname{diag}\underline{X}$, $\bar{x}=\operatorname{diag}\bar{X}$. ■

To solve (1) and (2), let us premultiply them first by an approximate inverse R of the matrix A to bring them to an equivalent fixed-point form

$$X=(E-RA)X-R(S*(\Delta|X|+D))+RB$$

$$X=(E-RA)X+R(S*(\Delta|X|+D))+RB$$

(E is the unit matrix). These nonlinear equations may be then solved iteratively by

$$\underline{X}^{k+1}=(E-RA)\underline{X}^k-R(S*(\Delta|\underline{X}^k|+D))+RB \tag{3}$$

$$\bar{X}^{k+1}=(E-RA)\bar{X}^k+R(S*(\Delta|\bar{X}^k|+D))+RB \tag{4}$$

($k=0,1,\ldots$) with the recommended starting point

$$\underline{X}^0=\bar{X}^0=RB. \tag{5}$$

To assure convergence, we shall assume the nonnegative matrix

$$G=|E-RA|+|R|\Delta$$

to satisfy

$$\rho(G)<1; \tag{6}$$

this is true if A^I is strongly regular (i.e. $\rho(|A^{-1}|\Delta)<1$; cf. Neumaier [1]) and R is a sufficiently close approximation of A^{-1}. We shall later show that (6) guarantees regularity. To be able to check A^I for inverse stability, we introduce also the matrix

$$F=(E-G)^{-1}$$

and the condition

$$GF|R|<|R|. \tag{7}$$

Then the following theorem holds

Theorem 2: *Let* (6) *and* (7) *hold. Then A^I is regular and inverse stable and for the sequences $\{\underline{X}^k\}$, $\{\bar{X}^k\}$ generated by* (3), (4), (5) *we have $\underline{X}^k\to\underline{X}$, $\bar{X}^k\to\bar{X}$, with*

$$|\underline{X}-\underline{X}^k|\le GF|\underline{X}^k-\underline{X}^{k-1}|\le G^kF|\underline{X}^1-\underline{X}^0|$$

$$|\bar{X}-\bar{X}^k|\le GF|\bar{X}^k-\bar{X}^{k-1}|\le G^kF|\bar{X}^1-\bar{X}^0|$$

for each $k\ge 1$.

Proof: First, for each $A'\in A^I$ we have

$$A'=R^{-1}(E-(E-RA+R(A-A')))$$

and since

$$\rho(E-RA+R(A-A'))\le\rho(G)<1,$$

we see that A' is regular and

$$|(A')^{-1}-R|\le\left(\sum_1^\infty G^j\right)|R|=GF|R|<|R|,$$

which means that A^I is inverse stable.

Next, arguing as in [2], we first obtain from (4) that

$$|\bar{X}^{k+1}-\bar{X}^k|\le G|\bar{X}^k-\bar{X}^{k-1}|$$

for each $k\ge 1$ and then by induction

$$|\bar{X}^{k+m}-\bar{X}^k|\le(G^m+\ldots+G)|\bar{X}^k-\bar{X}^{k-1}|\le GF|\bar{X}^k-\bar{X}^{k-1}|\le G^kF|\bar{X}^1-\bar{X}^0|$$

for each $m\ge 1$, so that $\{\bar{X}^k\}$ is a Cauchy sequence, hence $\bar{X}^k\to\bar{X}$ due to the uniqueness of the solution of (2). Taking $m\to\infty$ in the above inequality, we obtain the estimation for $|\bar{X}-\bar{X}^k|$. The proof for $\{\underline{X}_k\}$ is quite analogous. ■

Note that the conditions (6) and (7) under which the method works are satisfied if all the coefficients of A^{-1} are nonzero and Δ is sufficiently small, which is a frequent case. Also, under these conditions the sequences $\{\underline{X}^k\}$, $\{\bar{X}^k\}$ converge from arbitrary starting points, but the choice (5) seems to be the best since then each column of $\underline{X}^0, \bar{X}^0$ is equal to Rb, an approximate solution to $Ax=b$.

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