ON SENSITIVITY OF THE OPTIMAL VALUE OF A LINEAR PROGRAM*)

[illegible] we [illegible] the following problem. Given a linear program

(1) $$\min\{c^{\mathrm{T}}x;\ Ax = b,\ x \geqq 0\}$$

[illegible]ing an optimal solution $\bar{x}$, test how sensitive is the optimal value $k = c^{\mathrm{T}}\bar{x}$ subject to small [illegible] errors in the data A, b, c. We shall construct a ("condition") number measuring this kind of [illegible], involving only the initial data, the optimal solution $\bar{x}$ of (1) and the optimal solution $\bar{y}$ of the dual problem

(2) $$\max\{b^{\mathrm{T}}y;\ A^{\mathrm{T}}y \leqq c\}\,.$$

To begin with, for a given real number $\alpha > 0$ consider the family of perturbed problems

[illegible] $$\min\{c'^{\mathrm{T}}x;\ A'x = b',\ x \geqq 0\}$$

for which [illegible]relative errors of the data do not exceed α, i.e. for which

$$|A' - A| \leqq \alpha|A|$$

*) O sensitivitě optimální hodnoty úlohy lineárního programování. V stati je uvedeno „číslo podmíněnosti“ charakterizující vliv malých relativních chyb vstupních dat úlohy lineárního programování na relativní chybu její optimální hodnoty.

$$(5) \qquad |b' - b| \leqq \alpha|b|$$

$$(6) \qquad |c' - c| \leqq \alpha|c|$$

hold (the absolute value $|A|$ of a matrix $A = (a_{ij})$ is defined by $|A| = (|a_{ij}|)$; similarly for vectors). We shall assume that there exists an $\varepsilon > 0$ such that for each $\alpha \in [0, \varepsilon)$, each problem (3) satisfying (4), (5), (6) has an optimal solution and its optimal value will be denoted by $h(A', b', c')$. Now, for a given α, the maximal relative error of the optimal value under relative data errors of at most α is given by

$$\gamma_\alpha(A, b, c) = \max \left\{ \left| \frac{h(A', b', c') - h(A, b, c)}{h(A, b, c)} \right| ; \; A', b', c' \text{ satisfy } (4), (5), (6) \right\}.$$

This value depends on α, and is generally difficult to evaluate. Therefore we introduce the sensitivity coefficient

$$\gamma(A, b, c) = \lim_{\alpha \to 0_+} \frac{\gamma_\alpha(A, b, c)}{\alpha}.$$

Obviously, if α is small, then $\gamma_\alpha(A, b, c)$ is approximately equal to $\gamma(A, b, c)\,\alpha$, hence $\gamma(A, b, c)$ becomes a measure of sensitivity of the optimal value subject to relative errors of the data. We shall show that $\gamma(A, b, c)$ can be expressed in terms of the optimal solutions $\bar{x}$, $\bar{y}$ of the problems (1) and (2).

Theorem. Let the optimal solution $\bar{x}$ of (1) be nondegenerate, let the nonbasic relative cost coefficients be positive and let $c^{\mathrm{T}}\bar{x} \neq 0$. Then there holds

$$(7) \qquad \gamma(A, b, c) = \frac{|c|^{\mathrm{T}}\bar{x} + |b|^{\mathrm{T}}|\bar{y}| + |\bar{y}|^{\mathrm{T}}\,|A|\,\bar{x}}{|c^{\mathrm{T}}\bar{x}|}$$

where $\bar{y}$ is the optimal dual solution.

Proof. Under the assumptions stated, there exists an $\varepsilon > 0$ such that for each $\alpha \in [0, \varepsilon)$, any problem (3) satisfying (4), (5), (6) has a unique optimal basic solution with a common basis B. Let A'_B denote the basic part of A'. Then for the optimal solution x' of (3) we have

$$x'_B = A_B'^{-1} b' = [A_B(E - A_B^{-1}(A_B - A'_B))]^{-1} b' =$$

$$= [E - A_B^{-1}(A_B - A'_B)]^{-1} (\bar{x}_B + A_B^{-1}(b' - b)) = \bar{x}_B + A_B^{-1}(b' - b) +$$

$$+ A_B^{-1}(A_B - A'_B)\,\bar{x}_B + \sum_2^\infty (A_B^{-1}(A_B - A'_B))^j\,\bar{x}_B + \sum_1^\infty (A_B^{-1}(A_B - A'_B))^j A_B^{-1}(b' - b)$$

(provided α is small enough so that the inverse could be expanded into infinite series). Since $|A_B - A'_B| \leqq \alpha|A_B|$ and $|b' - b| \leqq \alpha|b|$, the remaining terms are of order $0(\alpha^2)$, so that we can write

$$x'_B = \bar{x}_B + A_B^{-1}(b' - b) + A_B^{-1}(A_B - A'_B)\,\bar{x}_B + 0(\alpha^2).$$

Hence for the optimal value of (3) we obtain

$$h(A', b', c') = c_B'^{\mathrm{T}} x'_B = (c_B + (c'_B - c_B))^{\mathrm{T}} x'_B =$$

$$= h(A, b, c) + \bar{y}^{\mathrm{T}}(b' - b) + (c'_B - c_B)^{\mathrm{T}}\,\bar{x}_B + \bar{y}^{\mathrm{T}}(A_B - A'_B)\,\bar{x}_B + 0(\alpha^2).$$

The maximal value of $\bar{y}^{\mathrm{T}}(b' - b)$ over b' satisfying (5) is equal to $\alpha|\bar{y}|^{\mathrm{T}}\,|b|$; similarly for the another two terms. Thus we have

$$\gamma_\alpha(A, b, c) = \frac{|c_B|^{\mathrm{T}}\,\bar{x}_B + |b|^{\mathrm{T}}\,|\bar{y}| + |\bar{y}|^{\mathrm{T}}\,|A_B|\,\bar{x}_B}{|c^{\mathrm{T}}\bar{x}|}\,\alpha + 0(\alpha^2),$$

hence taking the limit and using the fact that

$$|c_B|^T \bar{x}_B = |c|^T \bar{x}, \quad |A_B| \bar{x}_B = |A| \bar{x},$$

we get (7).

As an interesting consequence we obtain that

$$\gamma(A, b, c) \geqq 3 \tag{8}$$

for each problem satisfying the assumptions of our theorem. In fact, there holds $|c^T\bar{x}| \leqq |c|^T \bar{x}$ and from the duality theorem [1] we have $|c^T\bar{x}| = |b^T\bar{y}| \leqq |b|^T |\bar{y}|$ and $|c^T\bar{x}| = |\bar{y}^T A\bar{x}| \leqq \leqq |\bar{y}|^T |A| \bar{x}$, hence (8) follows from (7). Thus we can see that the maximal relative error of the optimal value is at least three times greater than the maximal relative error in the data. The minimal value of $\gamma(A, b, c) = 3$ is achieved e.g. if $A \geqq 0$, $b \geqq 0$, $c \geqq 0$ and $\bar{y} \geqq 0$.

The above result can be also extended to a linear programming problem in the standard form

$$\min \{c^T x; Ax \geqq b, x \geqq 0\} \tag{9}$$

and its dual problem

$$\max \{b^T y; A^T y \leqq c, y \geqq 0\}. \tag{10}$$

Defining γ_x and γ in the same way as above, we can see that the value of γ for (9) may be computed from (7) when considering the problem

$$\min \{c^T x; Ax - z = b, x \geqq 0, z \geqq 0\},$$

which is of the form (1), if we replace $|A| \bar{x}$ by $|(A, 0)| \binom{\bar{x}}{\bar{z}}$ (since the columns corresponding to z are not subject to perturbations), which results in the same formula (7), independent of z (the only difference is that $|\bar{y}|$ can be replaced simply by $\bar{y}$, since $\bar{y} \geqq 0$). Also, writing the dual problem (10) in the primal form

$$\min \{-b^T y; -A^T y \geqq -c, y \geqq 0\}$$

and applying the formula (7) to it, we obtain the same result as for the primal problem (9).

References

[1] Murty, K. G.: Linear and Combinatorial Programming. J. Wiley, New York 1976.

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