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AN ASYMPTOTIC RESULT FOR LINEAR INTERVAL SYSTEMS

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Abstract.

It is shown that the bounds $\underline{x}$ and $\bar{x}$ of the exact interval solution (hull) of a system of linear interval equations can be expanded into infinite series and some asymptotic conclusions are drawn.

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Let $A^I x = b^I$ be a linear interval system with an $n \times n$ interval matrix $A^I = [A - \Delta, A + \Delta]$ and a right-hand side interval n-vector $b^I = [b - \delta, b + \delta]$. The exact interval solution is defined as the narrowest interval $[\underline{x}, \bar{x}]$ enclosing the solution set $X = \{x'; A'x' = b', A' \in A^I, b' \in b^I\}$, i.e. $\underline{x} = (\underline{x}_i)$, $\bar{x} = (\bar{x}_i)$ must satisfy

$$\underline{x}_i = \min\{x'_i; x' \in X\}$$
$$\bar{x}_i = \max\{x'_i; x' \in X\} \quad (i = 1, \ldots, n).$$

In this note, we shall show how the vectors $\underline{x}$ and $\bar{x}$ can be expanded into infinite series. The result will be given under three assumptions (for $B = (b_{ij})$ we denote $|B| = (|b_{ij}|)$):

(i) A^I is inverse stable, i.e. $|(A')^{-1}| > 0$ for each $A' \in A^I$ (this means that each inverse matrix coefficient preserves its sign over A^I),

(ii) A^I is strongly regular, i.e. $\rho(|A^{-1}|\Delta) < 1$,

(iii) the whole solution set X is part of a single orthant $R^n_z = \{x' \in R^n; z_j x'_j \geqslant 0, \forall j\}$, where z is some ± 1-vector.

Before giving the main result, we introduce several notations. Put $x = A^{-1}b$, $e = (1, 1, \ldots, 1)^T \in R^n$ and let T_y denote the diagonal matrix with diagonal vector $y \in R^n$. For two $n \times n$ matrices $P = (P_{ij})$ and $R = (R_{ij})$ we denote $P * R = (P_{ij}R_{ij})$, the componentwise product of P and R, and $\operatorname{diag} P = (P_{11}, P_{22}, \ldots, P_{nn})^T$. Finally we introduce the sign matrix $S = (S_{ij})$ by

$$S_{ij} = \begin{cases} 1 & \text{if} \quad (A^{-1})_{ji} > 0 \\ -1 & \text{if} \quad (A^{-1})_{ji} < 0 \end{cases}$$

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(the case $(A^{-1})_{ji} = 0$ cannot occur due to (i)).

THEOREM. *Let (i), (ii), (iii) hold. Then we have*

$$\bar{x} = x + \operatorname{diag}\left(A^{-1} \sum_{j=0}^{\infty} M_j\right) \tag{1}$$

$$\underline{x} = x - \operatorname{diag}\left(A^{-1} \sum_{j=0}^{\infty} (-1)^j M_j\right) \tag{2}$$

where the matrices M_j are given by

$$M_0 = S * ((\Delta|x| + \delta)e^T) \tag{3}$$

$$M_j = S * (BM_{j-1}), \quad j = 1, 2, \ldots \tag{4}$$

with $B = \Delta T_z A^{-1}$.

PROOF. (a) Let $Y = \{y \in R^n; |y_j| = 1 \text{ for each } j\}$. As in the general theory [2], denote by x_y the solution of the system $(A - T_y \Delta T_z)x' = b + T_y \delta$, $\quad y \in Y$ (z is the same as in (iii), so that $T_z x_y \geqslant 0$ holds). From (ii) it follows that $\rho(A^{-1}T_y \Delta T_z) \leq \rho(|A^{-1}|\Delta) < 1$, hence $x_y = (A - T_y \Delta T_z)^{-1}(b + T_y \delta) = (E - A^{-1}T_y \Delta T_z)^{-1}(x + A^{-1}T_y \delta) = x + \sum_{j=1}^{\infty} (A^{-1}T_y \Delta T_z)^j x + \sum_{j=0}^{\infty} (A^{-1}T_y \Delta T_z)^j A^{-1} T_y \delta = x + \sum_{j=0}^{\infty} (A^{-1}T_y \Delta T_z)^j (A^{-1}T_y(\Delta|x| + \delta))$. Thus we have proved that

$$x_y = x + \sum_{j=0}^{\infty} (A^{-1}T_y \Delta T_z)^j (A^{-1}T_y(\Delta|x| + \delta)) \tag{5}$$

holds for each $y \in Y$.

(b) Let $y(i)$ denote the ith column of S. We shall prove by induction on j that for each $j \geqslant 0$ and $i \in \{1, \ldots, n\}$, $(A^{-1}M_j)_{\cdot i}$, the ith column of $A^{-1}M_j$, satisfies

$$(A^{-1}M_j)_{\cdot i} = (A^{-1}T_{y(i)}\Delta T_z)^j (A^{-1}T_{y(i)}(\Delta|x| + \delta)). \tag{6}$$

Let i be fixed. If $j = 0$, then from (3) we have $(A^{-1}M_0)_{\cdot i} = A^{-1}(M_0)_{\cdot i} = A^{-1}T_{y(i)}(\Delta|x| + \delta)$. Let (6) hold for some $j \geqslant 0$. Then by (4) we have $(A^{-1}M_{j+1})_{\cdot i} = A^{-1}(M_{j+1})_{\cdot i} = (A^{-1}T_{y(i)}\Delta T_z)^{j+1}(A^{-1}T_{y(i)}(\Delta|x| + \delta))$, which is (6).

(c) In view of assumption (i), we have $\bar{x}_i = (x_{y(i)})_i$, $\underline{x}_i = (x_{-y(i)})_i$, see [2], theorem 3. Hence, combining (5) and (6), we obtain for each $i \in \{1, \ldots, n\}$ that $\bar{x}_i = x_i + \sum_{j=0}^{\infty} (A^{-1}M_j)_{ii} = x_i + \left(\operatorname{diag}\left(A^{-1} \sum_{j=0}^{\infty} M_j\right)\right)_i$ and $\underline{x}_i = x_i + \sum_{j=0}^{\infty} (-1)^{j+1}(A^{-1}M_j)_{ii} = x_i - \left(\operatorname{diag}\left(A^{-1} \sum_{j=0}^{\infty} (-1)^j M_j\right)\right)_i$, which proves (1) and (2). ∎

Formulae (1) and (2) may be used in practical computations, but the rounding errors in A^{-1} may influence the result; therefore the method described in [3] is to be preferred. Nevertheless, the theorem implies some asymptotic consequences. Denote

$$\beta = \max\left\{\max_{ij} \Delta_{ij}, \max_{i} \delta_i\right\}.$$

Then we can easily prove by induction from (3), (4) that

$$A^{-1}M_j = O(\beta^{j+1})$$

holds for each $j \geq 0$, therefore for each $m \geq 0$ we have

$$\bar{x} = x + \operatorname{diag}\left(A^{-1}\sum_{j=0}^{m} M_j\right) + O(\beta^{m+2}) \tag{5}$$

$$\underline{x} = x - \operatorname{diag}\left(A^{-1}\sum_{j=0}^{m} (-1)^j M_j\right) + O(\beta^{m+2}) \tag{6}$$

and

$$\tfrac{1}{2}(\bar{x} - \underline{x}) = \operatorname{diag}\left(A^{-1}\sum_{\substack{j=0 \\ j \text{ even}}}^{m} M_j\right) + O(\beta^{m+2}). \tag{7}$$

For $m = 0$, we obtain

$$\bar{x} = x + |A^{-1}|(\Delta|x| + \delta) + O(\beta^2)$$

$$\underline{x} = x - |A^{-1}|(\Delta|x| + \delta) + O(\beta^2)$$

which is Miller's result in [1]. If we take $m = 1$, we get

$$\bar{x} = x + |A^{-1}|(\Delta|x| + \delta) + \operatorname{diag} N + O(\beta^3) \tag{8}$$

$$\underline{x} = x - |A^{-1}|(\Delta|x| + \delta) + \operatorname{diag} N + O(\beta^3) \tag{9}$$

where

$$N = A^{-1}M_1 = A^{-1}(S * (\Delta T_z A^{-1}(S * ((\Delta|x| + \delta)e^T)))).$$

Obviously, if β is small, then formulae (8), (9) can give acceptable estimates. Moreover, they imply

$$\tfrac{1}{2}(\bar{x} - \underline{x}) = |A^{-1}|(\Delta|x| + \delta) + O(\beta^3).$$

This is an interesting improvement of Miller's result, where the error is of the form $O(\beta^2)$.

REFERENCES

[1] W. Miller, *On an interval-arithmetic matrix method*, BIT 12 (1972), 213–219.
[2] J. Rohn, *Solving interval linear systems*, Freiburger Intervall-Berichte 84/7, 1–14.
[3] J. Rohn, *A two-sequence method for linear interval equations*, Computing 41 (1989), 137–140.