On Singular Matrices Contained in an Interval Matrix

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In this short note we investigate singular matrices contained in an interval matrix of a special form

$$A^I = \{A;\ A_c - qp^T \leqq A \leqq A_c + qp^T\} \tag{1}$$

where A_c is an $n \times n$ matrix and q, p are n-dimensional column vectors, $q \geqq 0$, $p > 0$ (the superscript "T" denotes transpose vector). We shall delineate a set S of matrices, $S \subset A^I$, such that each $B \in S$ is singular and if $Ax = 0$ for some $A \in A^I$, $x \neq 0$, then there exists a $B \in S$ such that $Bx = 0$.

First we introduce several notations. For $x = (x_i) \in R^n$, we denote $|x| = (|x_i|)$ and define its signature vector $\operatorname{sgn} x \in R^n$ by $(\operatorname{sgn} x)_i = 1$ if $x_i \geqq 0$ and $(\operatorname{sgn} x)_i = -1$ otherwise ($i = 1, \ldots, n$). $D_x = \operatorname{diag}\{x_1, \ldots, x_n\}$ denotes the $n \times n$ diagonal matrix whose diagonal is formed by the components of the vector x. With the positive vector p from (1) we associate the scaled vector norm

$$\|x\|_p = p^T|x| = \sum_i p_i|x_i| .$$

Let us now introduce the set of matrices

$$S = \{A_c - (A_c x)(\operatorname{sgn} x)^T D_p;\ -q \leqq A_c x \leqq q,\ \|x\|_p = 1\} .$$

In the following theorem we sum up some properties of the set S from which it can be seen that S may in certain sense serve as a set of representatives of singular matrices from A^I:

Theorem. Let A_c be an $n \times n$ matrix, $q, p \in R^n$, $q \geqq 0$, $p > 0$. Then we have:

(i) $S \subset A^I$,

(ii) each $B \in S$ is singular,

(iii) if $Ax = 0$ for some $A \in A^I$ and $x \neq 0$, then for $x_0 = x/\|x\|_p$ the matrix $B = A_c - (A_c x_0)(\operatorname{sgn} x_0)^T D_p$ belongs to S and satisfies $Bx = 0$,

(iv) $S \neq \emptyset$ if and only if A^I contains a singular matrix,

(v) if A_c is nonsingular, then $S \neq \emptyset$ if and only if $\|A_c^{-1} D_q y\|_p \geqq 1$ for some ± 1-vector y,

(vi) for each $B \in S$, the matrix $B - A_c$ is of rank at most 1.

Proof.

(i) If $B \in S$, then $B = A_c - (A_c x)(\operatorname{sgn} x)^T D_p$ for some x satisfying $-q \leqq A_c x \leqq q$ and $\|x\|_p = 1$, hence $|B - A_c| = |(A_c x)(\operatorname{sgn} x)^T D_p| \leqq |A_c x|\, p^T \leqq qp^T$, thus $B \in A^I$ due to (1).

(ii) For $B = A_c - (A_c x)(\operatorname{sgn} x)^T D_p$ with $\|x\|_p = 1$ we have $Bx = A_c x - (A_c x)(p^T|x|) = A_c x - (A_c x)\|x\|_p = 0$, hence B is singular.

(iii) If $Ax = 0$ for some $A \in A^I$ and $x \neq 0$, then $|A_c x| = |(A_c - A)x| \leqq qp^T|x| = q\|x\|_p$, hence $x_0 = x/\|x\|_p$ satisfies $-q \leqq A_c x_0 \leqq q$, $\|x_0\|_p = 1$ and for $B = A_c - (A_c x_0)(\operatorname{sgn} x_0)^T D_p$ we have $B \in S$ and $Bx = \|x\|_p Bx_0 = 0$ as proved in (ii).

(iv) The "if" part follows from (iii), the "only if" part from (i) and (ii).

(v) Obviously, $S \neq \emptyset$ if and only if the system of inequalities $-q \leqq A_c x \leqq q$ has a solution satisfying $\|x\|_p = 1$. Let $S \neq \emptyset$ and let x be such a solution; put $t = A_c x$, then $x = A_c^{-1}t$, where $-q \leqq t \leqq q$, and from the convexity of the norm we have that $1 = \|x\|_p = \|A_c^{-1}t\|_p \leqq \max\{\|A_c^{-1}D_q y\|_p;\ y \text{ is a } \pm 1\text{-vector}\}$ since each vertex of the box $\{t;\ -q \leqq t \leqq q\}$ is of the form $D_q y$ for some vector y whose each entry is equal to $+1$ or -1. Conversely, if $\|A_c^{-1}D_q y\|_p \geqq 1$ for some such a vector y, then the vector $x = (A_c^{-1}D_q y)/\|A_c^{-1}D_q y\|_p$ satisfies $|A_c x| \leqq q$ and $\|x\|_p = 1$, hence $S \neq \emptyset$.

(vi) For each $B \in S$ there holds $B - A_c = -(A_c x)(\operatorname{sgn} x)^T D_p$ for some x. If $A_c x = 0$, then $B - A_c$ is of rank 0, otherwise it is of rank 1. This concludes the proof.

The assertion (v) was proved in a slightly different manner in [2]. The proof was included here for completeness.

It follows from the assertion (iii) that each singular matrix $A \in A^I$ can be represented by the singular matrix $B = A_c - (A_c x)(\operatorname{sgn} x)^T D_p$ where x is a vector of unit norm satisfying $Ax = 0$ (notice that the matrix A itself is not used in the construction of B); hence the representation is unique if A has rank $n - 1$.

In a special case of interval matrices of the form

$$A^I = \{A;\ A_c - \beta ee^T \leqq A \leqq A_c + \beta ee^T\}$$

where $e = (1, 1, \ldots, 1)^T \in R^n$, we obtain, with $p = e$ and $q = \beta e$, this description of S (where the norm $\|x\|_e = \sum_i |x_i|$ is denoted, as usual, by $\|x\|_1$):

$$S = \{A_c - (A_c x)(\operatorname{sgn} x)^T;\ -\beta e \leqq A_c x \leqq \beta e,\ \|x\|_1 = 1\}\,.$$

Note, however, that even in this simple case testing whether $S \neq \emptyset$ (i.e. whether A^I contains a singular matrix) is an NP-complete problem; cf. [1].

References

[1] Poljak, S., Rohn, J.: Radius of Nonsingularity, to appear.

[2] Rohn, J.: Real Eigenvalues of an Interval Matrix with Rank One Radius, to appear in *Zeitschrift für Angewandte Mathematik und Mechanik.*

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Resumé

O SINGULÁRNÍCH MATICÍCH OBSAŽENÝCH V INTERVALOVÉ MATICI

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Je uveden explicitní popis jisté podmnožiny množiny všech singulárních matic obsažených v dané intervalové matici.