LINEAR INTERVAL EQUATIONS:

ENCLOSING AND NONSINGULARITY

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Abstract. We describe a family of convex polytopes with the following property: a convex set encloses the solution set of a system of linear interval equations if and only if it intersects all these polytopes. The result is applied to derive necessary and sufficient nonsingularity conditions for interval matrices.

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1. Enclosing the solution set

For a linear interval system

$$A^I x = b^I$$

where $A^I = \{A;\ A_c - \Delta \le A \le A_c + \Delta\}$ is an nxn interval matrix and $b^I = \{b;\ b_c - \delta \le b \le b_c + \delta\}$ is an interval n-vector ($\Delta \ge 0$, $\delta \ge 0$), the solution set is defined by

$$X = \{x;\ Ax = b,\ A \in A^I,\ b \in b^I\} .$$

Since X is generally a nonconvex set difficult to work with [6], numerous ingenious methods were invented (see e.g. survey paper [2]) for enclosing the solution set X, i.e. for finding an interval vector x^I such that $X \subset x^I$ holds. This paper is aimed at giving some necessary and sufficient conditions x^I must satisfy to enclose X, but the conditions are difficult to verify and therefore are rather of theoretical interest.

It is well-known [3] that the solution set X can be equivalently described as

$$X = \{x;\ |A_c x - b_c| \le \Delta |x| + \delta\}$$

where, for $x = (x_i)$, $|x|$ is defined by $|x| = (|x_i|)$. Define further $Y = \{y \in R^n;\ |y_j| = 1 \text{ for each } j\}$ and for each $y \in Y$, let T_y denote the diagonal matrix with diagonal vector y. We shall be interested here with the sets

$$P_y = \{p;\ T_y(A_c p - b_c) \geqslant \Delta |p| + \delta\}$$

for all $y \in Y$ (so that there are 2^n of them). We shall show later that the sets P_y are closely related to the problem of enclosing X. First we shall state some properties of the P_y's.

We shall assume A^I to be regular (i.e., each $A \in A^I$ to be nonsingular). In [5] we proved that if A^I is regular, then for each $y \in Y$, the nonlinear equation

$$T_y(A_c x - b_c) = \Delta |x| + \delta$$

has a unique solution x_y which belongs to X, and clearly also to P_y. Hence, if A^I is regular, then $P_y \neq \emptyset$ for each $y \in Y$; moreover, $X \cap P_y = \{x_y\}$. In the next proposition we show that the absolute value can be removed from the description of P_y:

<u>Proposition 1.</u> Let $y \in Y$. Then a vector p belongs to P_y if and only if it satisfies the system

$$\begin{aligned} T_y(A_c p - b_c) &\geqslant \Delta q + \delta \\ -q \leq p &\leq q \end{aligned} \qquad (1)$$

for some $q \in R^n$.

<u>Proof.</u> If $p \in P_y$, then $q = |p|$ satisfies (1). Conversely, if p and q satisfy (1), then $|p| \leq q$, hence $T_y(A_c p - b_c) \geqslant \Delta q + \delta \geqslant \Delta |p| + \delta$, so that $p \in P_y$.

With the help of this description, we can clarify the structure of P_y:

Theorem 1. Let A^I be regular and $\Delta \neq 0$. Then for each $y \in Y$, P_y is an unbounded convex polytope and x_y is a vertex of it.

Proof. (a) P_y is a convex polytope: According to Proposition 1, P_y is the p-projection of the solution set of (1) , which is a convex polytope; hence so is P_y.

(b) P_y is unbounded: In [5] , Theorem 5.1, we proved that if A^I is regular, then for each $y \in Y$ there exists a positive vector r_y such that $|A_c^{-1}T_y \Delta r_y| < r_y$ holds. For an arbitrary solution (p,q) of (1) consider the half-ray

$$(p,q) + \lambda\,(A_c^{-1}T_y\Delta r_y, r_y) \qquad (2)$$

for $\lambda \geq 0$. For each such a λ , we have

$$T_y(A_c(p + \lambda A_c^{-1}T_y\Delta r_y) - b_c) = T_y(A_c p - b_c) + \lambda\Delta r_y$$
$$\geq \Delta q + \delta + \lambda\Delta r_y = \Delta(q + \lambda r_y) + \delta$$

and

$$|p + \lambda A_c^{-1}T_y\Delta r_y| \leq |p| + \lambda\,|A_c^{-1}T_y\Delta r_y| \leq q + \lambda r_y,$$

hence the half-ray (2), $\lambda \geq 0$, belongs to the solution set of (1) so that the half-ray $\{p + \lambda A_c^{-1}T_y\Delta r_y;\ \lambda \geq 0\}$ is a part of P_y. Since $\Delta \neq 0$ and $r_y > 0$, we have that $A_c^{-1}T_y\Delta r_y \neq 0$, hence P_y is unbounded.

(c) x_y is a vertex of P_y: Assume for contrary that $x_y = \frac{1}{2}(p_1 + p_2)$ for some $p_1, p_2 \in P_y$, $p_1 \neq p_2$. Then we have

$$\Delta|x_y| + \delta = T_y(A_c x_y - b_c) \geq \Delta(\tfrac{1}{2}(|p_1| + |p_2|)) + \delta \geq \Delta|x_y| + \delta$$

which shows that $T_y(A_c p_1 - b_c) = \Delta|p_1| + \delta$, $T_y(A_c p_2 - b_c) =$ $= \Delta|p_2| + \delta$ holds, hence $p_1 = p_2 = x_y$ due to the uniqueness of solution of the equation $T_y(A_c x - b_c) = \Delta|x| + \delta$ stated above, which contradicts our assumption that $p_1 \neq p_2$.

This concludes the proof.

Next we shall investigate conditions under which the sets P_y, $y \in Y$, are mutually disjoint:

Proposition 2. Let $y, z \in Y$. Then, a vector p belongs to $P_y \cap P_z$ if and only if it satisfies

$$y_j(A_c p - b_c)_j \geq (\Delta|p| + \delta)_j \quad \text{for each } j \text{ with } y_j = z_j$$
$$(A_c p - b_c)_j = (\Delta|p| + \delta)_j = 0 \text{ for each } j \text{ with } y_j = -z_j.$$

Proof. The ''if'' part is obvious. Conversely, if $p \in P_y \cap P_z$, then for each j with $y_j = -z_j$ we have $y_j(A_c p - b_c)_j \geq \geq (\Delta|p| + \delta)_j \geq 0$ and $z_j(A_c p - b_c)_j = -y_j(A_c p - b_c)_j \geq \geq (\Delta|p| + \delta)_j \geq 0$, hence $(A_c p - b_c)_j = 0$ and consequently $(\Delta|p| + \delta)_j = 0$.

Corollary. Let either of the two conditions hold:

(a) $\delta > 0$,

(b) $\Delta > 0$ and $|b_c| > 0$.

Then all the sets P_y, $y \in Y$, are mutually disjoint.

Proof. Assume $p \in P_y \cap P_z$, $y \neq z$. Then there exists a j with $y_j \neq z_j$, hence $(\Delta|p| + \delta)_j = 0$ and $(A_c p)_j = (b_c)_j$ due to Proposition 2. However, each of the conditions (a), (b) guarantees that $\Delta|p| + \delta > 0$ for each $0 \neq p \in R^n$, which is a contradiction.

We shall now approach the problem of enclosing the solution set X. We shall first formulate (also for purposes of section 2) this more general statement in which regularity of A^I is not assumed:

Proposition 3. Let $p_y \in P_y$ for each $y \in Y$. Then for each $A \in A^I$ and $b \in b^I$, the equation $Ax = b$ has a solution which belongs to $\mathrm{Conv}\{p_y;\ y \in Y\}$.

Proof. According to an existence theorem proved in [7], it will suffice to prove that

$$T_y A p_y \geq T_y b \qquad (3)$$

holds for each $y \in Y$ and each $A \in A^I$, $b \in b^I$ since the theorem states that in such a case $Ax = b$ has a solution belonging to $\mathrm{Conv}\{p_y;\ y \in Y\}$. To prove (3), let us write

$$T_y(Ap_y - b) = T_y(A_c p_y - b_c) + T_y[(A - A_c)p_y + (b_c - b)] .$$

Since

$$|T_y[(A - A_c)p_y + (b_c - b)]| \leq \Delta|p_y| + \delta,$$

we have

$$T_y(Ap_y - b) \geq T_y(A_c p_y - b_c) - (\Delta|p_y| + \delta) \geq 0$$

(since $p_y \in P_y$), hence (3) holds and the proof is complete.

Our main result concerning enclosing the solution set X can be formulated as follows:

Theorem 2. Let A^I be regular. Then a convex set C encloses the solution set X (i.e., $X \subset C$) if and only if

$$C \cap P_y \neq \emptyset \qquad (4)$$

holds for each $y \in Y$.

Proof. ''Only if'': Since A^I is regular and $X \subset C$, we have $x_y \in X \cap P_y \subset C \cap P_y$ for each $y \in Y$. ''If'': Conversely, let (4) hold for each $y \in Y$. Take a $p_y \in C \cap P_y$ for each y. Due to Proposition 3, for each $A \in A^I$, $b \in b^I$ the unique solution of the equation $Ax = b$ belongs to Conv $\{p_y;\ y \in Y\}$. Hence we have

$$X \subset \text{Conv}\ \{p_y;\ y \in Y\} \subset C,$$

so that C encloses X.

In the special case of $C = x^I$ we get a criterion for interval enclosures.

According to Proposition 3, a convex enclosure of the (generally nonconvex) set X can be constructed by picking a $p_y \in P_y$ for each $y \in Y$ and then by forming the convex hull of the p_y's; an interval enclosure $x^I = [\underline{x}, \bar{x}]$ can be given by

$$\underline{x}_i = \min\ \{(p_y)_i;\ y \in Y\}$$

$$\bar{x}_i = \max\ \{(p_y)_i;\ y \in Y\}.$$

This, however, is a merely theoretical possibility since Y consists of 2^n elements and generally finding a $p_y \in P_y$ is also not an easy task. Therefore we shall not pursue these ideas any further and instead we shall be interested in the second section in using an analogue of the sets P_y to formulate some necessary and sufficient regularity conditions for interval matrices.

2. Regularity conditions

For a square interval matrix $A^I = [A_c - \Delta\ ,\ A_c + \Delta]$ we introduce the sets

$$P_y^o = \{p;\ T_y A_c p > \Delta |p|\} \qquad (5)$$

(sharp inequality componentwise), $y \in Y$. Elementary properties of these sets are summed up in this

Proposition 4. For an arbitrary square A^I, the sets P_y^o have the following properties:

(i) P_y^o is an open convex cone,

(ii) $P_y^o \cap P_z^o = \emptyset$ for $y, z \in Y$, $y \neq z$,

(iii) $P_{-y}^o = -P_y^o$ for each $y \in Y$,

(iv) the number of nonempty P_y^o's is even.

Proof. (i) P_y^o is open since the inequality in (5) is sharp; convexity can be established either directly, or as in Theorem 1; if $p \in P_y^o$ and $\alpha > 0$, then $\alpha p \in P_y^o$ by (5).

(ii) If $y \neq z$, then $y_j = -z_j$ for some j and for $p \in P_y^o \cap P_z^o$ we would have $y_j(A_c p)_j > (\Delta|p|)_j \geq 0$ and $y_j(A_c p)_j < 0$, a contradiction.

(iii) Obviously, if $p \in P_y^o$, then $-p \in P_{-y}^o$, hence $P_{-y}^o = -P_y^o$.

(iv) Follows from (iii) since the nonempty sets can be grouped into pairs P_y^o, P_{-y}^o .

Now we have this regularity criterion:

Theorem 3. A square interval matrix A^I is regular if and only if $P_y^o \neq \emptyset$ for each $y \in Y$.

Proof. ''Only if'': Consider the right-hand side vector $b^I = [-e, e]$ where $e = (1,1,\dots,1)^T \in R^n$. According to the result from [5] quoted in section 1, for each $y \in Y$ the equation

$$T_y(A_c x - 0) = \Delta|x| + e$$

has a solution x_y which then satisfies $T_y A_c x_y > \Delta|x_y|$, hence $x_y \in P_y^o$.

''If'': Let $p_y \in P_y^o$, $y \in Y$, so that $T_y A_c p_y - \Delta|p_y| > 0$. Take an arbitrary vector $b \in R^n$, then there exists a positive real number α such that $\alpha(T_y A_c p_y - \Delta|p_y|) \geq T_y b$, hence $T_y(A_c(\alpha p_y) - b) \geq \Delta|\alpha p_y|$, so that αp_y belongs to the set P_y from section 1 for the linear interval system $A^I x = [b, b]$. Hence, due to Proposition 3, the equation $Ax = b$ has a solution for each $A \in A^I$. Since b was arbitrary, it follows that each $A \in A^I$ is nonsingular, hence A^I is regular.

In accordance with Proposition 4, we can give this result also the following form:

Theorem 4. A square interval matrix $A^I = [A_c - \Delta, A_c + \Delta]$ is regular if and only if the solution set of the inequality

$$|A_c p| > \Delta |p| \tag{6}$$

consists of exactly 2^n mutually disjoint nonempty open convex cones.

Proof. Denote by P the solution set of (6). Let $p \in P$; put $y_i = 1$ if $(A_c p)_i > 0$ and $y_i = -1$ otherwise, then $T_y(A_c p) = |A_c p| > \Delta |p|$, hence $p \in P_y^o$, and conversely, if $p \in P_y^o$, i.e. $T_y A_c p > \Delta |p|$, then $|A_c p| = T_y A_c p > \Delta |p|$, so that $p \in P$. Hence $P = \bigcup_{y \in Y} P_y^o$.

''Only if'': If A^I is regular, then $P_y^o \neq \emptyset$ by Theorem 3, hence P is a union of 2^n mutually disjoint nonempty convex cones P_y^o, $y \in Y$.

''If'': If P consists of 2^n mutually disjoint nonempty open convex cones, then each such a cone must, due to its convexity, be a part of some P_y^o. On the other hand, since each P_y^o is convex and thus connected, it cannot be equal to a union of more disjoint nonempty open cones than one. Hence each such a nonempty open convex cone is equal to some P_y^o. Thus $P_y^o \neq \emptyset$ for each $y \in Y$ and A^I is regular by Theorem 3.

If A^I is singular (by definition, not regular), then the number $\mathcal{K}$ of nonempty cones forming the solution set of (6) is even (Proposition 4), but $\mathcal{K} < 2^n$. The case of $\mathcal{K} = 0$ is not excluded.

In the printouts referred to in the examples to follow, an interval matrix A^I in question is written, for technical reasons, in the form $A^I x = 0$, but this has nothing to do with linear systems.

Examples. The four open cones P_y^o for the regular interval matrix from the example by Barth and Nuding in section 1 are depicted in Fig. 3. The structure of P is indeed regular here since the axes of each two neighbouring cones form a 90^o angle.

The sets P_y^o for another regular matrix, forming, in the area depicted, something like a Maltese cross, are shown on Fig. 4. This example was included to demonstrate that a set P_y^o does not necessarily belong to a single orthant as it was the case in the foregoing example.

Fig. 5 depicts the case of a singular interval matrix where two cones are empty (i.e. $\pi = 2$).

Finally, on Fig. 6 we have a situation where all four cones vanish ($\pi = 0$). In this case the inequality (6) has the form

$$|x_1| > 2|x_1|$$
$$|x_2| > 2|x_2| \ ,$$

so that it does not have any solution.

The regularity criterion given in Theorem 3 is, of course, hardly of use in practice. No easily verifiable necessary and sufficient regularity condition is known to date, however, which is explainable by a recently proved result [4] stating that the problem of verifying regularity of an interval matrix is NP-complete.

References

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BARTH-NUDING (1974)

```
< 2, 4>*X(1) + <-2, 1>*X(2) = <-2, 2>
<-1, 2>*X(1) + < 2, 4>*X(2) = <-2, 2>
```

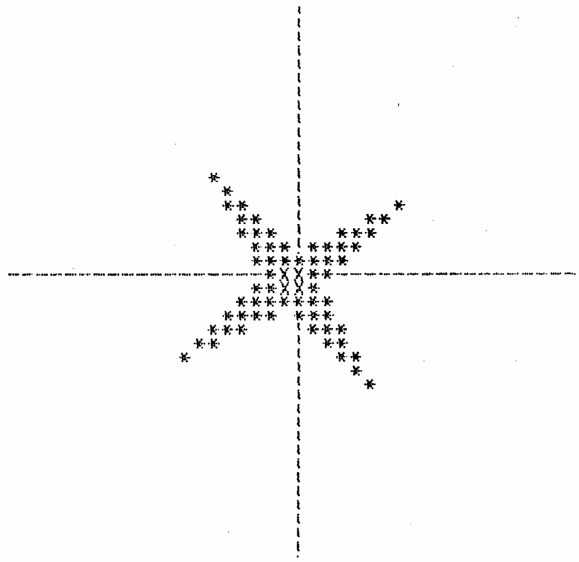

FIG. 1. THE SOLUTION SET X
DEPICTED AREA: <-10,10>X<-10,10>

BARTH-NUDING (1974)

```
< 2, 4>*X(1) + <-2, 1>*X(2) = <-2, 2>
<-1, 2>*X(1) + < 2, 4>*X(2) = <-2, 2>
```

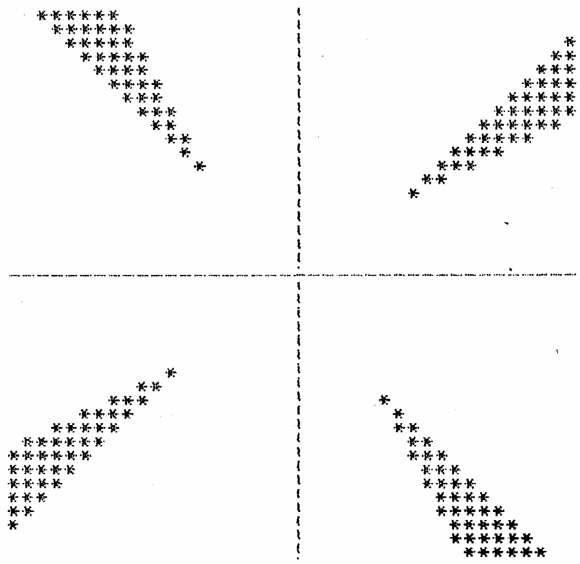

FIG. 2. THE SETS PY
DEPICTED AREA: <-10,10>X<-10,10>

BARTH-NUDING (1974)

< 2, 4>*X(1) + <-2, 1>*X(2) = < 0, 0>
<-1, 2>*X(1) + < 2, 4>*X(2) = < 0, 0>

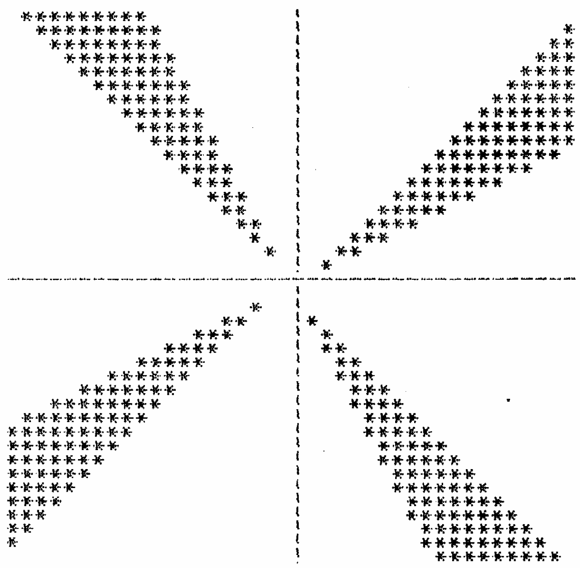

FIG. 3. REGULARITY CRITERION: THE FOUR OPEN CONES
DEPICTED AREA: <-10,10>X<-10,10>

EXAMPLE

```
<-4,-2>*X(1) + < 1, 3>*X(2) = < 0, 0>
<-4,-2>*X(1) + <-4,-2>*X(2) = < 0, 0>
```

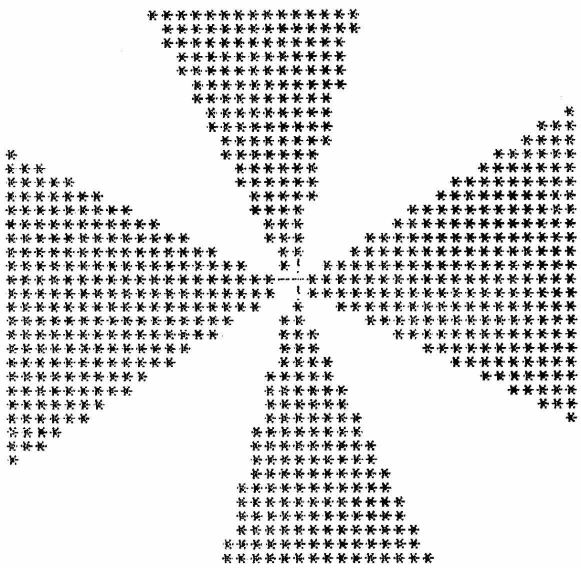

FIG. 4. REGULARITY: A CONE MAY INTERSECT MORE THAN ONE ORTHANT
DEPICTED AREA: <-10,10>X<-10,10>

EXAMPLE

```
<1,2>*X(1) + <3,4>*X(2) = <0,0>
<4,5>*X(1) + <5,6>*X(2) = <0,0>
```

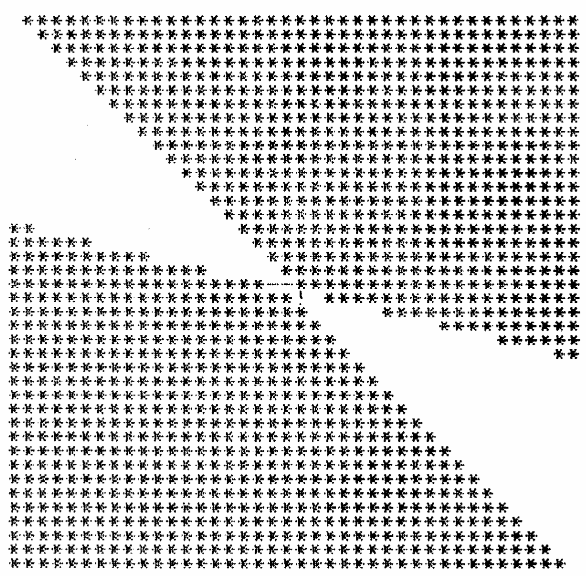

FIG. 5. SINGULAR INTERVAL MATRIX: TWO CONES ONLY
DEPICTED AREA: <-20,20>X<-20,20>

EXAMPLE

<-1, 3>*X(1) + < 0, 0>*X(2) = < 0, 0>
< 0, 0>*X(1) + <-1, 3>*X(2) = < 0, 0>

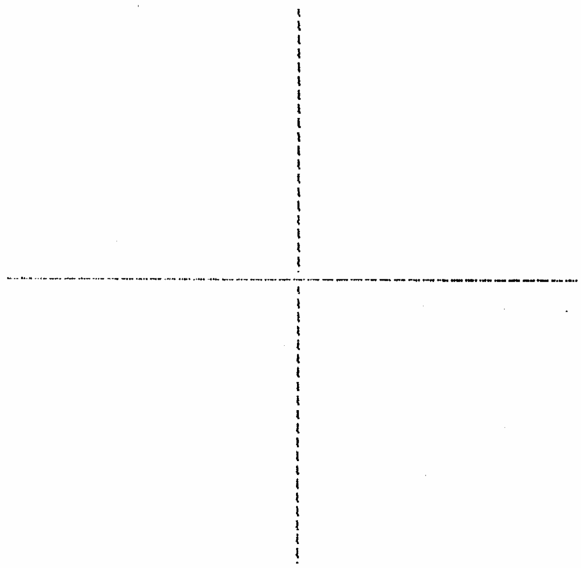

FIG. 6. SINGULAR INTERVAL MATRIX: ALL CONES EMPTY
DEPICTED AREA: <-10,10>X<-10,10>