Computing 43, 93–95 (1989)

Computing

A Farkas-Type Theorem for Linear Interval Equations

J. Rohn, Prague

Received December 27, 1988; revised April 17, 1989

Abstract — Zusammenfassung

A Farkas-Type Theorem for Linear Interval Equations. We give a Farkas-type necessary and sufficient condition for a system of linear interval equations to have a nonnegative solution, and derive a consequence of it.

AMS Subject Classifications: 15A06, 15A39, 65G10

Key words: linear interval equations, nonnegative solution

Ein Satz von Farkasschen Type für lineare Intervallgleichungssysteme. Notwendige und hinreichende Bedingungen für die Existenz einer nichtnegativen Lösung cines linearen Intervallgleichungssystems werden angegeben.

The classical Farkas theorem says that a system of linear equations $Ax = b$ has a nonnegative solution if and only if for each y, $A^T y \geq 0$ implies $b^T y \geq 0$. In this short note, we give an interval version of this theorem. The result was already stated in [2], where, however, its actual meaning was hidden under a burdensome notation and was proved in a rather complicated manner via the duality theorem of interval linear programming. Here, we restate the theorem in a more compact form, give a simple proof of it and derive a consequence showing an interesting property of linear interval systems.

Let $\underline{A}$, $\bar{A}$ be two $m \times n$ matrices satisfying $\underline{A} \leq A$ and $\underline{b}$, $\bar{b}$ two vectors in R^m with $\underline{b} \leq \bar{b}$. We introduce the interval matrix $A^I = \{A; \underline{A} \leq A \leq \bar{A}\}$ and the interval vector $b^I = \{b; \underline{b} \leq b \leq \bar{b}\}$. A vector $x \in R^n$ is called a solution of the system of linear interval equations

$$A^I x = b^I \tag{1}$$

if it satisfies $Ax = b$ for some $A \in A^I$, $b \in b^I$. We shall be interested in nonnegative solutions of (1), i.e. solutions satisfying $x \geq 0$. The following theorem gives a Farkas-type necessary and sufficient condition for the system (1) to have a nonnegative solution; notice the difference in quantifiers:

Theorem 1. *A system* (1) *has a nonnegative solution if and only if there holds*

$$(\forall y)(A^T y \geq 0 \text{ for each } A \in A^I \Rightarrow b^T y \geq 0 \text{ for some } b \in b^I) \tag{2}$$

Proof. (i) To prove the "only if" part, assume that $A_0x = b_0$, $x \geq 0$ holds for some $A_0 \in A^I$, $b_0 \in b^I$. Then, if a vector y satisfies $A^Ty \geq 0$ for each $A \in A^I$, then also $A_0^Ty \geq 0$, hence $b_0^Ty \geq 0$ due to the Farkas theorem applied to the system $A_0x = b_0$, which proves (2).

(ii) Conversely, assume (2) to be satisfied. We shall first show that then there holds

$$(\forall y_1 \geq 0)(\forall y_2 \geq 0)(\underline{A}^Ty_1 - \bar{A}^Ty_2 \geq 0 \Rightarrow \bar{b}^Ty_1 - \underline{b}^Ty_2 \geq 0). \qquad (3)$$

In fact, let $\underline{A}^Ty_1 - \bar{A}^Ty_2 \geq 0$ for some $y_1 \geq 0$, $y_2 \geq 0$. Then for each A with $\underline{A} \leq A \leq \bar{A}$ we have $\underline{A}^Ty_1 \leq A^Ty_1$ and $A^Ty_2 \leq \bar{A}^Ty_2$, hence $A^T(y_1 - y_2) \geq \underline{A}^Ty_1 - \bar{A}^Ty_2 \geq 0$. Now (2) implies existence of a $b_0 \in b^I$ such that $b_0^T(y_1 - y_2) \geq 0$. Since $\underline{b} \leq b_0 \leq \bar{b}$, from nonnegativity of both y_1 and y_2 we obtain that $\underline{b}^Ty_2 \leq b_0^Ty_2$ and $b_0^Ty_1 \leq \bar{b}^Ty_1$, implying $\bar{b}^Ty_1 - \underline{b}^Ty_2 \geq b_0^T(y_1 - y_2) \geq 0$, which completes the proof of (3).

Now, (3) can be easily checked to be the Farkas condition for the system of linear inequalities

$$\begin{aligned} \underline{A}x &\leq \bar{b} \\ -\bar{A}x &\leq -\underline{b} \\ x &\geq 0 \end{aligned} \qquad (4)$$

to have a solution (introducing slack variables to (4) in order to bring it to a system of linear equations provides for nonnegativity of y_1, y_2 in (3)). But it follows from the result by Oettli and Prager in [1] that the system (4) describes the set of nonnegative solutions to (1), which is thus nonempty, and the proof is complete. ■

As a consequence, we obtain the following result.

Theorem 2. *A system* (1) *does not have a nonnegative solution if and only if there exists a* $y \in R^m$ *such that the equation*

$$y^TAx = y^Tb \qquad (5)$$

does not have a nonnegative solution for any $A \in A^I$, $b \in b^I$.

Proof. "Only if": Assuming that (1) does not have a nonnegative solution, we obtain from Theorem 1 that there exists a $y \in R^m$ such that for each $A \in A^I$ and $b \in b^I$ there holds $A^Ty \geq 0$, $b^Ty < 0$, hence (5) cannot have a nonnegative solution since in such a case the left-hand side in (5) were nonnegative while the right-hand one is strictly negative.

"If": Suppose (1) has a nonnegative solution x, i.e. $A_0x = b_0$ for some $A_0 \in A^I$, $b_0 \in b^I$. Then $y^TA_0x = y^Tb_0$, hence (5) has a nonnegative solution, which is a contradiction. ■

Of course, Theorem 1 is only of theoretical interest. In practice, checking for a nonnegative solution of (1) may be performed by verifying that the system of linear inequalities (4) has a solution, which may be done e.g. by phase I of the simplex algorithm. In case of nonexistence of a nonnegative solution, the vector y from

Theorem 2 can be computed simply as $y = y_1^* - y_2^*$, where y_1^*, y_2^* is an optimal solution of the linear programming problem

$$\min\{\bar{b}^T y_1 - \underline{b}^T y_2; \underline{A}^T y_1 - \bar{A}^T y_2 \geq 0, 0 \leq y_1 \leq e, 0 \leq y_2 \leq e\} \tag{6}$$

where e is the vector of all units. In fact, we know already that in this case there exist $y_1 \geq 0$, $y_2 \geq 0$ such that $\underline{A}^T y_1 - \bar{A}^T y_2 \geq 0$, $\bar{b}^T y_1 - \underline{b}^T y_2 < 0$ hold. By norming y_1, y_2 if necessary, we may assume them to satisfy $0 \leq y_1 \leq e$, $0 \leq y_2 \leq e$. These constraints assure that (6) has an optimal solution y_1^*, y_2^* satisfying

$$A^T(y_1^* - y_2^*) \geq \underline{A}^T y_1^* - \bar{A}^T y_2^* \geq 0 \qquad \text{for each } A \in A^I$$

and

$$b^T(y_1^* - y_2^*) \leq \bar{b}^T y_1^* - \underline{b}^T y_2^* < 0 \qquad \text{for each } b \in b^I,$$

so that $y = y_1^* - y_2^*$ is the vector wanted.

In this note, we were interested in conditions for *some* system $Ax = b$ with $A \in A^I$, $b \in b^I$ to have a nonnegative solution. The system (1) can be also studied from another point of view, asking for conditions under which *each* system $Ax = b$ with $A \in A^I$, $b \in b^I$ has a nonnegative solution. Such conditions were given in [3].

Acknowledgment

The presentation was improved due to the referee's suggestions.

References

[1] Oettli, W., and Prager, W.: Compatibility of Approximate Solution of Linear Equations with Given Error Bounds for Coefficients and Right-Hand Sides, Numerische Mathematik 6(1964), 405–409.
[2] Rohn, J.: Duality in Interval Linear Programming, in: Interval Mathematics 1980 (K. L. E. Nickel, Ed.), Academic Press, New York 1980, 521–529.
[3] Rohn, J.: Strong Solvability of Interval Linear Programming Problems, Computing 26(1981), 79–82.

J. Rohn
Faculty of Mathematics and Physics
Charles University
Malostranské nám. 25
CS-118 00 Prague
Czechoslovakia

Verleger: Springer-Verlag KG, Mölkerbastei 5, A-1010 Wien. Herausgeber: Prof. Dr. Hans J. Stetter, Institut für Angewandte und Numerische Mathematik der Technischen Universität Wien, Wiedner Hauptstraße 6—10, A-1040 Wien. — Redaktion: Wiedner Hauptstraße 6 10, A-1040 Wien. Satz und Umbruch: Asco Trade Typesetting Limited, Hong Kong; Reproduktion und Offsetdruck: Novographic, Ing. W. Schmid, Maurer-Lange-Gasse 64, A-1238 Wien. — Verlagsort: Wien. Herstellungsort: Wien. Printed in Austria.