Systems of Linear Interval Equations

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ABSTRACT

This paper presents theory and methods for computing the exact bounds on the solution of a system of n linear equations in n variables whose coefficients and right-hand sides vary in some real intervals. Finite and iterative methods are given, based on results from linear complementarity theory. Also regularity conditions, regularity testing, and computing the exact inverse of an interval matrix are dealt with.

0. INTRODUCTION

The problem of solving a system of linear interval equations is formulated as follows. Assume that the coefficients and right-hand sides of a system of n linear equations in n variables are not determined exactly, but are only known to lie within some real intervals (obtained as a result of roundoff, truncation, or data errors). Such a system of linear interval equations represents a family of ordinary linear systems which can be obtained from it by fixing coefficients and right-hand-side values in the prescribed intervals. Each of these systems, under a regularity assumption, has a unique solution, and all these solutions constitute a so-called solution set X. Now the basic problem treated in this paper is how to compute the numbers

$$\underline{x}_i = \min\{x_i;\ x \in X\},$$

$$\bar{x}_i = \max\{x_i;\ x \in X\} \tag{0}$$

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$(i = 1,\ldots,n)$, describing the exact ranges of the components of the solution if coefficients and right-hand sides are allowed to vary independently of each other in the given intervals.

Since the pioneer work by Oettli and Prager [29] in 1964, this problem has received much attention. The main source of difficulties connected with computing the exact values of $\underline{x}_i$ and $\bar{x}_i$ $(i = 1,\ldots,n)$ is the complicated structure of the solution set X, which is generally nonconvex. Since the intersection of X with each orthant is, however, a convex polyhedron, Oettli [28] proposed using a linear programming procedure in each orthant to determine $\underline{x}_i, \bar{x}_i$; this is seemingly the only general method known so far. Otherwise methods for computing the exact values of $\underline{x}_i, \bar{x}_i$ have been constructed only under special assumptions related to the inverse nonnegativity of the coefficient matrix (Barth and Nuding [4], Beeck [8], Garloff [14]). Most authors were therefore concerned with obtaining sufficiently close outer estimates of the solution set X, and several ingenious methods were found in this direction; for a detailed description, see Alefeld and Herzberger [3], Deif [11], and Neumaier [26].

Our approach, developed in [31–40], is based on the fact that $\operatorname{Conv} X$ is a convex polyhedron, so that each minimum (maximum) in (0) is achieved at some of its vertices. In the main Theorem 2.2 we show that each vertex of $\operatorname{Conv} X$ is a unique solution of a nonlinear equation of type $x = D|x| + d$, so that solving 2^n of these equations provides us, at least theoretically, with a tool for determining $\underline{x}_i, \bar{x}_i$ by finite means. In Section 3, devoted to methods for solving the aforementioned nonlinear equations, we first give a general Algorithm 3.1, prove its finiteness (Theorem 3.1), examine its computational complexity, and finally give an iterative method, which is not general, but is suitable for problems with small coefficient deviations. The problem of reducing the number of vertices to be computed is handled in Section 4. In Theorem 4.5 we show that in the case of the so-called inverse stable interval matrices (where each inverse matrix is of the same sign pattern), computing only $2n$ (instead of 2^n) vertices is needed. This gives a practically applicable method for solving systems of linear interval equations. The last two sections, 5 and 6, give some results concerning interval matrices (i.e. matrices with interval coefficients). In Section 5 we prove various necessary and sufficient regularity conditions (Theorem 5.1) and give some methods for testing regularity. An iterative method for computing the exact inverse of an interval matrix is described in Section 6. A number of examples are included to illustrate various features of our algorithms.

The main theorems, 2.2 and 3.1, may be proved in two ways. Firstly, we may prove them by rearranging the basic nonlinear equation to a linear complementarity problem and using the results by Samelson, Thrall, and Wesler [30] and Murty [23]; this is possible because regular interval matrices are closely related to real P-matrices (Theorem 1.2; also, Assertion (B1) of Theorem 5.1). Secondly, an elementary proof may be given based only on the auxiliary Theorem 1.1. We have incorporated both these approaches.

We sum up here briefly our basic notation. The coefficients of a real $m \times n$ matrix A are denoted by A_{ij}, its columns by $A_{.j}$ $(i = 1,\dots,m,$ $j = 1,\dots,n)$. The matrix $|A|$ is defined by $|A|_{ij} = |A_{ij}|$ for each i, j; A^T denotes the transpose of A, and $\det A$ its determinant. Given two matrices $\underline{A}$, $\overline{A}$ of the same size, we write $\underline{A} \leqslant \overline{A}$ $(\underline{A} < \overline{A})$ if $\underline{A}_{ij} \leqslant \overline{A}_{ij}$ $(\underline{A}_{ij} < \overline{A}_{ij})$ for each i, j. An interval matrix A^I is defined by $A^I = [\underline{A}, \overline{A}] = \{A;\ \underline{A} \leqslant A \leqslant \overline{A}\}$. We shall often use the center matrix $A_c = \frac{1}{2}(\underline{A} + \overline{A})$ and the radius $\Delta = \frac{1}{2}(\overline{A} - \underline{A})$. In this notation we have $\underline{A} = A_c - \Delta$, $\overline{A} = A_c + \Delta$, and $\Delta \geqslant 0$, so that $A^I = [A_c - \Delta, A_c + \Delta]$. Given a compact (in particular, finite) set of matrices B, we introduce matrices $\min B$, $\max B$ componentwise by

$$(\min B)_{ij} = \min\{A_{ij};\ A \in B\},$$

$$(\max B)_{ij} = \max\{A_{ij};\ A \in B\};$$

hence $[\min B, \max B]$ is the narrowest interval matrix containing B. In particular, for each real matrix A we introduce $A^+ = \max\{A, 0\}$, $A^- = \max\{-A, 0\}$; then $A^+ \geqslant 0$, $A^- \geqslant 0$, $A = A^+ - A^-$, $|A| = A^+ + A^-$ (0 is the zero matrix). This notation also applies to vectors, which are always considered as one-column matrices, except when in subscript, where e.g. we write $x_{(-1,1)}$ instead of $x_{(-1,1)^T}$, for typographical reasons. For each $x \in R^n$, we define its signature vector $\operatorname{sgn} x$ by $(\operatorname{sgn} x)_i = 1$ if $x_i \geqslant 0$ and $(\operatorname{sgn} x)_i = -1$ otherwise.

1. PREREQUISITES

A square interval matrix A^I is called regular if each $A \in A^I$ is nonsingular; otherwise it is called singular. Various necessary and sufficient regularity conditions will be given later, in Section 5. Here we shall prove two preliminary results (the second of them being of independent interest) which will be used in the proofs of the main Theorems 2.2 and 3.1.

THEOREM 1.1. *Let A^I be regular, and let $Ax = A'x'$ for some $A, A' \in A^I$, $x \neq x'$. Then there exists a $j \in \{1,\dots,n\}$ such that $A_{.j} \neq A'_{.j}$ and $x_j x'_j > 0$.*

Proof. Assume to the contrary that there exist $A, A' \in A^I$, $x \neq x'$ such that $Ax = A'x'$ and for each j with $A_{.j} \neq A'_{.j}$ one has $x_j x'_j \leqslant 0$. Denote

$J = \{j;\ A_{\cdot j} \neq A'_{\cdot j},\ x_j x'_j < 0\}$, and define a matrix $\tilde{A}$ as follows. If $j \in J$, put

$$\tilde{A}_{\cdot j} = \frac{x_j}{x_j - x'_j} A_{\cdot j} - \frac{x'_j}{x_j - x'_j} A'_{\cdot j}.$$

Since $x_j/(x_j - x'_j) = x_j^2/(x_j^2 - x_j x'_j) > 0$ and similarly $-x'_j/(x_j - x'_j) = -x_j x'_j/(x_j^2 - x_j x'_j) > 0$, $\tilde{A}_{\cdot j}$ is a convex combination of $A_{\cdot j}$ and $A'_{\cdot j}$; therefore $\tilde{A}_{\cdot j} \in [\underline{A}_{\cdot j}, \bar{A}_{\cdot j}]$. If $j \notin J$, then either $A_{\cdot j} = A'_{\cdot j}$, or $A_{\cdot j} \neq A'_{\cdot j}$ and $x_j x'_j = 0$; in both cases we may choose an $\tilde{A}_{\cdot j} \in \{A_{\cdot j}, A'_{\cdot j}\}$ such that $x_j A_{\cdot j} - x'_j A'_{\cdot j} = (x_j - x'_j)\tilde{A}_{\cdot j}$. Hence $\tilde{A} \in A^I$, and we have $\tilde{A}(x - x') = \sum_j (x_j - x'_j)\tilde{A}_{\cdot j} = \sum_j (x_j A_{\cdot j} - x'_j A'_{\cdot j}) = Ax - A'x' = 0$, implying that $\tilde{A}$ is singular, a contradiction. ■

The second result relates regular interval matrices to real P-matrices. Recall that a square matrix is said to be a P-matrix if all its principal minors are positive.

THEOREM 1.2. *Let A^I be regular. Then for each $A_1, A_2 \in A^I$, both $A_1^{-1}A_2$ and $A_1 A_2^{-1}$ are P-matrices.*

Proof. Let $A_1, A_2 \in A^I$. Take an $x \in R^n$, $x \neq 0$, and set $x' = A_1^{-1}A_2 x$. Then either $x = x'$, implying $x_j x'_j = x_j^2 > 0$ for some j, or $x \neq x'$, in which case again $x_j x'_j > 0$ for some j in view of Theorem 1.1. Hence $A_1^{-1}A_2$ is a P-matrix, due to the characterization by Fiedler and Pták [12]. Applying this result to the transposed interval matrix $A^{IT} = \{A^T;\ A \in A^I\}$, we see that $(A_1 A_2^{-1})^T = (A_2^T)^{-1} A_1^T$ is a P-matrix, hence so is $A_1 A_2^{-1}$. ■

Later, in Theorem 5.1, we shall show that this result can also be reversed: we shall delineate a set of 2^n matrices of the form $A_1^{-1}A_2$ such that if all of them are P-matrices, then A^I is regular.

2. THE CONVEX-HULL THEOREM

In this section we prove the basic theoretical result of this paper, the convex-hull Theorem 2.2. But first we give some notation. We denote $e = (1, 1, \ldots, 1)^T \in R^n$, $f = -e$, and $Y = \{y;\ |y| = e,\ y \in R^n\}$, so that Y has 2^n elements. For each $z \in R^n$, denote $T_z = \operatorname{diag}\{z_1, \ldots, z_n\}$, the diagonal matrix with diagonal vector z. Given an $n \times n$ interval matrix $A^I =$

$[A_c - \Delta, A_c + \Delta]$ and an interval n-vector $b^I = [b_c - \delta, b_c + \delta]$, we introduce for any vectors y, $z \in R^n$ the notation

$$A_{yz} = A_c - T_y \Delta T_z,$$

$$b_y = b_c + T_y \delta.$$

With a few exceptions, we shall use this notation only for $y, z \in Y$. In this case, as can be readily verified, we have for each $i, j \in \{1, \dots, n\}$

$$(A_{yz})_{ij} = \begin{cases} (A_c - \Delta)_{ij} & \text{if} \quad y_i z_j = 1, \\ (A_c + \Delta)_{ij} & \text{if} \quad y_i z_j = -1, \end{cases}$$

$$(b_y)_i = \begin{cases} (b_c + \delta)_i & \text{if} \quad y_i = 1, \\ (b_c - \delta)_i & \text{if} \quad y_i = -1, \end{cases}$$

so that $A_{yz} \in A^I$ and $b_y \in b^I$.

Let $A^I = [A_c - \Delta, A_c + \Delta]$ be an $n \times n$ interval matrix and

$$b^I = [b_c - \delta, b_c + \delta]$$

be an interval n-vector. Since no values in A^I or b^I are preferred, each x satisfying $Ax = b$ for some $A \in A^I$, $b \in b^I$ is considered to be a solution of the (formally written) system of linear interval equations

$$A^I x = b^I. \tag{2.1}$$

Hence we introduce the solution set X of (2.1) by

$$X = \{x;\ Ax = b,\ A \in A^I,\ b \in b^I\}.$$

The description of X is due to Oettli and Prager [29]. We give here a (new) proof of their result for completeness. Notice that regularity of A^I is not assumed.

THEOREM 2.1. *We have* $X = \{x;\ |A_c x - b_c| \leqslant \Delta|x| + \delta\}$.

Proof. If $x \in X$, then $Ax = b$ for some $A \in A^I$, $b \in b^I$, which gives $|A_c x - b_c| = |(A_c - A)x + b - b_c| \leqslant \Delta|x| + \delta$. Conversely, let $|A_c x - b_c| \leqslant$

$|x|+\delta$ for some x. Define a $y \in R^n$ by

$$y_i = \begin{cases} \dfrac{(A_c x - b_c)_i}{(\Delta|x|+\delta)_i} & \text{if} \quad (\Delta|x|+\delta)_i > 0, \\ 1 & \text{otherwise} \end{cases}$$

$(i = 1, \dots, n)$; then $|y| \leqslant e$ and $A_c x - b_c = T_y(\Delta|x|+\delta)$. Setting $z = \operatorname{sgn} x$ and substituting $|x| = T_z x$, we get $A_{yz}x = (A_c - T_y \Delta T_z)x = b_c + T_y\delta = b_y$. Since $|y| \leqslant e$, we have $|T_y \Delta T_z| \leqslant \Delta$ and $|T_y \delta| \leqslant \delta$, so that $A_{yz} \in A^I$ and $b_y \in b^I$, implying $x \in X$. ■

It follows from this theorem that for regular A^I, the intersection of X with each orthant is a convex polyhedron (Beeck [6]); hence X, being a union of convex polyhedra, is generally nonconvex (for examples, see Barth and Nuding [4], Nickel [27], Hansen [16]). However, $\operatorname{Conv} X$ (the convex hull of X) is a convex polyhedron; therefore it is equal to the convex hull of its vertices. We shall show in the next theorem that each vertex of $\operatorname{Conv} X$ satisfies the nonlinear equation

$$|A_c x - b_c| = \Delta|x| + \delta. \tag{2.2}$$

We shall show that this equation can be decomposed into 2^n simpler equations. Let x solve (2.2); put $y = \operatorname{sgn}(A_c x - b_c)$, then $y \in Y$, and from $|A_c x - b_c| = T_y(A_c x - b_c) = \Delta|x| + \delta$, using the fact that $T_y^{-1} = T_y$, we obtain

$$A_c x - b_c = T_y(\Delta|x| + \delta). \tag{2.3}$$

Conversely, if x satisfies (2.3) for some $y \in Y$, then, taking absolute values on both sides of (2.3), we obtain that x solves (2.2). Thus we have replaced (2.2) by 2^n equations of the type (2.3) (for all $y \in Y$). We shall bring (2.3) into three equivalent forms. First, substituting $x = x^+ - x^-$, $|x| = x^+ + x^-$ in (2.3) and using our notation $A_c - T_y\Delta = A_{ye}$, $A_c + T_y\Delta = A_{yf}$, we obtain

$$x^+ = A_{ye}^{-1}A_{yf}x^- + A_{ye}^{-1}b_y, \tag{2.4}$$

which is a linear complementarity problem; we shall use this form for the proof of the existence and uniqueness of a solution of (2.3). Second, setting $z = \operatorname{sgn} x$ and substituting $|x| = T_z x$ in (2.3), we get another equivalent form

$$A_{yz}x = b_y,$$

$$T_z x \geqslant 0, \qquad z \in Y, \tag{2.5}$$

which will be used in Section 3 for the formulation of a general algorithm for solving (2.3). Third, by a simple rearrangement of (2.3) we have

$$x = D_y|x| + d_y \tag{2.6}$$

(where we have denoted $D_y = A_c^{-1}T_y\Delta$, $d_y = A_c^{-1}b_y$), which is a fixed-point equation to be used in Section 3 for solving (2.3) by an iterative method.

Our basic result is now formulated as follows:

THEOREM 2.2. *Let A^I be regular. Then for each $y \in Y$, the equation* (2.3) (*equivalently*, (2.4), (2.5) *or* (2.6)) *has exactly one solution $x_y \in X$, and*

$$\operatorname{Conv} X = \operatorname{Conv}\{x_y;\ y \in Y\}.$$

Proof. Let $y \in Y$; since $A_{ye}^{-1}A_{yf}$ is a P-matrix by Theorem 1.2, the linear complementarity problem (2.4) has exactly one solution x_y according to the well-known result proved independently by Samelson, Thrall, and Wesler [30], Ingleton [18], and Murty [22]. From the equivalent equation (2.5) it follows that $x_y \in X$, which gives $\operatorname{Conv}\{x_y;\ y \in Y\} \subset \operatorname{Conv} X$. To prove the converse inclusion, take an $x \in X$, so that $A_0 x = b_0$ for some $A_0 \in A^I$, $b_0 \in b^I$. We shall prove that the system of linear equations

$$\sum_{y \in Y} \lambda_y (A_0 x_y) = b_0,$$

$$\sum_{y \in Y} \lambda_y = 1 \tag{2.7}$$

has a nonnegative solution $\lambda_y \in R^1$, $y \in Y$. In view of the Farkas lemma [42], it suffices to show that for each $p \in R^n$ and $p_0 \in R^1$, if $p^T A_0 x_y + p_0 \geqslant 0$ for each $y \in Y$, then $p^T b_0 + p_0 \geqslant 0$. Thus let $p^T A_0 x_y + p_0 \geqslant 0$ for each $y \in Y$.

Put $y = -\operatorname{sgn} p$; then $|p| = -T_y p$, hence $|p^T(A_0 - A_c)| \leqslant |p^T|\Delta = -p^T T_y \Delta$, implying $p^T A_{yf} \leqslant p^T A_0 \leqslant p^T A_{ye}$; similarly, $p^T b_y \leqslant p^T b_0$. Now we have

$$p^T b_0 \geqslant p^T b_y = p^T\left(A_{ye}x_y^+ - A_{yf}x_y^-\right) \geqslant p^T A_0 x_y^+ - p^T A_0 x_y^- = p^T A_0 x_y \geqslant -p_0,$$

so that $p^T b_0 + p_0 \geqslant 0$. Thus (2.7) has a solution $\lambda_y \geqslant 0$, $y \in Y$, so that

$$A_0\left(\sum_{y \in Y} \lambda_y x_y\right) = b_0 = A_0 x,$$

and the nonsingularity of A_0 gives $x = \sum_{y \in Y} \lambda_y x_y$. Hence $X \subset \operatorname{Conv}\{x_y;\ y \in Y\}$, implying $\operatorname{Conv} X \subset \operatorname{Conv}\{x_y;\ y \in Y\}$, which completes the proof. ∎

In Section 3 we shall present another proof of the existence and uniqueness of a solution of (2.3), making no use of results on P-matrices and based purely on Theorem 1.1.

The next theorem shows that under mild assumptions, all the x_y's are different:

THEOREM 2.3. *Let A^I be regular, and let either of the following assumptions hold*:

(a) $\delta > 0$,
(b) $\Delta > 0$ *and* $0 \notin b^I$.

Then for each $y, y' \in Y$, $y \neq y'$ *implies* $x_y \neq x_{y'}$.

Proof. Each of the two assumptions implies that $\Delta|x| + \delta > 0$ for each $x \in X$. Let $x_y = x_{y'}$ for some y, $y' \in Y$. Then $T_y(\Delta|x_y| + \delta) = A_c x_y - b_c = A_c x_{y'} - b_c = T_{y'}(\Delta|x_y| + \delta)$, and since $\Delta|x_y| + \delta > 0$, it follows that $y = y'$. ∎

The interval vector $x^I = [\underline{x}, \bar{x}]$, where

$$\underline{x} = \min\{x;\ x \in X\},$$

$$\bar{x} = \max\{x;\ x \in X\}$$

(min, max to be understood componentwise) is called the interval solution to $A^I x = b^I$. Obviously, x^I is the narrowest interval vector containing the solution set X, and generally neither $\underline{x} \in X$ nor $\bar{x} \in X$ holds. The interval solution is a rather natural term, describing the exact range of each component of the solution of a system of linear equations $Ax = b$ if A, b vary over A^I, b^I.

So far methods for computing x^I have been developed only for special cases (Barth and Nuding [4], Beeck [8], Oettli [28]); most papers are concerned with obtaining a sufficiently close outer estimation of x^I (for a survey of results, see Neumaier [26], Deif [11], Alefeld and Herzberger [3]). Our approach is based on the following simple consequence of Theorem 2.2:

THEOREM 2.4. *Let A^I be regular. Then we have*

$$\underline{x} = \min\{x_y;\ y \in Y\},$$

$$\bar{x} = \max\{x_y;\ y \in Y\}. \tag{2.8}$$

Proof. Since a linear function attains its minimum over a convex polyhedron at some of its vertices, for each i, $1 \leqslant i \leqslant n$, we have $\underline{x}_i = \min\{x_i;\ x \in X\} = \min\{x_i;\ x \in \operatorname{Conv} X\} = \min\{(x_y)_i;\ y \in Y\}$. Similarly for $\bar{x}_i$. ∎

To be able to use Theorem 2.4 for practical computations, we must solve two problems: (1) how to compute the x_y's, (2) how to reduce the number of x_y's to be computed for determining $\underline{x}, \bar{x}$. These problems will be handled separately in the next two sections.

3. COMPUTATION OF THE x_y'S

In this section, we present a general finite method for computing the x_y's, examine its computational complexity, and then give an iterative method, whose convergence, however, is guaranteed only under an additional assumption.

In principle, a finite method for computing the x_y's is at hand, since the linear complementarity problem (2.4) with a P-matrix $A_{ye}^{-1}A_{yf}$ may be solved by any of the standard algorithms (Cottle and Dantzig [10], Lemke [21]). However, the necessity of inverting A_{ye} first makes this approach disadvantageous. We shall therefore use the idea of Murty's algorithm in [23] to solve the system (2.5) directly. Our "sign-accord algorithm" solves the systems $A_{yz}x = b_y$ for different z's until the condition $T_z x \geqslant 0$ (i.e. $z_j x_j \geqslant 0$ for each j) is met. As shown below, in many practical cases solving only one system $A_{yz}x = b_y$ is sufficient to find x_y. (Recall that $d_y = A_c^{-1}b_y$.)

ALGORITHM 3.1 (Computing x_y for a given $y \in Y$).

Step 0. Select a $z \in Y$ (recommended: $z = \operatorname{sgn} d_y$).
Step 1. Solve $A_{yz}x = b_y$.
Step 2. If $T_z x \geqslant 0$, set $x_y := x$ and terminate.
Step 3. Otherwise find

$$k = \min\{ j;\ z_j x_j < 0 \}.$$

Step 4. Set $z_k := -z_k$ and go to step 1.

THEOREM 3.1. *Let A^I be regular. Then the algorithm is finite for each $y \in Y$ and for an arbitrary starting $z \in Y$ in step* 0.

Proof. We shall prove the finiteness of the sequence of k's defined in step 3 of the algorithm by induction, showing that each k can occur there at most 2^{n-k} times ($k = n,\dots,1$).

Case $k = n$: Assume that n appears at least twice in the sequence, and let z, x, z', x' correspond to its two nearest occurrences. Then $z_j x_j \geqslant 0$, $z'_j x'_j \geqslant 0$ for $j = 1,\dots, n-1$, and $z_n z'_n = -1$, $z_n x_n < 0$, $z'_n x'_n < 0$; hence $z_j x_j z'_j x'_j \geqslant 0$ for each j, $j = 1,\dots, n$. But according to Theorem 1.1 ($x \neq x'$, since $x_n x'_n < 0$), there exists a j such that $z_j z'_j = -1$ and $x_j x'_j > 0$, implying $z_j x_j z'_j x'_j < 0$, a contradiction.

Case $k < n$: Again let z, x and z', x' correspond to two nearest occurrences of k, so that $z_j x_j z'_j x'_j \geqslant 0$ for $j = 1,\dots, k$. Then Theorem 1.1 implies the existence of a j with $z_j z'_j = -1$, $x_j x'_j > 0$; hence $z_j x_j z'_j x'_j < 0$, so that $j > k$. Hence between any two occurrences of k there is an occurrence of some $j > k$ in the sequence; this means that k cannot occur more than $1 + (2^{n-k-1} + \cdots + 2 + 1) = 2^{n-k}$ times. ∎

REMARK. The method employed here gives an alternative proof of the existence of a solution of (2.5), avoiding any use of results on linear complementarity problems or P-matrices. Also, the uniqueness of the solution of (2.5) may be proved in this way. In fact, if $A_{yz}x = b_y$, $T_z x \geqslant 0$, $A_{yz'}x' = b_y$, and $T_{z'}x' \geqslant 0$ for some $x \neq x'$, then Theorem 1.1 assures the existence of a j with $z_j z'_j = -1$ and $x_j x'_j > 0$, implying $z_j x_j z'_j x'_j < 0$, contrary to $z_j x_j \geqslant 0$, $z'_j x'_j \geqslant 0$.

The efficiency of the algorithm depends obviously on the number of systems $A_{yz}x = b_y$ to be solved during repeated returns to step 1 before arriving at x_y. The reason for the recommendation made in step 0 is explained in the next theorem.

THEOREM 3.2. *Let A^I be regular, and let*

$$|D_y|x_y|| < |x_y| \tag{3.1}$$

hold for some $y \in Y$. Then Algorithm 3.1, when started from $z = \operatorname{sgn} d_y$ in step 0, solves only one system of linear equations to find x_y.

COMMENT. If $\Delta \to 0$ and $\delta \to 0$, then the left-hand side in (3.1) tends to 0, while the right-hand side tends to $|x_c|$, where $x_c = A_c^{-1}b_c$. Hence if $|x_c| > 0$ and Δ, δ are sufficiently small, then (3.1) holds for each $y \in Y$.

Proof. x_y satisfies $A_{yz}x_y = b_y$, $T_z x_y \geqslant 0$ for some $z \in Y$. From the equivalent equation (2.6) we obtain $|x_y - d_y| = |D_y|x_y|| < |x_y|$, which implies that $(x_y)_j(d_y)_j > 0$ for each j, so that $z = \operatorname{sgn} x_y = \operatorname{sgn} d_y$. Hence when started from this z in step 0, after solving the system $A_{yz}x = b_y$ in step 1, the algorithm stops with $T_z x \geqslant 0$ in step 2. ■

The recommendation made in step 0 is an important part of the algorithm. According to our computational experience, termination after solving only one system of linear equations occurs in most examples from the literature.

In the examples, we follow the usual convention of writing a system $A^I x = b^I$ in the form

$$\sum_{j=1}^{n} \left[\underline{A}_{ij}, \overline{A}_{ij}\right] x_j = \left[\underline{b}_i, \overline{b}_i\right], \qquad i = 1, \ldots, n.$$

EXAMPLE 3.1 (Nickel [27]).

$$\begin{aligned} [2,4]x_1 + [-2,-1]x_2 &= [8,10], \\ [2,5]x_1 \qquad + [4,5]x_2 &= [5,40]. \end{aligned}$$

Here for each $y \in Y$, only one system must be solved when starting from the recommended value $z = \operatorname{sgn} d_y$, as depicted in this tableau (results rounded to five decimals):

y_1	y_2	z_1	z_2	$(x_y)_1$	$(x_y)_2$
1	1	1	1	10	5
-1	1	1	1	4	8
1	-1	1	-1	3.46154	-3.07692
-1	-1	1	-1	1.61538	-0.76923

Hence for the interval solution $x^I = [\underline{x}, \bar{x}]$ we obtain, according to (2.8), $\underline{x} = (1.61538, -3.07692)^T$, $\bar{x} = (10,8)^T$.

EXAMPLE 3.2. Consider the system of linear interval equations

$$[1,1000]x_1 + [1,1000]x_2 = [1,2],$$

$$[-1000, -1]x_1 + [1,1000]x_2 = [3,4].$$

Although the left-hand-side intervals are very wide, the algorithm still preserves the one-system termination property for $y \in \{(1,1)^T, (-1,1)^T, (-1,-1)^T\}$. For $y = (1,-1)^T$, with starting $z = \operatorname{sgn} d_y = (-1,1)^T$, it gives $x = (0.001995, 0.004995)^T$, where $z_1 x_1 < 0$. Setting $z_1 := -z_1$ and returning to step 1, $x_y = (1.995005, 0.004995)^T$ is obtained, rounded to six decimals.

Theorem 3.1 implies that no z can reappear in the course of the algorithm (otherwise it would cycle infinitely); hence at most 2^n systems must be solved for each $y \in Y$. We shall show that this upper estimate can really be attained for each $n \geqslant 2$ when the algorithm is improperly initialized in step 0. For this purpose, consider for $n \geqslant 2$ the system of linear interval equations (a modification of Murty's example in [24])

$$[E - \Delta_n, E + \Delta_n]x = [f, e], \tag{3.2}$$

where E is the $n \times n$ unit matrix and

$$(\Delta_n)_{ij} = \begin{cases} 2 & \text{if} \quad j = i+1, \quad 1 \leqslant i \leqslant n-1 \\ 0 & \text{otherwise} \end{cases} \qquad (i, j = 1, \ldots, n).$$

Then, we have:

THEOREM 3.3. *Let $n \geqslant 2$. Then for each $y \in Y$, the number $p_y(z)$ of systems Algorithm* 3.1 *must solve to find x_y for* (3.2) *when started from a*

$z \in Y$ in step 0 is given by

$$p_y(z) = 1 + \sum_{j=1}^{n} \left(1 - \prod_{i=j}^{n} y_i z_i\right) 2^{j-2}.$$

Proof. First we find by back substitutions that for each $y, z \in Y$, the solution x of the system $A_{yz}x = b_y$ satisfies

$$x_j = y_j \sum_{m=0}^{n-j} 2^m \prod_{i=j+1}^{j+m} y_i z_i \qquad (j = 1, \dots, n).$$

Hence

$$z_j x_j = \sum_{m=0}^{n-j} 2^m \prod_{i=j}^{j+m} y_i z_i,$$

and since the last term prevails, we have

$$\operatorname{sgn}(z_j x_j) = \operatorname{sgn}\left(\prod_{i=j}^{n} y_i z_i\right) = \prod_{i=j}^{n} y_i z_i \qquad (j = 1, \dots, n).$$

We shall carry out the proof of the formula for $p_y(z)$ by induction on $p_y(z)$. If $p_y(z) = 1$, then Algorithm 3.1, after solving $A_{yz}x = b_y$, terminates in step 2 with $T_z x \geqslant 0$. Hence for each j we have

$$\prod_{i=j}^{n} y_i z_i = \operatorname{sgn}(z_j x_j) = 1,$$

so that the formula holds. Now assume that the formula is valid for all $y, z \in Y$ with $p_y(z) \leqslant s$, $s \geqslant 1$, and let $p_y(z) = s + 1$ for some $y, z \in Y$. Let z' be the updated value of z after passing for the first time through step 4 of the algorithm. Then $z'_k = -z_k$, $z'_j = z_j$ for $j \neq k$,

$$\prod_{i=j}^{n} y_i z'_i = -\prod_{i=j}^{n} y_i z_i = -\operatorname{sgn}(z_j x_j) = -1 \qquad \text{for} \quad j < k,$$

$$\prod_{i=j}^{n} y_i z'_i = -\operatorname{sgn}(z_k x_k) = 1 \qquad \text{for} \quad j = k,$$

and

$$\prod_{i=j}^{n} y_i z_i' = \prod_{i=j}^{n} y_i z_i \qquad \text{for} \quad j > k;$$

hence by the inductive assumption,

$$p_y(z) = 1 + p_y(z') = 2 + \sum_{j=1}^{n} \left(1 - \prod_{i=j}^{n} y_i z_i'\right) 2^{j-2}$$

$$= 1 + \sum_{j=1}^{n} \left(1 - \prod_{i=j}^{n} y_i z_i\right) 2^{j-2},$$

which completes the proof. ■

COROLLARY 3.1. *For the system of linear interval equations* (3.2) *with* $n \geqslant 2$ *we have*:

(i) *for each* $y \in Y$, *if* $z = (y_1, \ldots, y_{n-1}, -y_n)^T$, *then* $p_y(z) = 2^n$;
(ii) *for each* $y \in Y$, *if* $z = \operatorname{sgn} d_y$, *then* $p_y(z) = 1$;
(iii) *for each* $y \in Y$ *and each* $s \in \{1, \ldots, 2^n\}$ *there exists a* $z \in Y$ *such that* $p_y(z) = s$;
(iv) *for each* $z \in Y$ *and each* $s \in \{1, \ldots, 2^n\}$ *there exists a* $y \in Y$ *such that* $p_y(z) = s$.

Proof. (i): If $z = (y_1, \ldots, y_{n-1}, -y_n)^T$, then $\prod_{i=j}^{n} y_i z_i = -1$ for each j, and $p_y(z) = 1 + \sum_{j=1}^{n} 2^{j-1} = 2^n$.

(ii): If $z = \operatorname{sgn} d_y = y$, then $\prod_{i=j}^{n} y_i z_i = 1$ for each j, and $p_y(z) = 1$.

(iii) + (iv): For each $s \in \{1, \ldots, 2^n\}$ there exist $\beta_j \in \{0, 1\}$, $j = 1, \ldots, n$, such that $s = 1 + \sum_{j=1}^{n} \beta_j 2^{j-1}$. Hence if y is given and if we define z inductively from $\prod_{i=j}^{n} y_i z_i = 1 - 2\beta_j$ ($j = n, n-1, \ldots, 1$), then $p_y(z) = s$; similarly if z is given. ■

EXAMPLE 3.3. Let us demonstrate the behavior of the algorithm for (3.2) with $n = 3$, $y = (1, 1, -1)^T$, and starting $z = (1, 1, 1)^T$ [case (i) above]. Algorithm 3.1 produces this sequence of z's and x's (notice that each k appears

2^{n-k} times, as estimated in the proof of Theorem 3.1):

k	z_1	z_2	z_3	x_1	x_2	x_3
—	1	1	1	−1	−1	−1
1	−1	1	1	−1	−1	−1
2	−1	−1	1	3	−1	−1
1	1	−1	1	3	−1	−1
3	1	−1	−1	−5	3	−1
1	−1	−1	−1	−5	3	−1
2	−1	1	−1	7	3	−1
1	1	1	−1	7	3	−1

We shall now turn to a class of interval matrices $A^I = [A_c - \Delta, A_c + \Delta]$ whose radius Δ is of the form

$$\Delta = qr^T \tag{3.3}$$

for some nonnegative vectors $q, r \in R^n$ (i.e. $\Delta_{ij} = q_i r_j$ for each i, j). These interval matrices were introduced by Hansen [16]. We shall call them matrices with rank-one radius, since, with the exception of the trivial case $\Delta = 0$, the radius Δ satisfying (3.3) is of rank one. Interval matrices from this class appear often in the literature (e.g. Hansen [15], Albrecht [1], Alefeld and Herzberger [2], Jahn [19]). We shall show that for interval matrices with rank-one radius the sign-accord Algorithm 3.1 takes on an attractively simple form as a consequence of the following Theorem 3.4. We use an additional symbol

$$g_y = A_c^{-1} T_y q \qquad (y \in Y).$$

THEOREM 3.4. *Let A^I be a regular interval matrix satisfying* (3.3). *Then for each $y, z \in Y$, the solution to $A_{yz}x = b_y$ is given by*

$$x = \alpha_{yz} g_y + d_y, \tag{3.4}$$

where

$$\alpha_{yz} = \frac{r^T T_z d_y}{1 - r^T T_z g_y}. \tag{3.5}$$

Proof. Let x solve $A_{yz}x = b_y$. Put $\alpha_{yz} = r^T T_z x$; then from $A_{yz}x = (A_c - T_y q r^T T_z)x = A_c x - \alpha_{yz} T_y q = b_y$ we obtain (3.4). Then premultiplying (3.4) by $r^T T_z$ yields (3.5). ■

REMARK. It will be proved in Section 5 that if a regular interval matrix satisfies (3.3), then $r^T T_z g_y < 1$ for each $y, z \in Y$.

In view of this result, we may replace step 1 of Algorithm 3.1 by the formulae (3.4), (3.5). Even more, setting $g := T_z g_y$, $d := T_z d_y$, we obtain this simplified description:

ALGORITHM 3.2 (Computing x_y for matrices satisfying (3.3)).

Step 0. Set $d := |d_y|$ and $g := T_z g_y$, where $z = \operatorname{sgn} d_y$.
Step 1. Compute

$$\alpha = \frac{r^T d}{1 - r^T g}.$$

Step 2. If $\alpha g + d \geqslant 0$, set $x_y := \alpha g_y + d_y$ and terminate.
Step 3. Otherwise find

$$k = \min\{ j;\ (\alpha g + d)_j < 0 \}.$$

Step 4. Set $g_k := -g_k$, $d_k := -d_k$, and go to step 1.

An example will be given in Section 4.

So far we have dealt only with finite methods for computing x_y. To guarantee convergence of iterative methods we are about to describe, we shall assume the nonnegative matrix

$$D = |A_c^{-1}|\Delta$$

to satisfy

$$\rho(D) < 1 \tag{3.6}$$

(ρ denotes the spectral radius). This condition is important also in another respects. As proved by Beeck [8], (3.6) implies regularity of A^I; according to our experience, (3.6) should be recommended as the first trial when testing regularity. In our previous Examples 3.1 to 3.3, the values of $\rho(D)$ were

0.544, 1.996, and 0, respectively. This verifies *ex post* the regularity of A^I in Examples 3.1 and 3.3, but A^I in Example 3.2 is also regular, since $\det A \geqslant 2$ for each $A \in A^I$ there; this simultaneously shows that (3.6) is not a necessary regularity condition.

For iterative methods for computing x_y we may employ the equivalent fixed-point equation (2.6):

$$x = D_y|x| + d_y$$

(where $D_y = A_c^{-1}T_y\Delta$), which may be solved by Jacobi iterations

$$\begin{aligned} x_y^0 &= d_y, \\ x_y^{i+1} &= D_y|x_y^i| + d_y \qquad (i = 0, 1, \ldots) \end{aligned} \tag{3.7}$$

(under our assumption, convergence will be assured for an arbitrary x_y^0, but the choice made seems to be generally the best). Further, using the decomposition $D_y = L_y + Q_y$, where L_y is a lower triangular matrix with zero diagonal coefficients and Q_y is an upper triangular matrix, we may construct the Gauss-Seidel iterations

$$\begin{aligned} \tilde{x}_y^0 &= d_y, \\ \tilde{x}_y^{i+1} &= L_y|\tilde{x}_y^{i+1}| + Q_y|\tilde{x}_y^i| + d_y \qquad (i = 0, 1, \ldots). \end{aligned} \tag{3.8}$$

For the convergence proof for both (3.7) and (3.8) under (3.6), we introduce additional notation. If (3.6) holds, then $(E - D)^{-1} \geqslant 0$; hence the matrix

$$C = D(E - D)^{-1} = \sum_{j=1}^{\infty} D^j$$

is nonnegative. Let $D = L + Q$ be an analogous decomposition of D into triangular matrices. Then $\tilde{D} = (E - L)^{-1}Q$ is again nonnegative and satisfies $\rho(\tilde{D}) \leqslant \rho(D)$. [Proof: Denote $\mu = \rho(\tilde{D})$. The case of $\mu = 0$ is trivial; otherwise $\tilde{D}x = \mu x$ for some $x \neq 0$ [42], implying $Ax = \mu x$ for $A = \mu L + Q \geqslant 0$. If $\mu \geqslant 1$, then $(1/\mu)A \leqslant D$; hence $1 \leqslant \rho((1/\mu)A) \leqslant \rho(D)$, contrary to (3.6). Thus $\mu < 1$, so that $A \leqslant D$, giving $\mu \leqslant \rho(A) \leqslant \rho(D)$.]

We again introduce

$$\tilde{C} = \tilde{D}(E - \tilde{D})^{-1} = \sum_{j=1}^{\infty} \tilde{D}^j.$$

THEOREM 3.5. *Let (3.6) hold. Then for each $y \in Y$, both the sequences $\{x_y^i\}_{i=0}^{\infty}$, $\{\tilde{x}_y^i\}_{i=0}^{\infty}$ given by (3.7), (3.8) tend to x_y, and for each $i \geqslant 1$ we have*

$$|x_y - x_y^i| \leqslant C|x_y^i - x_y^{i-1}| \leqslant CD^i|d_y|, \tag{3.9}$$

$$|x_y - \tilde{x}_y^i| \leqslant \tilde{C}|\tilde{x}_y^i - \tilde{x}_y^{i-1}| \leqslant \tilde{C}\tilde{D}^i|d_y|. \tag{3.10}$$

Proof. Let $y \in Y$ and $i \geqslant 1$. Then, since $|D_y| \leqslant D$, from (3.7) we obtain

$$|x_y^{i+1} - x_y^i| \leqslant D|x_y^i - x_y^{i-1}|, \tag{3.11}$$

and by induction, $|x_y^{i+1} - x_y^i| \leqslant D^i|x_y^1 - x_y^0| \leqslant D^{i+1}|d_y|$. Next, for each $m \geqslant 1$ we have $|x_y^{i+m} - x_y^i| \leqslant |x_y^{i+m} - x_y^{i+m-1}| + \cdots + |x_y^{i+1} - x_y^i| \leqslant (D^m + \cdots + D)|x_y^i - x_y^{i-1}| \leqslant C|x_y^i - x_y^{i-1}| \leqslant CD^i|d_y|$. Since $D^i \to 0$ as $i \to \infty$ due to (3.6), this shows that $\{x_y^i\}_{i=0}^{\infty}$ is a Cauchy sequence; hence from (3.7) it follows that $x_y^i \to x_y$. Taking $m \to \infty$ in the above-proved inequality, we obtain (3.9).

To prove an analogous result for the Gauss-Seidel sequence, we first obtain from (3.8) that

$$|\tilde{x}_y^{i+1} - \tilde{x}_y^i| \leqslant L|\tilde{x}_y^{i+1} - \tilde{x}_y^i| + Q|\tilde{x}_y^i - \tilde{x}_y^{i-1}|$$

(since $|L_y| \leqslant L$, $|Q_y| \leqslant Q$), which, in view of the nonnegative invertibility of $E - L$, gives

$$|\tilde{x}_y^{i+1} - \tilde{x}_y^i| \leqslant \tilde{D}|\tilde{x}_y^i - \tilde{x}_y^{i-1}|.$$

In this way we have obtained an analogue of (3.11), and the rest of the proof follows the same line, with x_y^i, D, C being replaced by $\tilde{x}_y^i, \tilde{D}, \tilde{C}$, respectively. ∎

Using the vector and matrix norm $\|x\|_\infty = \max_j |x_j|$, $\|A\|_\infty = \max_i \Sigma_j |A_{ij}|$, from (3.9) we have $\|x_y - x_y^i\|_\infty \leqslant \|C\|_\infty \|x_y^i - x_y^{i-1}\|_\infty$. Hence

$$\|x_y^i - x_y^{i-1}\|_\infty \leqslant \frac{\varepsilon}{\|C\|_\infty}$$

is an appropriate stopping rule (the case $C = 0$ is trivial, since then $x_y = d_y$ for each $y \in Y$); similarly for $\{\tilde{x}_y^i\}_{i=0}^{\infty}$. Since $\rho(\tilde{D}) \leqslant \rho(D)$, the Gauss-Seidel sequence (3.8) is generally to be preferred. The sequences $\{x_y^i\}_{i=0}^{\infty}$, $\{\tilde{x}_y^i\}_{i=0}^{\infty}$

TABLE 1

i	$(\tilde{x}_y^i)_1$	$(\tilde{x}_y^i)_2$
0	8.3413	4.9178
1	9.3897	4.9686
2	9.7753	4.9884
3	9.9173	4.9957
4	9.9695	4.9984
5	9.9888	4.9994
6	9.9959	4.9998
7	9.9985	4.9999
8	9.9994	5.0000
9	9.9998	5.0000
10	9.9999	5.0000
11	10.0000	5.0000

sometimes converge even if the spectral condition (3.6) is violated, but the convergence is very slow, and using iterative methods cannot be recommended in this case.

EXAMPLE 3.4. In Nickel's example [Example 3.1, where $\rho(D) = 0.544$], for $y = (1,1)^T$ with stopping rule $\|\tilde{x}_y^i - \tilde{x}_y^{i-1}\|_\infty < 10^{-4}$, we obtain the sequence $\{\tilde{x}_y^i\}$ in Table 1 (rounded to four decimals). For $y = (-1,1)^T$, $y = (1,-1)^T$, $y = (-1,-1)^T$ we need 6, 7, and 10 iterations, respectively, under the same stopping rule.

EXAMPLE 3.5. For the extremely disadvantageous Example 3.2 [$\rho(D) = 1.996$], 1040 iterations are necessary for $y = (1,1)^T$ under the stopping rule $\|x_y^i - x_y^{i-1}\|_\infty < 10^{-3}$ to obtain the approximation $(0.001998, 3.49926)^T$, while the exact result is $x_y = (0.001998, 3.998)^T$.

4. INTERVAL SOLUTION

In Section 2 we derived formulae (2.8) for computing the interval solution $x^I = [\underline{x}, \bar{x}]$ of a system of linear interval equations $A^I x = b^I$:

$$\underline{x} = \min\{x_y;\ y \in Y\},$$

$$\bar{x} = \max\{x_y;\ y \in Y\}.$$

In the present section we shall show that in many cases there is no need for computing all the 2^n vectors x_y. Theorem 4.2 below forms a basis for a reduction of the set Y in the above formulae to some possibly small subset Y_0. We shall prove it as a consequence of a more general result concerning the optimization problem (with $c^I = [\underline{c}, \bar{c}]$)

$$\max\{ c^T x;\ Ax = b,\ A \in A^I,\ b \in b^I,\ c \in c^I \} \tag{4.1}$$

and its "dual" problem

$$\max\{ b^T p;\ A^T p = c,\ A \in A^I,\ b \in b^I,\ c \in c^I \}. \tag{4.2}$$

Like the vectors x_y $(y \in Y)$ for the system $A^I x = b^I$, there exist vectors p_z $(z \in Y)$ for the system $A^{IT} p = c^I$ appearing in (4.2). We have this "duality theorem":

THEOREM 4.1. *Let A^I be regular. Then the optimization problems* (4.1), (4.2) *have a pair of optimal solutions x_y, p_z, $y, z \in Y$, satisfying $T_z x_y \geqslant 0$, $T_y p_z \geqslant 0$, and the optimal values of both the problems are equal.*

Proof. (a) Assume first that $\delta > 0$. Then (4.1) has an optimal solution x_y for some $y \in Y$. Set $z = \operatorname{sgn} x_y$; then, since $c^T x_y \leqslant c_z^T x_y$ for each $c \in c^I$, $c_z^T x_y$ is the optimal value of (4.1). Let p solve $A_{yz}^T p = c_z$. Assume to the contrary that $T_y p \geqslant 0$ does not hold, so that $y_k p_k < 0$ for some k. Define $y' \in Y$ by $y'_k = -y_k$, $y'_j = y_j$ for $j \neq k$, and set $x' = A_{yz}^{-1} b_{y'}$. Then $c_z^T x_y = p^T b_y < p^T b_{y'} = c_z^T x' \leqslant c_z^T x_y$, a contradiction. Hence $T_y p \geqslant 0$ and $A_{yz}^T p = c_z$, which means that p is equal to p_z for the system $A^{IT} p = c^I$; moreover, $c_z^T x_y = b_y^T p_z$. Assume $b_y^T p_z$ is not the optimal value of (4.2), so that $b_y^T p_z < b^T p$ for some $b \in b^I$ and $p = (A^T)^{-1} c$, $A \in A^I$, $c \in c^I$. Then for $x = A^{-1} b$ we have $c^T x = b^T p > b_y^T p_z = c_z^T x_y$, contrary to the optimality of x_y. Hence p_z is an optimal solution of (4.2), and we have $T_z x_y \geqslant 0$, $T_y p_z \geqslant 0$, and $c_z^T x_y = b_y^T p_z$.

(b) Now let $\delta_j = 0$ for some $j \in \{1, \ldots, n\}$. For each $i = 1, 2, \ldots,$ put $\delta^i = \delta + (1/i)e$, $b_i^I = [b_c - \delta^i, b_c + \delta^i]$, and consider the pair

$$\max\left\{ c^T x;\ Ax = b,\ A \in A^I,\ b \in b_i^I,\ c \in c^I \right\}, \tag{4.1$_i$}$$

$$\max\left\{ b^T p;\ A^T p = c,\ A \in A^I,\ b \in b_i^I,\ c \in c^I \right\}. \tag{4.2$_i$}$$

The first part of the proof assures the existence of $y_i, z_i \in Y$ and of optimal solutions $x^i_{y_i}, p^i_{z_i}$ to (4.1$_i$), (4.2$_i$) such that $T_{z_i} x^i_{y_i} \geqslant 0$, $T_{y_i} p^i_{z_i} \geqslant 0$, $c^T_{z_i} x^i_{y_i} = b^T_{y_i} p^i_{z_i}$ $(i = 1, 2, \ldots)$. Since Y is finite, there exist $y, z \in Y$ such that $y_{i_j} = y$, $z_{i_j} = z$ for infinitely many i_j. Letting $i_j \to \infty$ and using compactness of solution sets, we obtain the desired result. ∎

Applying this result to the problem of evaluating $\underline{x}_i$ $(\bar{x}_i)$, we get these conditions (e_i is the ith column of the unit matrix E):

Theorem 4.2. *Let A^I be regular. Then for each i, $1 \leqslant i \leqslant n$, we have:*

(i) *there exist $y, z \in Y$ such that $\underline{x}_i = (x_y)_i$, $T_z x_y \geqslant 0$, and $T_y(A^{-1}_{yz})^T e_i \leqslant 0$,*
(ii) *there exist $y, z \in Y$ such that $\bar{x}_i = (x_y)_i$, $T_z x_y \geqslant 0$, and $T_y(A^{-1}_{yz})^T e_i \geqslant 0$.*

Proof. Since $\bar{x}_i$ is the optimal value of (4.1) for $c^I = [e_i, e_i]$, Theorem 4.1 implies the existence of $y, z \in Y$ such that $\bar{x}_i = (x_y)_i$, $T_z x_y \geqslant 0$, and $T_y p_z = T_y(A^{-1}_{yz})^T e_i \geqslant 0$, which is (ii). Assertion (i) is obtained if we apply (ii) to the system $A^I x = -b^I = [-b_c - \delta, -b_c + \delta]$ whose interval solution $[\underset{\sim}{x}, \tilde{x}]$ satisfies $\tilde{x} = -\underline{x}$. ∎

The condition $T_y(A^{-1}_{yz})^T e_i \leqslant 0$ may be equivalently written as

$$\left(A^{-1}_{yz}\right)_{ij} y_j \leqslant 0 \qquad \text{for} \quad j = 1, \ldots, n; \tag{4.3}$$

the condition $T_y(A^{-1}_{yz})^T e_i \geqslant 0$, analogously, as

$$\left(A^{-1}_{yz}\right)_{ij} y_j \geqslant 0 \qquad \text{for} \quad j = 1, \ldots, n. \tag{4.4}$$

These conditions are, however, generally not sufficient for $\underline{x}_i$ $(\bar{x}_i)$ to be achieved at x_y:

Example 4.1. In the well-known example by Barth and Nuding [4]

$$[2,4]x_1 + [-2,1]x_2 = [-2,2],$$

$$[-1,2]x_1 + [2,4]x_2 = [-2,2],$$

(4.4) is satisfied for $i = 2$ by both $y = z = (1,1)^T$ and $y = z = (-1,1)^T$, but $(x_{(1,1)})_2 = 3 < 4 = (x_{(-1,1)})_2 = \bar{x}_2$.

The formulae (4.3), (4.4) show that y_j can be determined if $(A^{-1})_{ij}$ preserves its signature over A^I. Returning to the problem formulated at the beginning of this section, assume we know an interval matrix $[\underset{\sim}{B}, \tilde{B}]$ such that

$$A^{-1} \in [\underset{\sim}{B}, \tilde{B}] \qquad \text{for each} \quad A \in A^I. \tag{4.5}$$

If the spectral condition (3.6) is satisfied, then with $C = D(E-D)^{-1}$ as above, we may simply put

$$\underset{\sim}{B} = A_c^{-1} - C|A_c^{-1}|,$$

$$\tilde{B} = A_c^{-1} + C|A_c^{-1}|$$

(to be proved later). For each $i \in \{1,\dots,n\}$ define

$$Y_i = \left\{y;\ y \in Y,\ y_j = -1 \text{ if } \tilde{B}_{ij} < 0,\ y_j = 1 \text{ if } \underset{\sim}{B}_{ij} > 0\ (j = 1,\dots,n)\right\}.$$

Denoting $-Y_i = \{y;\ -y \in Y_i\}$, we have:

THEOREM 4.3. *Let A^I be regular and let $[\underset{\sim}{B}, \tilde{B}]$ satisfy* (4.5). *Then for each i, $1 \leqslant i \leqslant n$,*

$$\underline{x}_i = (x_y)_i \qquad \textit{for some} \quad y \in -Y_i,$$

$$\bar{x}_i = (x_y)_i \qquad \textit{for some} \quad y \in Y_i.$$

Proof. According to Theorem 4.2, $\bar{x}_i = (x_y)_i$ for some $y, z \in Y$ satisfying (4.4). If $\tilde{B}_{ij} < 0$, then $(A_{yz}^{-1})_{ij} < 0$ and from (4.4) we obtain $y_j = -1$; if $\underset{\sim}{B}_{ij} > 0$, then the same reasoning gives $y_j = 1$, hence $y \in Y_i$. Analogous reasoning holds for $\underline{x}_i$. ■

Introducing $Y_0 = \cup_{i=1}^{n}[Y_i \cup (-Y_i)]$, we may rewrite the assertions of Theorem 4.3 as

$$\underline{x} = \min\{x_y;\ y \in Y_0\},$$

$$\bar{x} = \max\{x_y;\ y \in Y_0\}; \tag{4.6}$$

hence in (2.8), Y has been replaced by Y_0. These formulae, combined with methods for computing the x_y's in Section 3, are recommended for practical computations, where $\rho(D)$ is usually small.

We shall now focus our attention on cases where y satisfying $\underline{x}_i = (x_y)_i$ can be given explicitly. To this end, we introduce this definition: a regular interval matrix A^I is called inverse stable if for each $i, j \in \{1,\dots,n\}$, either $(A^{-1})_{ij} \leqslant 0$ for each $A \in A^I$, or $(A^{-1})_{ij} \geqslant 0$ for each $A \in A^I$. We have this verifiable criterion of inverse stability:

THEOREM 4.4. *Let A^I satisfy* (3.6), *and let*

$$C|A_c^{-1}| \leqslant |A_c^{-1}| \tag{4.7}$$

(*with $C = D(E-D)^{-1}$*). *Then A^I is inverse stable.*

COMMENT. Since $C \to 0$ as $\Delta \to 0$, one may again argue that (4.7) holds if $|A_c^{-1}| > 0$ and Δ is sufficiently small.

Proof. Let $A \in A^I$. Since $A = A_c[E - A_c^{-1}(A_c - A)]$, in the light of (3.6) we have $|A^{-1} - A_c^{-1}| \leqslant (\Sigma_{j=1}^{\infty} D^j)|A_c^{-1}| = C|A_c^{-1}| \leqslant |A_c^{-1}|$. Hence, if $(A_c^{-1})_{ij} \geqslant 0$, then $(A^{-1})_{ij} \geqslant 0$, and similarly $(A_c^{-1})_{ij} \leqslant 0$ implies $(A^{-1})_{ij} \leqslant 0$, showing that A^I is inverse stable. ∎

REMARK. From the inequality $|A^{-1} - A_c^{-1}| \leqslant C|A_c^{-1}|$ it follows that (4.5) holds for $\underset{\sim}{B} = A_c^{-1} - C|A_c^{-1}|$, $\tilde{B} = A_c^{-1} + C|A_c^{-1}|$, as stated above.

For an inverse stable interval matrix A^I we can define vectors $y(i) \in Y$, $i = 1,\dots,n$, by

$$(y(i))_j = \begin{cases} 1 & \text{if } (A^{-1})_{ij} \geqslant 0 \text{ for each } A \in A^I \\ -1 & \text{otherwise} \end{cases} \qquad (j = 1,\dots,n).$$

We have this result:

THEOREM 4.5. *Let A^I be inverse stable. Then for each i, $1 \leqslant i \leqslant n$,*

$$\underline{x}_i = (x_{-y(i)})_i,$$

$$\bar{x}_i = (x_{y(i)})_i.$$

Proof. Let $i \in \{1,\dots,n\}$. To simplify notation, set $y := y(i)$. Let $x \in X$, so that $Ax = b$ for some $A \in A^I$, $b \in b^I$. First we prove that the vector

$$h = T_y A(x_y - x)$$

is nonnegative. In fact, since $-T_y(A-A_c)x_y \leqslant |T_y(A-A_c)x_y| \leqslant \Delta|x_y|$ and $-T_y(b_c-b) \leqslant |T_y(b_c-b)| \leqslant \delta$, we get $h=T_y[Ax_y-b-(A_cx_y-b_c)+T_y(\Delta|x_y|+\delta)]=T_y(A-A_c)x_y+\Delta|x_y|+T_y(b_c-b)+\delta \geqslant 0$. Hence in view of the definition of $y(i)$ we obtain $(x_y-x)_i=(A^{-1}T_yh)_i=\Sigma_j(A^{-1})_{ij}y_jh_j \geqslant 0$; hence $(x_y)_i \geqslant x_i$. Since x was independent of X, we conclude that $(x_y)_i=(x_{y(i)})_i=\bar{x}_i$. The proof for $\underline{x}_i$ is analogous. ■

Thus in the case of inverse stability, instead of 2^n only at most $2n$ vectors x_y need be computed [for $y \in \{y(1),\ldots,y(n), -y(1),\ldots, -y(n)\}$]. If it is known beforehand that the whole of solution set X lies in a single orthant, say $\operatorname{sgn} x = z$ for each $x \in X$, then $x_{y(i)}$ is the solution to $A_{y(i)z}x=b_{y(i)}$, and similarly $x_{-y(i)}$ solves $A_{-y(i)z}x=b_{-y(i)}$. These explicit systems of linear equations were given already by Hansen [15].

EXAMPLE 4.2. This example, due to Albrecht [1], has since been studied by Oettli [28], Hansen [15], and Cope and Rust [9]. Here $A_cx=b_c$ has the form

$$
\begin{aligned}
4.33\,x_1-1.12\,x_2-1.08\,x_3+1.14\,x_4&=3.52,\\
-1.12\,x_1+4.33\,x_2+0.24\,x_3-1.22\,x_4&=1.57,\\
-1.08\,x_1+0.24\,x_2+7.21\,x_3-3.22\,x_4&=0.54,\\
1.14\,x_1-1.22\,x_2-3.22\,x_3+5.43\,x_4&=-1.09,
\end{aligned}
$$

and $\Delta_{ij}=\delta_i=0.005$ for $i,j=1,\ldots,4$. Since $\rho(D)=0.008$ and (4.7) holds, A^I is inverse stable, and from the sign pattern of A_c^{-1} we deduce $y(1)=(1,1,1,-1)^T$, $y(2)=y(3)=(1,1,1,1)^T$, $y(4)=(-1,1,1,1)^T$. Since A^I is of rank-one radius, we may use Algorithm 3.2, which in each of the six cases requires computing α only once to determine x_y, as summed up in this tableau (rounded to five decimals):

y_1	y_2	y_3	y_4	$(x_y)_1$	$(x_y)_2$	$(x_y)_3$	$(x_y)_4$
1	1	1	1	1.05092	0.56888	0.11636	−0.22183
−1	1	1	1	1.04327	0.56715	0.11560	−0.22107
−1	−1	−1	1	1.04083	0.55860	0.10908	−0.22636
1	1	1	−1	1.05171	0.56701	0.11294	−0.22990
1	−1	−1	−1	1.04923	0.55842	0.10640	−0.23517
−1	−1	−1	−1	1.04161	0.55672	0.10568	−0.23437

This gives the interval solution

$$\underline{x} = (1.04083, 0.55672, 0.10568, -0.23517)^T,$$

$$\bar{x} = (1.05171, 0.56888, 0.11636, -0.22107)^T.$$

Using alternatively the iterative method (3.7), in all the cases 5-digit accuracy was achieved already at x_y^2. Oettli [28] had to solve eight linear programming problems to find $\underline{x}$, $\bar{x}$.

For an even greater decrease in the number of x_y's to be computed, consider regular interval matrices satisfying

$$T_z A^{-1} T_y \geqslant 0 \qquad \text{for each} \quad A \in A^I \tag{4.8}$$

for some (fixed) $y, z \in Y$. The criterion below shows that only two matrices need be tested to verify (4.8):

THEOREM 4.6. *Let $y, z \in Y$. Then a regular interval matrix A^I satisfies* (4.8) *if and only if*

$$T_z A_{yz}^{-1} T_y \geqslant 0,$$

$$T_z A_{-y,z}^{-1} T_y \geqslant 0.$$

Proof. A^I satisfies (4.8) iff $(T_y A T_z)^{-1} \geqslant 0$ for each $A \in A^I$, i.e. iff the interval matrix $\{T_y A T_z;\ A \in A^I\} = [T_y A_{yz} T_z, T_y A_{-y,z} T_z]$ is inverse nonnegative, which, according to Kuttler's theorem in [20], re-proved in [41], is equivalent to $(T_y A_{yz} T_z)^{-1} \geqslant 0$, $(T_y A_{-y,z} T_z)^{-1} \geqslant 0$. ∎

For interval matrices satisfying (4.8) only two vectors need be computed:

THEOREM 4.7. *Let A^I be a regular interval matrix satisfying* (4.8). *Then*

$$\underline{x} = \min\{x_{-y}, x_y\},$$

$$\bar{x} = \max\{x_{-y}, x_y\}.$$

In particular, if $z = e$, then $\underline{x} = x_{-y}$, $\bar{x} = x_y$.

Proof. Follows from Theorem 4.5, since A^I is inverse stable with $y(i) = z_i y$ for each $i \in \{1,\dots,n\}$. ■

This result was proved, without using the x_y's, by Beeck [8] for $y = z = e$, and by Garloff [14] for $y = z = (1, -1,\dots,(-1)^{n-1})^T$. Under additional assumptions on b^I, x_{-y} and x_y may be expressed explicitly [7, 41].

EXAMPLE 4.3 (Theoretical). For $n = 2$, consider a linear interval system $A^I x = b^I$ with A^I regular and $\underline{A} \geqslant 0$. Then A^I satisfies (4.8) with $y = z = (1, -1)^T$ if $\det A_c > 0$, and with $y = (1, -1)^T$, $z = -y$ if $\det A_c < 0$. In both cases

$$\underline{x} = \min\{x_{(1,-1)}, x_{(-1,1)}\},$$

$$\bar{x} = \max\{x_{(1,-1)}, x_{(-1,1)}\}.$$

Cf. e.g. Hansen's example in [16].

5. REGULARITY

This section is devoted to the problem of regularity of square interval matrices. First we give some necessary and sufficient regularity conditions; then we describe a sequence of tests for verifying regularity (singularity) in practical examples.

Necessary and sufficient regularity conditions are summed up in the following theorem. In addition to the notation already introduced, we denote $D_{yz} = A_c^{-1} T_y \Delta T_z$, $y, z \in Y$, and $\rho_0(A) = \max\{|\lambda|;\ \lambda$ is a real eigenvalue of $A\}$; we set $\rho_0(A) = 0$ if no real eigenvalue exists. To simplify formulations, inverse matrices are always assumed to exist when spoken of.

THEOREM 5.1. *Let A^I be an $n \times n$ interval matrix. Then the following conditions are mutually equivalent:*

(R) *A^I is regular,*
(A1) *$A_{yz} x = y$, $T_z x \geqslant 0$ has a (unique) solution for each $y \in Y$,*
(A2) *$A_{ye} x_1 - A_{yf} x_2 = y$, $x_1 \geqslant 0$, $x_2 \geqslant 0$ has a solution for each $y \in Y$,*
(A3) *$B = D_y |B| + A_c^{-1}$ has a (unique) matrix solution for each $y \in Y$,*
(B1) *$A_{ye}^{-1} A_{yf}$ is a P-matrix for each $y \in Y$,*

(B2) $A_{ye}^{-1}A_{yf}x > 0,\ x > 0$ *has a solution for each* $y \in Y$,

(B3) $A_{ye}^{-1}x > 0,\ A_{yf}^{-1}x > 0$ *has a solution for each* $y \in Y$,

(B4) $|D_y x| < x$ *has a solution for each* $y \in Y$,

(C1) $(\det A_{yz})(\det A_{y'z'}) > 0$ *for each* $y, z, y', z' \in Y$,

(C2) $(\det A_{yz})(\det A_{y'z}) > 0$ *for each* $y, y', z \in Y$ *such that* y *and* y' *differ just in one entry*,

(C3) $\rho_0(D_{yz}) < 1$ *for each* $y, z \in Y$,

(C4) $(A_c A_{yz}^{-1})_{ii} > \frac{1}{2}$ *for each* $y, z \in Y,\ i \in \{1,\dots,n\}$,

(C5) $(A_c A^{-1})_{ii} > \frac{1}{2}$ *for each* $A \in A^I,\ i \in \{1,\dots,n\}$,

(C6) $\det A \neq 0$ *for each matrix* $A \in A^I$ *having the following form for some* $y, z \in Y,\ k, m \in \{1,\dots,n\}$:

$$A_{ij} = \begin{cases} (A_{yz})_{ij} & \text{if either } i \neq k \text{ or} \\ & i = k \text{ and } j \in \{1,\dots,m-1\}, \\ (A_{-y,z})_{ij} & \text{if } i = k \text{ and } j \in \{m+1,\dots,n\}, \end{cases}$$

$$A_{km} \in [\underline{A}_{km}, \overline{A}_{km}],$$

(C7) $\det A \neq 0$ *for each* $A \in A^I$ *for which there exist* $k, m \in \{1,\dots,n\}$ *such that*

$$A_{ij} \in \{\underline{A}_{ij}, \overline{A}_{ij}\} \qquad \text{for each} \quad (i,j) \neq (k,m),$$
$$A_{km} \in [\underline{A}_{km}, \overline{A}_{km}].$$

COMMENT. Before embarking upon the proof, we shall comment on the items. Conditions (A1)–(A3) are related to Theorem 2.2, (B1)–(B4) follow from Theorem 1.2, and (C2)–(C7) are consequences of Baumann's criterion (C1) in [5], whose proof is included here for completeness. In (A1) and (A3), the word "unique" may be omitted (in both cases, if a solution exists for each $y \in Y$, then it is unique for each $y \in Y$). In (C3), ρ_0 cannot be replaced by ρ, as can be demonstrated with Hudak's example (Example 5.3 below), which is regular despite $\rho(D_{ee}) > 1$. (C6) and (C7) show that if A^I is singular, then it also contains a real singular matrix of a very special form. They cannot be weakened: neither regularity of all the A_{yz}'s nor regularity of all matrices A satisfying $A_{ij} \in \{\underline{A}_{ij}, \overline{A}_{ij}\}$ for each i, j is sufficient for regularity of A^I (counterexample: $[-E, E]$). Another set of regularity conditions may be obtained from Theorem 5.1 by applying it to the transpose A^{IT}. Some conditions are only theoretical; some [as (A1), (A3), (C3)] will be used later.

Proof. We shall carry out the proof along this scheme:

$$\begin{tikzcd}
 & (A1) \arrow[d, Rightarrow] & (A3) \arrow[d, Leftrightarrow] & (C3) \arrow[r, Rightarrow] & (C1) \arrow[dl, Rightarrow] & \\
(B3) \arrow[dr, Rightarrow] & (A2) \arrow[r, Rightarrow] & (R) \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & (C2) \arrow[l, Rightarrow] \arrow[r, Leftrightarrow] & (C4) \arrow[r, Leftrightarrow] & (C5) \\
(B4) \arrow[r, Leftrightarrow] & (B2) \arrow[u, Rightarrow] & (B1) \arrow[l, Rightarrow] & (C6) \arrow[u, Rightarrow] \arrow[r, Leftrightarrow] & (C7) &
\end{tikzcd}$$

(R) $\Rightarrow$ (A1): Follows from Theorem 2.2 with $b^I = [-e, e]$.

(A1) $\Rightarrow$ (A2): If x solves $A_{yz}x = y$, $T_z x \geqslant 0$, then for $x_1 = x^+$, $x_2 = x^-$ we have $A_{ye}x_1 - A_{yf}x_2 = A_{yz}x = y$, $x_1 \geqslant 0$, $x_2 \geqslant 0$.

(A2) $\Rightarrow$ (R) by contradiction: Assume A^I is not regular, so that $A^T p = 0$ for some $p \neq 0$. Setting $y = -\operatorname{sgn} p$, we have $A^T_{ye}p \geqslant A^T p = 0$, $A^T_{yf}p \leqslant A^T p = 0$, and $p^T y = -|p^T|e < 0$; hence $A_{ye}x_1 - A_{yf}x_2 = y$ cannot have a nonnegative solution, by the Farkas lemma.

(R) $\Rightarrow$ (A3): Let $y \in Y$. From Theorem 2.2 applied to systems $A^I x = [e_j, e_j]$, $j = 1, \ldots, n$, we obtain that for each j the equation $x = D_y|x| + A_c^{-1}e_j$ has a unique solution x^j. Defining a matrix B by $B_{\cdot j} = x^j$ $(j = 1, \ldots, n)$, we see that B solves uniquely the equation $B = D_y|B| + A_c^{-1}$.

(A3) $\Rightarrow$ (R) by contradiction: If A^I is singular, then $A^T p = 0$ for some $p \neq 0$; assume w.l.o.g. that $p_k < 0$ for some k. Then for $y = -\operatorname{sgn} p$ we get, as in the proof of the implication (A2) $\Rightarrow$ (R), that $A^T_{ye}p \geqslant 0$, $A^T_{yf}p \leqslant 0$, $p^T e_k < 0$; hence $A_{ye}x_1 - A_{yf}x_2 = e_k$ cannot have a nonnegative solution. However, from $B = D_y|B| + A_c^{-1}$ and its equivalent form $A_{ye}B^+ - A_{yf}B^- = E$ we see that such a solution exists $(x_1 = B^+_{\cdot k},\ x_2 = B^-_{\cdot k})$.

(R) $\Rightarrow$ (B1) was proved in Theorem 1.2.

(B1) $\Rightarrow$ (B2) follows from a well-known property of P-matrices [13].

(B2) $\Rightarrow$ (A2): If $A_{ye}^{-1}A_{yf}x > 0$, $x > 0$, then there exists a real number $\alpha > 0$ such that $\alpha A_{ye}^{-1}A_{yf}x + A_{ye}^{-1}y > 0$. Then for $x_2 = \alpha x$, $x_1 = A_{ye}^{-1}A_{yf}x_2 + A_{ye}^{-1}y$ we have $A_{ye}x_1 - A_{yf}x_2 = y$, $x_1 > 0$, $x_2 > 0$.

(B2) $\Leftrightarrow$ (B3): If $A_{ye}^{-1}A_{yf}x > 0$, $x > 0$, then for $x' = A_{yf}x$ we have $A_{ye}^{-1}x' > 0$, $A_{yf}^{-1}x' > 0$, and vice versa.

(B2) $\Leftrightarrow$ (B4): It can be easily verified that $A_{ye}^{-1}A_{yf} = (E - D_y)^{-1}(E + D_y) = 2(E - D_y)^{-1} - E$. Let $A_{ye}^{-1}A_{yf}x > 0$ and $x > 0$. Then, setting $x' = (E -$

$D_y)^{-1}x$, we have $A_{ye}^{-1}A_{yf}x = x' + D_yx' > 0$, $x = x' - D_yx' > 0$; hence $|D_yx'| < x'$. Conversely, if $|D_yx'| < x'$, then for $x = (E - D_y)x'$ we have $A_{ye}^{-1}A_{yf}x > 0$, $x > 0$.

(R) $\Rightarrow$ (C3) by contradiction: Assume that $D_{yz}x = A_c^{-1}T_y\Delta T_z x = \lambda x$ for some $x \neq 0$, λ real, $|\lambda| \geqslant 1$, $y, z \in Y$. Then $[A_c - (1/\lambda)T_y\Delta T_z]x = 0$; hence $A_c - (1/\lambda)T_y\Delta T_z$ is singular; but $|(1/\lambda)T_y\Delta T_z| \leqslant \Delta$; therefore $A_c - (1/\lambda)T_y\Delta T_z \in A^I$, a contradiction.

(C3) $\Rightarrow$ (C1): Consider, for given $y, z \in Y$, a function of one real variable $\varphi(\alpha) = \det(A_c - \alpha T_y\Delta T_z)$. If $\varphi(\alpha) = 0$ for some α, then $1/\alpha$ is a real eigenvalue of D_{yz}; hence $|\alpha| > 1$. Thus φ has no root in $[0,1]$, and hence $(\det A_c)(\det A_{yz}) = \varphi(0)\varphi(1) > 0$, so that each $\det A_{yz}$ is nonzero and has the same signature as $\det A_c$.

(C1) $\Rightarrow$ (C2) is obvious.

(C2) $\Rightarrow$ (R) by contradiction: Assume A^I is singular, $Ax = 0$ for some $A \in A^I$, and $x \neq 0$. Then from the construction given in the proof of Theorem 2.1 it follows that there exists a singular matrix $A_{t_0z} \in A^I$ for $z = \operatorname{sgn} x$ and some $t_0 \in [f, e]$. Since the function $\chi(t) = \det A_{tz}$, $t \in [f, e]$, is linear in each variable t_j, $j = 1, \ldots, n$, and satisfies $\chi(t_0) = 0$, there exist $y, y' \in Y$ such that $\det A_{yz} \leqslant 0$, $\det A_{y'z} \geqslant 0$. If $y = y'$, then $\det A_{yz} = 0$ and we are done. Otherwise there exists a sequence $y_0, y_1, \ldots, y_m$ of vectors from Y such that $y_0 = y$, $y_m = y'$, and each pair y_j, y_{j+1} differs in just one entry. Then there must exist a $k \in \{0, \ldots, m-1\}$ such that $(\det A_{y_kz})(\det A_{y_{k+1}z}) \leqslant 0$, for otherwise $(\det A_{yz})(\det A_{y'z}) > 0$, a contradiction.

(C2) $\Leftrightarrow$ (C4): If y and y' differ just in the ith entry, then $A_{y'z} = A_{yz} + (T_y - T_{y'})\,\Delta T_z = A_{yz} + (T_y - T_{y'})\,T_y(A_c - A_{yz}) = [E + (E - T_{y'}T_y)\,(A_cA_{yz}^{-1} - E)]A_{yz}$. Since $E - T_{y'}T_y$ has all entries zero except the iith, which is equal to 2, we obtain $\det[E + (E - T_{y'}T_y)(A_cA_{yz}^{-1} - E)] = 2(A_cA_{yz}^{-1})_{ii} - 1$, implying

$$\det A_{y'z} = \left[2\left(A_cA_{yz}^{-1}\right)_{ii} - 1\right]\det A_{yz}.$$

Hence $(\det A_{y'z})(\det A_{yz}) > 0$ if and only if $2(A_cA_{yz}^{-1})_{ii} - 1 > 0$.

(C4) $\Rightarrow$ (C5): Let $A \in A^I$ and $i \in \{1, \ldots, n\}$. Then from Theorem 2.2 applied to the system $A^Ix = [e_i, e_i]$ it follows that $A^{-1}e_i$ is a convex combination of $A_{yz}^{-1}e_i$ for (y, z) from some subset of $Y \times Y$. Hence also $A_cA^{-1}e_i$ is a convex combination of $A_cA_{yz}^{-1}e_i$ for those y, z, and since $(A_cA_{yz}^{-1})_{ii} > \frac{1}{2}$ for each $y, z \in Y$, it follows $(A_cA^{-1})_{ii} > \frac{1}{2}$.

(C5) $\Rightarrow$ (C4) is obvious.

(R) $\Rightarrow$ (C6) also holds obviously.

(C6) $\Rightarrow$ (C2) by contradiction: Assume $(\det A_{yz})(\det A_{y'z}) \leqslant 0$ for some $y, y' \in Y$ such that $y'_k = -y_k$, $y'_j = y_j$ for $j \neq k$. If $\det A_{yz} = 0$ or $\det A_{y'z} = 0$, then we are done. Thus assume that $(\det A_{yz})(\det A_{y'z}) < 0$, and define matrices A^s, $s = 0,\dots,n$, as follows: $A^0 = A_{y'z}$, and by induction $(A^s)_{ij} = (A^{s-1})_{ij}$ for $(i,j) \neq (k,s)$, $(A^s)_{ks} = (A_{yz})_{ks}$, $s = 1,\dots,n$. Then $A^n = A_{yz}$; therefore there must exist an $m \in \{1,\dots,n\}$ such that $(\det A^{m-1})(\det A^m) \leqslant 0$, for otherwise $(\det A_{y'z})(\det A_{yz}) > 0$. Now, since A^{m-1} and A^m differ just in the (k,m)th coefficient, we may choose a value from $[\underline{A}_{km}, \overline{A}_{km}]$ for which the resulting determinant is zero, thus obtaining a singular matrix of the form described in (C6).

(C6) $\Rightarrow$ (C7): If (C6) is satisfied, then from (C6) $\Rightarrow$ (C2), (C2) $\Rightarrow$ (R) we know that A^I is regular; hence (C7) holds.

(C7) $\Rightarrow$ (C6) holds obviously, since if A is of the form described in (C6), then $A_{ij} \in \{\underline{A}_{ij}, \overline{A}_{ij}\}$ for each $(i,j) \neq (k,m)$. ∎

Theorem 5.1 has some consequences concerning the matrix $D = |A_c^{-1}|\Delta$; assertion (i) was proved in [8] by another means.

Corollary 5.1. *Let A^I be a square interval matrix. Then we have*:

(i) *if* (3.6) *holds, then A^I is regular*;
(ii) *if A^I is regular and satisfies* (4.8), *then* (3.6) *holds*;
(iii) *if $D_{jj} \geqslant 1$ for some j, then A^I is singular.*

Proof. (i): If (3.6) holds, then in view of $|D_{yz}| \leqslant D$ we have (see [42]) that $\rho_0(D_{yz}) \leqslant \rho(D_{yz}) \leqslant \rho(D) < 1$ for each $y, z \in Y$. Hence A^I is regular by (C3).

(ii): If A^I is regular and satisfies (4.8) for some $y, z \in Y$, then, since $\rho(AB) = \rho(BA)$, we have $\rho(D) = \rho(T_z A_c^{-1} T_y \Delta) = \rho(D_{yz}) = \rho_0(D_{yz}) < 1$.

(iii): Assume that $D_{jj} \geqslant 1$ for some j and that A^I is regular. Then according to (B4), for $y = \operatorname{sgn}(A_c^{-1})^T e_j$ there exists a solution to $|D_y x| < x$, which then satisfies $x_j \leqslant (Dx)_j = (D_y x)_j < x_j$, a contradiction. ∎

We shall now approach the problem of testing regularity of a given interval matrix A^I. This problem seems to be generally difficult, and we shall describe a hierarchy of tests. The two simplest tests are provided by assertions (i), (iii) of Corollary 5.1. If neither of them is conclusive, then, before resorting to necessary and sufficient conditions, we propose testing singularity by the following algorithm, which, however, may fail. For a given matrix $A \in A^I$, denote

$$K_i = \left\{ j;\ (A_c - A)_{ij} A_{ji}^{-1} < 0 \right\}$$

and

$$\psi_i = \sum_{j \in K_i} (A_c - A)_{ij} A_{ji}^{-1}$$

$(i=1,\ldots,n)$. It would be more correct to write $K_i(A)$, $\psi_i(A)$, but we omit the argument for the sake of simplicity.

ALGORITHM 5.1 (Testing singularity; may fail).

Step 0. Select a matrix A such that $|A-A_c|=\Delta$ [recommended: $A_{ij} = \underline{A}_{ij}$ if $(A_c^{-1})_{ji} \geqslant 0$, and $A_{ij}=\overline{A}_{ij}$ otherwise].
Step 1. Compute A^{-1}.
Step 2. If $K_i=\varnothing$ for each i, terminate. The algorithm fails.
Step 3. Otherwise find k such that $K_k \neq \varnothing$ and $\psi_k=\min\{\psi_j;\ K_j\neq\varnothing\}$.
Step 4. If $\psi_k \leqslant -\frac{1}{2}$, terminate. A^I is singular.
Step 5. Otherwise set $A_{kj}:=(2A_c-A)_{kj}$ for each $j\in K_k$ and go to step 1.

The idea of the algorithm becomes clear if we realize that (1) in step 0 it starts from a matrix A satisfying $A_{ij}\in\{\underline{A}_{ij}, \overline{A}_{ij}\}$ for each $i,j\in\{1,\ldots,n\}$, and the updating in step 5 preserves this property; (2) the new matrix A' formed from the current matrix A by updating in step 5 satisfies

$$\det A' = (1+2\psi_k)\det A.$$

Thus if $\psi_k \leqslant -\frac{1}{2}$, then $(\det A')(\det A)\leqslant 0$, implying that A^I is singular; moreover, since A', A differ only in some elements of the kth row, a singular real matrix of the form (C7) may be found by a method similar to that used in the proof of (C6) $\Rightarrow$ (C2) above. If $\psi_k > -\frac{1}{2}$, then $|\det A'| < |\det A|$ and the choice of k made in step 3 guarantees the steepest descent of the absolute value of the determinant. Therefore the algorithm is finite. Since A', A differ only in elements of one row, $(A')^{-1}$ may be obtained from A^{-1} by using a single pivoting procedure. In our experience, the algorithm, especially when started as recommended in step 0, usually detects singularity in a few steps.

In the examples to follow, we write, as is customary, an interval matrix $A^I=[\underline{A},\overline{A}]$ as $A^I=([\underline{A}_{ij},\overline{A}_{ij}])_{i,j=1}^n$.

EXAMPLE 5.1. Consider the interval matrix

$$\begin{pmatrix} [2,3] & [4,5] & [1,2] \\ [-6,-5] & [-3,-2] & [3,4] \\ [-4,0] & [-5,-4] & [2,3] \end{pmatrix}.$$

Algorithm 5.1, started, for illustrative purposes, from a matrix A satisfying $A_{ij}=\overline{A}_{ij}$ if $(A_c^{-1})_{ji}\geqslant 0$ and $A_{ij}=\underline{A}_{ij}$ otherwise (contrary to the recommendation made in step 0), produces a sequence of matrices with determinant values 192, 48, 16, -2. Employing the method from the proof of (C6) $\Rightarrow$ (C2) in Theorem 5.1, we obtain a singular matrix of the form (C7):

$$\begin{pmatrix} 2 & 5 & 1.25 \\ -5 & -3 & 4 \\ -4 & -4 & 2 \end{pmatrix}.$$

A failure of the algorithm in step 2 cannot, however, be identified with regularity, as the following example shows.

EXAMPLE 5.2. Consider the obviously singular interval matrix

$$\begin{pmatrix} [0,4] & [1,1] \\ [1,1] & [0,4] \end{pmatrix}.$$

Here in step 0 the algorithm sets

$$A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and fails immediately in step 2, since no single coefficient change in this matrix can decrease the value of $|\det A|$.

If both the tests using the matrix D and Algorithm 5.1 fail, then we must turn to some necessary and sufficient conditions in Theorem 5.1. Among the options, we have evaluating 2^{2n-1} determinants (since $A_{-y,-z} = A_{yz}$) using Baumann's criterion (C1), or verifying (A2) by solving 2^{n-1} linear programs. According to our experience, we prefer the (still burdensome) criterion (A1). If x solves $A_{yz}x = y$, $T_z x \geqslant 0$, then $A_{-y,-z}(-x) = -y$, $T_{-z}(-x) \geqslant 0$; hence only 2^{n-1} systems need be tested, e.g. for those $y \in Y$ satisfying $y_n = 1$. For any such y, the system $A_{yz}x = y$, $T_z x \geqslant 0$ may be solved by Algorithm 3.1. If A^I is regular, then a solution is found in a finite number of steps; if A^I is singular, then the algorithm fails to work for some y: it either arrives at a singular matrix A_{yz}, or cycles (returns to the same z after several steps). In case of singularity, cycling often occurs as a result of a choice of the same k in step 3 of Algorithm 3.1 in two successive passes; then a singular matrix $A \in A^I$ may be found by the method used in the proof of Theorem 1.1. Of course, these procedures are of small practical importance.

EXAMPLE 5.3 (Hudak [17]). For the interval matrix

$$\begin{pmatrix} [31,41] & [-43,-43] & [49,49] \\ [-31,-31] & [31,41] & [-35,-35] \\ [25,25] & [-35,-35] & [28,38] \end{pmatrix}$$

one has $\rho(D) = 1.722$, $\max D_{jj} = 0.135$, and Algorithm 5.1 fails. Using (A1), for each $y \in Y$, $y_3 = 1$, Algorithm 3.1 requires solving one system of linear equations to find a solution. Therefore A^I is regular.

EXAMPLE 5.4. Let us apply the general method based on (A1) to the interval matrix from Example 5.1. For $y = (1, -1, 1)^T$ Algorithm 3.1 produces this sequence of z's and x's:

z_1	z_2	z_3	x_1	x_2	x_3
1	−1	1	−23.5	13.5	−19.5
−1	−1	1	0.4123	−0.1053	0.2895
1	−1	1	−23.5	13.5	−19.5

Hence cycling occurs and A^I is singular.

Finally, we shall again turn to the class of interval matrices with rank-one radius. In this case necessary and sufficient regularity conditions may be given a simpler form (cf. Hudak [17]) in terms of the matrix

$$A_0 = T_r A_c^{-1} T_q$$

(again, $A^I = [A_c - qr^T, A_c + qr^T]$).

THEOREM 5.2. *Let A^I be a square interval matrix with rank-one radius. Then the following conditions are equivalent:*

(R) *A^I is regular,*
(R1) *$z^T A_0 y < 1$ for each $y, z \in Y$,*
(R2) *$\|A_0 y\|_1 < 1$ for each $y \in Y$.*

Proof. (R) $\Rightarrow$ (R1) by contradiction: Assume $z^T A_0 y \geqslant 1$ for some $y, z \in Y$. Then for $x = A_c^{-1} T_y q$ we have $D_{yz} x = (z^T A_0 y) x$; hence $\rho_0(D_{yz}) \geqslant 1$, and A^I is singular due to (C3).

(R1) $\Rightarrow$ (R) by contradiction: If A^I is singular, then $\rho_0(D_{yz}) \geqslant 1$ for some $y, z \in Y$. Set $\lambda = \rho_0(D_{yz})$; then either λ or $-\lambda$ is an eigenvalue of D_{yz}. Assume w.l.o.g. (since $D_{-y,z} = -D_{yz}$) that $D_{yz}x = \lambda x$, $x \neq 0$; then $A_c^{-1}T_y q(r^T T_z x) = \lambda x$. Since $r^T T_z x \neq 0$ in view of $\lambda \neq 0$, premultiplying this equation by $r^T T_z$ yields $\lambda = r^T T_z A_c^{-1} T_y q = z^T A_0 y \geqslant 1$.

(R1) $\Rightarrow$ (R2): For each $y \in Y$, setting $z = \operatorname{sgn}(A_0 y)$, we obtain $\|A_0 y\|_1 = z^T A_0 y < 1$.

(R2) $\Rightarrow$ (R1): For each $z, y \in Y$ we have $z^T A_0 y \leqslant \|A_0 y\|_1 < 1$. ■

COROLLARY 5.2. *Let A^I be a square interval matrix with rank-one radius such that $e^T|A_0|e < 1$. Then A^I is regular.*

Proof. In this case, for each $y \in Y$ we have $\|A_0 y\|_1 \leqslant e^T|A_0|e < 1$; hence A^I is regular. ■

In analogy to Algorithm 5.1, criterion (R1) may be used for construction of an ascent algorithm for testing singularity [which sets $z := \operatorname{sgn}(A_0 y)$ or $y := \operatorname{sgn}(A_0^T z)$ until either $z^T A_0 y \geqslant 1$ or no further increase in this way is possible]. This algorithm may also fail; we will not go into details.

6. INVERSE INTERVAL MATRIX

In this last section we give some results about the inverse interval matrix $(A^I)^{-1} = [\underline{B}, \overline{B}]$ defined for a regular interval matrix A^I by

$$\underline{B} = \min\{A^{-1};\ A \in A^I\},$$

$$\overline{B} = \max\{A^{-1};\ A \in A^I\}$$

(min, max to be understood componentwise). As stated in Theorem 5.1, Assertion (A3), for each $y \in Y$ the equation

$$B = D_y|B| + A_c^{-1} \tag{6.1}$$

has a unique matrix solution B_y. We shall formulate our results in terms of these matrices B_y, $y \in Y$.

THEOREM 6.1. *Let A^I be regular. Then for each $A \in A^I$ there exist nonnegative diagonal matrices L_y, $y \in Y$, satisfying $\Sigma_{y \in Y} L_y = E$ such that*

$$A^{-1} = \sum_{y \in Y} B_y L_y.$$

Proof. Let $A \in A^I$ and $j \in \{1, \dots, n\}$. Since for each $y \in Y$, $(B_y)_{\cdot j}$ is equal to x_y for the system $A^I x = [e_j, e_j]$ [cf. Equation (2.6)], it follows from Theorem 2.2 that $(A^{-1})_{\cdot j}$ is equal to a convex combination of vectors $(B_y)_{\cdot j}$, $y \in Y$. Thus if we define $(L_y)_{jj}$ to be the coefficient of $(B_y)_{\cdot j}$ in this convex combination and $(L_y)_{ik} = 0$ for $i \neq k$, then $\Sigma_Y L_y = E$ and $A^{-1} = \Sigma_Y B_y L_y$. ■

THEOREM 6.2. *Let A^I be regular. Then for the inverse interval matrix $(A^I)^{-1} = [\underline{B}, \overline{B}]$ we have*

$$\underline{B} = \min\{ B_y;\ y \in Y \},$$

$$\overline{B} = \max\{ B_y;\ y \in Y \}. \tag{6.2}$$

Proof. Let $j \in \{1, \dots, n\}$. According to Theorem 6.1, for each $A \in A^I$, $(A^{-1})_{\cdot j}$ is equal to a convex combination of $(B_y)_{\cdot j}$, $y \in Y$; hence $\underline{B}_{\cdot j} \geqslant \min\{(B_y)_{\cdot j};\ y \in Y\}$. However, since $(B_y)_{\cdot j}$ is equal to x_y for the system $A^I x = [e_j, e_j]$, it follows from (2.5) that $(B_y)_{\cdot j} = (A_{yz}^{-1})_{\cdot j}$ for some $z \in Y$ (which may, in general, depend upon j); hence $\underline{B}_{\cdot j} = \min\{(B_y)_{\cdot j};\ y \in Y\}$. The proof for $\overline{B}$ is analogous. ■

Now assume we know an interval matrix $[\underset{\sim}{B}, \tilde{B}]$ such that $A^{-1} \in [\underset{\sim}{B}, \tilde{B}]$ for each $A \in A^I$ (i.e. $[\underline{B}, \overline{B}] \subset [\underset{\sim}{B}, \tilde{B}]$), and let

$$Y_0 = \bigcup_{i=1}^{n} [Y_i \cup (-Y_i)],$$

where Y_i, $i = 1, \dots, n$, are constructed as in Section 4. Then in the formulae (6.2), Y may be replaced by Y_0:

THEOREM 6.3. *Let A^I be regular, and let Y_0 be constructed as described. Then we have*

$$\underline{B} = \min\{ B_y;\ y \in Y_0 \},$$

$$\overline{B} = \max\{ B_y;\ y \in Y_0 \}.$$

Proof. The assertion is a direct consequence of Theorem 4.3 applied to systems $A^I x = [e_j, e_j]$, $j = 1, \ldots, n$, since Y_0 is constructed from an estimation of $(A^I)^{-1}$ and thus is independent of the right-hand sides. ■

As a consequence we obtain that if A^I is inverse stable, then at most $2n$ matrices B_y need be computed to determine $\underline{B}$, $\overline{B}$ [for $y \in \{y(1), \ldots, y(n), -y(1), \ldots, -y(n)\}$]. Although $(B_y)_{\cdot j}$ can always be computed by Algorithm 3.1 applied to the system $A^I x = [e_j, e_j]$, it requires a great amount of computation, and iterative methods are preferable in this case. The equation (6.1) may again be solved either by Jacobi iterations

$$B_y^0 = A_c^{-1},$$

$$B_y^{i+1} = D_y |B_y^i| + A_c^{-1} \qquad (i = 0, 1, \ldots)$$

or by Gauss-Seidel iterations

$$\tilde{B}_y^0 = A_c^{-1},$$

$$\tilde{B}_y^{i+1} = L_y |\tilde{B}_y^{i+1}| + Q_y |\tilde{B}_y^i| + A_c^{-1} \qquad (i = 0, 1, \ldots).$$

For the latter method, $\tilde{B}_y^{i+1}$ must be evaluated column by column. If $\rho(D) < 1$, then $B_y^i \to B_y$ and $\tilde{B}_y^i \to B_y$ due to Theorem 3.5.

EXAMPLE 6.1. The interval matrix

$$A^I = \begin{pmatrix} [2.215, 2.225] & [5.275, 5.285] & [3.465, 3.475] \\ [7.345, 7.355] & [2.895, 2.995] & [6.125, 6.225] \\ [4.565, 4.575] & [2.345, 2.355] & [6.455, 6.465] \end{pmatrix}$$

is regular $[\rho(D) = 0.022]$ and inverse stable. For each $y \in Y_0 = \{(1, -1, 1)^T, (-1, -1, 1)^T, (-1, 1, -1)^T, (1, 1, 1)^T\}$, computing B_y^2 was sufficient to obtain the inverse interval matrix with 10^{-4} accuracy:

$$(A^I)^{-1} = \begin{pmatrix} [-0.0630, -0.0519] & [\ \ 0.3251, \ \ 0.3368] & [-0.2968, -0.2743] \\ [\ \ 0.2446, \ \ 0.2465] & [\ \ 0.0179, \ \ 0.0208] & [-0.1527, -0.1482] \\ [-0.0531, -0.0443] & [-0.2461, -0.2363] & [\ \ 0.4025, \ \ 0.4206] \end{pmatrix}.$$

For interval matrices with rank-one radius, a simplified procedure may also be used (as before, $g_y = A_c^{-1} T_y q$):

THEOREM 6.4. *Let A^I be a regular interval matrix with rank-one radius. Then for each $y \in Y$*

$$B_y = g_y p_y^T + A_c^{-1},$$

where p_y is the unique solution of the equation

$$p^T = r^T \left| g_y p^T + A_c^{-1} \right|. \tag{6.3}$$

Proof. Let $y \in Y$. Put $p_y^T = r^T |B_y|$; then from (6.1) we have $B_y = g_y p_y^T + A_c^{-1}$. Taking the absolute values of both sides and premultiplying by r^T, we get that P_y solves (6.3). Let p be any solution to (6.3). Define $B = g_y p^T + A_c^{-1}$. Then from (6.3) we have $B = D_y |B| + A_c^{-1}$; hence $B = B_y$, implying $g_y p^T = g_y p_y^T$ and $p = p_y$. ∎

Thus in the case of interval matrices with rank-one radius, the matrix equation (6.1) may be replaced by the vector equation (6.3), which may be again solved by either of the two iterative methods.

Finally, only two matrices need be computed if A^I satisfies (4.8):

THEOREM 6.5. *Let A^I be a regular interval matrix satisfying* (4.8) *for some $y, z \in Y$. Then we have*

$$\underline{B} = \min\{B_{-y}, B_y\},$$

$$\overline{B} = \max\{B_{-y}, B_y\}.$$

In particular, if $z = e$, then $\underline{B} = B_{-y}$, $\overline{B} = B_y$.

Proof. Follows directly from Theorem 4.7. ∎

REMARK. As stated in the proof of Theorem 6.2, for each $y \in Y$, $j \in \{1,\dots,n\}$ there exists a $z \in Y$ such that $(B_y)_{\cdot j} = (A_{yz}^{-1})_{\cdot j}$. Hence Theorems 6.1 and 6.2 may be also formulated in terms of matrices A_{yz}^{-1}, $y, z \in Y$. If A^I is inverse stable, then using Theorem 4.5, for each $i, j \in \{1,\dots,n\}$ we may explicitly determine vectors $y, z \in Y$ for which $\underline{B}_{ij} = (A_{yz}^{-1})_{ij}$, etc.

The author thanks the referee for a number of suggestions that helped to improve the text of this paper.

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Received 1 September 1987; final manuscript accepted 20 October 1988