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A SHORT PROOF OF FINITENESS OF MURTY'S PRINCIPAL PIVOTING ALGORITHM

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We give a short proof of the finiteness of Murty's principal pivoting algorithm for solving the linear complementarity problem $y = Mz + q$, $y^{\mathrm{T}}z = 0$, $y \geqslant 0$, $z \geqslant 0$ with P-matrix M.

Key words: Linear complementarity problem, P-matrix.

Murty proposed in [2] a principal pivoting algorithm for solving a linear complementarity problem

$$y = Mz + q, \tag{1}$$

$$y^{\mathrm{T}}z = 0, \tag{2}$$

$$y \geqslant 0, \quad z \geqslant 0, \tag{3}$$

with an $n \times n$ P-matrix M. The algorithm starts with $y = q$, $z = 0$ and in the subsequent steps maintains (1) and (2) while working toward reaching (3). At each step, exactly one of the variables y_j, z_j is in the basis for $j = 1, \dots, n$. If in the current step the updated right-hand side vector $\bar{q}$ is not nonnegative, we compute

$$k = \min\{j;\ \bar{q}_j < 0\} \tag{4}$$

and introduce into basis the nonbasic variable in the pair y_k, z_k, with pivot in the kth row. Murty showed in [2] that if M is a P-matrix, then the pivot choice is correct (the pivot is then nonzero due to Tucker's result [4]) and proved that after a finite number of steps $\bar{q}$ becomes nonnegative and the algorithm terminates with a solution to (1)–(3). We shall reprove here the finiteness of his algorithm using this auxiliary result:

Lemma. *Let M be a P-matrix and let (y^1, z^1) and (y^2, z^2) satisfy* (1), (2), $(y^1, z^1) \neq (y^2, z^2)$. *Then there exists an $i \in \{1, \dots, n\}$ such that*

$$y_i^1 z_i^2 < 0 \quad \text{or} \quad y_i^2 z_i^1 < 0 \tag{5}$$

holds.

Proof. From $y^1 = Mz^1 + q$, $y^2 = Mz^2 + q$ it follows that $y^1 - y^2 = M(z^1 - z^2)$ and $z^1 \neq z^2$ (otherwise $(y^1, z^1) = (y^2, z^2)$); hence, due to a characterization of P-matrices by Fiedler and Pták [1], there exists an $i \in \{1, \ldots, n\}$ with $(y_i^1 - y_i^2)(z_i^1 - z_i^2) > 0$. But, from (2), we have $(y_i^1 - y_i^2)(z_i^1 - z_i^2) = -y_i^1 z_i^2 - y_i^2 z_i^1$, whence (5) holds. □

Theorem. *Let M be a P-matrix. Then each $k \in \{1, \ldots, n\}$ can be chosen by rule* (4) *at most 2^{n-k} times during the algorithm. Hence Murty's algorithm is finite, giving the unique solution to* (1)–(3) *in at most $2^n - 1$ steps.*

Proof. We shall prove the assertion by induction on $k = n, n-1, \ldots, 1$.

Case $k = n$. Assume n appears at least twice in the sequence of k's and let (y^1, z^1), (y^2, z^2) be the solutions corresponding to its first and second appearance, respectively. Then $y_i^1 z_i^2 \geq 0$, $y_i^2 z_i^1 \geq 0$ for each $i < n$ by (4) and one of the numbers $y_n^1 z_n^2$, $y_n^2 z_n^1$ is positive, while the second one is zero. Hence $y_i^1 z_i^2 \geq 0$, $y_i^2 z_i^1 \geq 0$ for each $i = 1, \ldots, n$, contradicting the lemma.

Case $k < n$. Consider any two consecutive appearances of k in the sequence and let (y^1, z^1), (y^2, z^2) be the respective solutions. Then $(y^1, z^1) \neq (y^2, z^2)$ and arguing as above, we get that $y_i^1 z_i^2 \geq 0$, $y_i^2 z_i^1 \geq 0$ for each $i \leq k$. Hence, according to the lemma, there exists an $i > k$ such that (5) holds, which means that the variable y_i is basic in one of the solutions (y^1, z^1), (y^2, z^2) while z_i is basic in the second one of them, implying that this i, $i > k$, must have been chosen by rule (4) in some of the pivot steps between the two appearances of k. This shows, in view of the inductive assumption, that k cannot appear more than $\sum_{i=k+1}^{n} 2^{n-i} + 1 = 2^{n-k}$ times, which concludes the inductive proof. Hence the algorithm is finite and cannot take more than $\sum_{k=1}^{n} 2^{n-k} = 2^n - 1$ steps. The obtained solution to (1)–(3) is unique since the existence of another solution would contradict (3) in view of (5). □

Note that in [3] Murty constructed for any $n \geq 1$ an LCP of size n for which the algorithm takes exactly $2^n - 1$ steps.

References

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